CLEMSON UNIVERSITY

Department of Mechanical Engineering

PhD Qualifiers exam (Spring 2010)

Mathematics

ID # ________________________

Solve all six problems.

Detail your approach to finding the solutions. We want to see your thought process even though you may not reach the final solution. Giving results without details will not be given credit. All problems carry equal weight.

This is a closed-book and closed calculator, two-hour examination.

No calculator of any kind allowed.

Problem 1: Find the 3 values of $(-8i)^{1/3}$, that is, the cube roots of $-8i$. Express each complex root in Cartesian form $x + iy$. Also exhibit the roots in the complex plane. Answers may include trigonometric functions of angles expressed as fractions of π.

Problem 2: Find the general solution of the ordinary differential equation

$$y'' - 4y' + 4y = x\cos(x)$$

Problem 3: Solve the system of equations using Jacobi's method with initial values $(x_1, x_2, x_3) = (0, 0, 0)$ and 2 iterations.

$$\begin{aligned}
10x_1 + x_2 - x_3 &= 20 \\
x_1 + 15x_2 + x_3 &= 13 \\
-x_1 + x_2 + 20x_3 &= 17
\end{aligned}$$

Repeat using the Gauss-Seidel method with the same initial values, but using only 1 iteration.

If the same number of iterations were carried out, which method would give a better approximation to the exact solution (Do not compute the exact solution).

Problem 4: Using the Laplace Transform method, solve the differential equation

$$\frac{dy}{dt} - y = e^{at},$$

for $t > 0$, and initial condition $y(0) = -1$.

The Laplace Transform of a function $f(t)$ is defined as a function of the variable s, by the integral

$$\tilde{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, d \ ,$$

and

$$s\tilde{f}(s) - f(0) = L\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty e^{-st} \frac{df(t)}{dt} dt\,.$$

The solution can be obtained by direct integration, partial fractions and the definition of the Laplace Transform.

Problem 5: Find a normal (perpendicular) vector to the plane formed by the three points (1,2,1), (-1,1,3), and (-2,-2,-2).

Then find the equation of the plane of the form $\mathrm{ax} + \mathrm{by} + \mathrm{cz} + \mathrm{d} = 0$, containing the three points.

Problem 6: The path of a particle moving in the *xy* plane is specified by the parametric equations $x = f(t)$, and $y = g(t)$, where t is time. It is required to determine the time at which the particles trajectory will intercept a curve specified by the equation $\varphi(x, y) = 0$. If the approximate time is t_0 and the approximate coordinates of the interception are (x_0, y_0), use first-order Taylor series expansions to linearize the relations,

$$
\begin{aligned}
&x_0 + \Delta x - f(t_0 + \Delta t) = 0, \\
&y_0 + \Delta y - g(t_0 + \Delta t) = 0, \\
&\varphi(x_0 + \Delta x, y_0 + \Delta y) = 0,
\end{aligned}
$$

solve for Δt, and show that this Newton iterative method yields a new estimate $t = t_0 + \Delta t$, where

$$
\Delta t = -\frac{\varphi_0 + (f_0 - x_0)\dfrac{\partial \varphi_0}{\partial x} + (g_0 - y_0)\dfrac{\partial \varphi_0}{\partial y}}{\dfrac{\partial f_0}{\partial t}\dfrac{\partial \varphi_0}{\partial x} + \dfrac{\partial g_0}{\partial t}\dfrac{\partial \varphi_0}{\partial y}}
$$

In the above equation, the zero subscripts indicate evaluation at $t = t_0$, $x = x_0$, and $y = y_0$.