VALUE-DRIVEN DESIGN UNDER UNCERTAINTY

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A MOTIVATING EXAMPLE (REQUIREMENT-BASED DESIGN)

- Resources $\times a$ $\times b$
- Allocation

$\times a_1$ $\times b_1$ — Subsystem 1

$$\min_{x_1} \; c_1 \cdot x_1$$
$$\text{s.t. } d_1 \cdot x_1 \le a_1,\quad e_1 \cdot x_1 \le b_1,\quad W_1 \cdot x_1 \le h_1$$

$\times a_2$ $\times b_2$ — Subsystem 2

$$\min_{x_2} \; c_2 \cdot x_2$$
$$\text{s.t. } d_2 \cdot x_2 \le a_2,\quad e_2 \cdot x_2 \le b_2,\quad W_2 \cdot x_2 \le h_2$$

Overall objective: $\min c_1 \cdot x_1 + c_2 \cdot x_2$

- Surplus

$$d_1 \cdot x_1^* = a_1, \quad e_1 \cdot x_1^* < b_1,$$
$$d_2 \cdot x_2^* < a_2, \quad e_2 \cdot x_2^* = b_2.$$

- Reallocation may yield a better solution!

$\times (a_1 + \epsilon)$ $\times (b_1 - \epsilon)$ — Subsystem 1

$\times (a_2 - \epsilon)$ $\times (b_2 + \epsilon)$ — Subsystem 2

- Potential drawback of Requirement-Based Design:
  1. Initial allocation may far from global optimality.
  2. How to reallocate resources?

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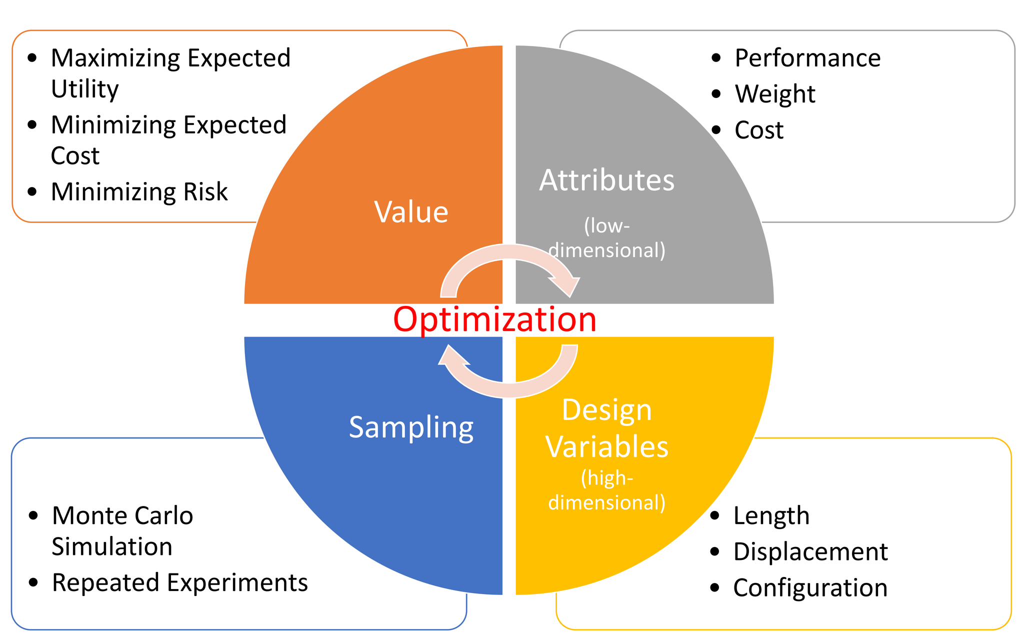
- Maximizing value instead of meeting requirements



CLASSIC TRUST-REGION ALGORITHM

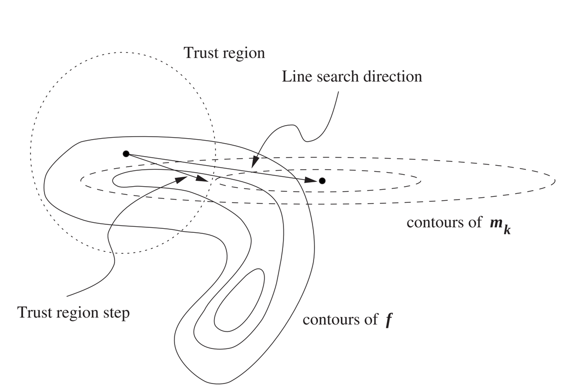
Given $\bar{\Delta} > 0$, $\Delta_0 \in (0, \bar{\Delta})$, $0 < \eta_1 < \eta_2 < 1$, x_0.

1: **for** $t = 0, 1, \ldots$ **do**
2: Solve subproblem

$$s^t := \arg\min_{\|s\| \le \Delta^t} \{ m^t(s) := \nabla f(x^t)^\top s + \frac{1}{2} s^\top B^t s \},$$

and evaluate $\rho^t = \frac{f(x^t) - f(x^t + s^t)}{-m^t(s^t)}$.

3: **if** $\rho^t < \eta_1$ **then** $\Delta^{t+1} = \Delta^t / 2$,
4: **else if** $\rho^t > \eta_2$ **then** $\Delta^{t+1} = \min(2\Delta^t, \bar{\Delta})$,
5: **else** $\Delta^{t+1} = \Delta^t$.
6: **end if**
7: **if** $\rho^t > \eta_1$ **then** $x^{t+1} = x^t + s^t$,
8: **else** $x^{t+1} = x^t$.
9: **end if**
10: **end for**



ALGORITHMIC FRAMEWORK

$$\min_{x_1, \ldots, x_I} \mathbb{E}[F(y_1(x_1, \xi_1), \ldots, y_I(x_I, \xi_I), \xi_0)]$$

Sample Average Approximation

$$\min_{x_1, \ldots, x_I} \frac{1}{N_y} \sum_{j=1}^{N_y} \left(\frac{1}{N_F} \sum_{k=1}^{N_F} F(y_1(x_1, \xi_1^j), \ldots, y_I(x_I, \xi_I^j), \xi_0^{jk}) \right)$$

$$\left(y_1(x_1, \xi_1^j), \ldots, y_I(x_I, \xi_I^j) \right), \quad j = 1, \ldots, N_y$$

$$\frac{1}{N_F} \sum_{k=1}^{N_F} F\left(y(x, \xi^j), \xi_0^{jk} \right),$$
$$\frac{1}{N_F} \sum_{k=1}^{N_F} \nabla_{y_i} F\left(y(x, \xi^j), \xi_0^{jk} \right),$$
$$\frac{1}{N_F} \sum_{k=1}^{N_F} \nabla^2_{y_i} F\left(y(x, \xi^j), \xi_0^{jk} \right)$$
$$i = 1, \ldots, I, \quad j = 1, \ldots, N_y$$

Subsampling N_y out of $N_1 \times \cdots N_I$

$$\left(y_1(x_1, \xi_1^{j_1}), \ldots, y_I(x_I, \xi_I^{j_I}) \right), \quad 1 \le j_i \le N_i, \; i = 1, \ldots, I$$

$$y_1(x_1, \xi_1^1), \cdots, y_1(x_1, \xi_1^{N_1}) \quad \ldots \quad y_I(x_I, \xi_I^1), \cdots, y_I(x_I, \xi_I^{N_I})$$

$$\nabla_{x_1} y_1(x_1, \xi_1^{j_1}), \; \nabla^2_{x_1} y_1(x_1, \xi_1^{j_1}), \; 1 \le j_1 \le N_1 \quad \ldots \quad \nabla_{x_I} y_I(x_I, \xi_I^{j_I}), \; \nabla^2_{x_I} y_I(x_I, \xi_I^{j_I}), \; 1 \le j_I \le N_I$$

Trust-Region subproblem 1 ... Trust-Region subproblem I

- Low-dimensional public data communication.
- Each subsystem generates samples and solves Trust-Region subproblems **in parallel**, coordinated by the system designer.

SAMPLE AVERAGE APPROXIMATION AND SAMPLE SIZE CONTROL

Statistical properties of SAA estimator

- Sufficient condition for consistency:

$$\theta \le \frac{N_i}{N_j} \le \theta^{-1}, \theta \in (0, 1)$$

- Necessary and sufficient condition for Central Limit Theorem:

$$N_y \le k N_i, i = 1, \ldots, I$$
$$\Rightarrow \text{convergence rate} \frac{1}{\sqrt{N_y}}$$

Optimal sample sizes allocation (minizing variance-cost ratio)

$$\min \sum_{i=1}^{I} C_i N_i + C_y N_F N_y \qquad \text{[minimize cost]}$$
$$\text{s.t. } \mathbb{V}\text{ar}[\hat{f}(x^{t+1}) - \hat{f}(x^t)] \le \Delta, \qquad [\Delta \sim \text{improvement}^2]$$
$$\theta \le \frac{N_i}{N_j} \le \theta^{-1}, \; \forall i, j = 1, \ldots, I, F, \qquad \text{[Consistency]}$$
$$1 \le N_y \le k N_i, \; \forall i = 1, \ldots, I, \qquad \text{[CLT]}$$
$$\Rightarrow N_i = \begin{cases} N_y / k, & i \in I^=, \\ \sqrt{\lambda / C_i} \sigma_i, & i \in I \setminus I^=. \end{cases}$$

STOCHATIC TRUST-REGION ALGORITHM WITH CUBIC REGULARIZATION

- Choose block-diagonal $\hat{B}^t = \mathrm{diag}(\hat{B}_1^t, \ldots, \hat{B}_I^t)$, then the i-th Trust-Region subproblem is

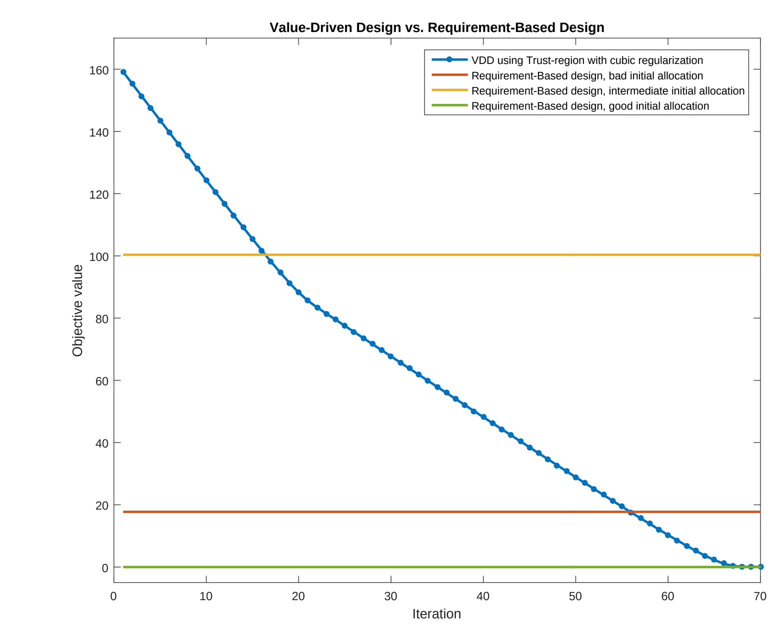
$$\min_{s_i \in \mathbb{R}^{n_i}} s_i^\top \hat{g}_i^t + \frac{1}{2} s_i^\top \hat{B}_i^t s_i + \frac{1}{3} \zeta^t \|s_i\|^3,$$

where $\hat{g}_i^t$, $\hat{B}_i^t$ are sample average approximations.

- Converges with probability one.
- Cubic regularization: better convergence rate $O(\epsilon^{-2/3})$ to obtain $\mathbb{E}[\|\hat{g}^t\|] < \epsilon$.

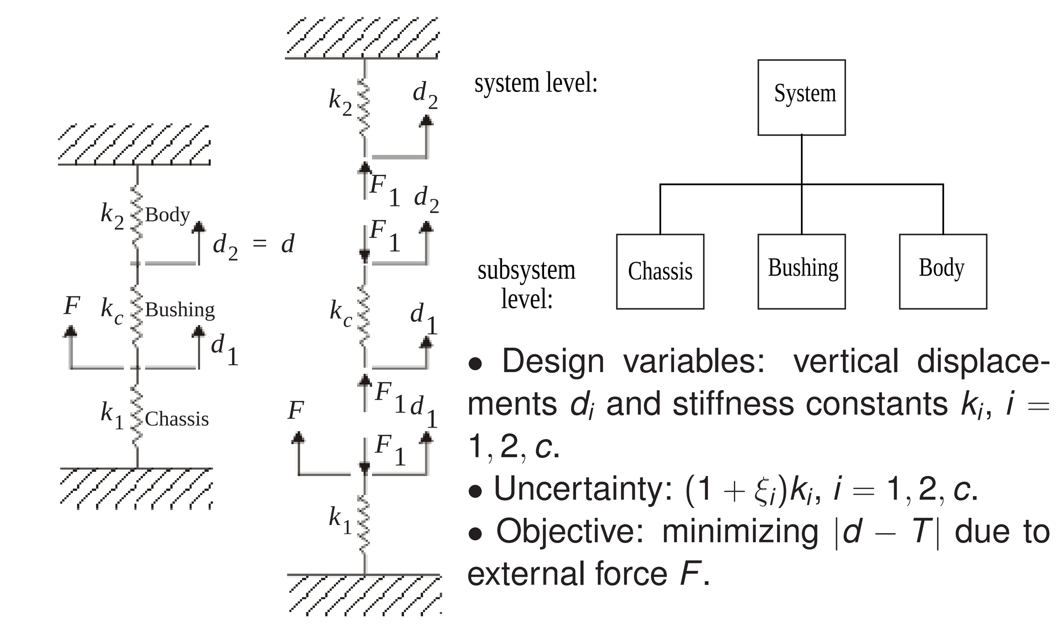
PRELIMINARY NUMERICAL RESULTS

I. Comparison with Requirement-Based Design

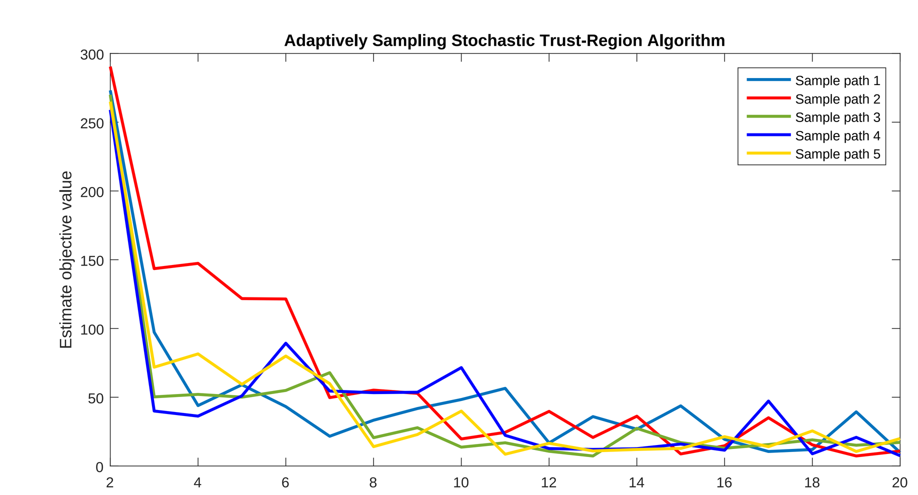


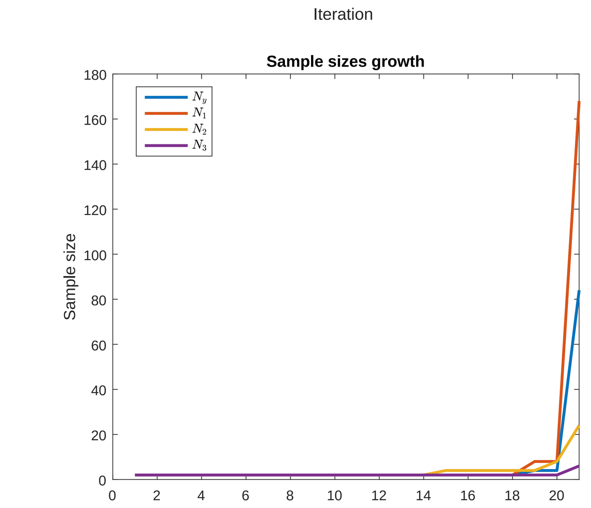
- Requirement-based design: sensitive to initial resource allocation
- Value-driven design: always find a good design

II. Three-spring systems design under uncertainty



- Design variables: vertical displacements d_i and stiffness constants k_i, $i = 1, 2, c$.
- Uncertainty: $(1 + \xi_i) k_i$, $i = 1, 2, c$.
- Objective: minimizing $|d - T|$ due to external force F.





- The algorithm finds a good design with a few iterations.
- Sample sizes grow fast when higher accuracy is needed.

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