STOCHASTIC OPTIMIZATION FOR DESIGN UNDER UNCERTAINTY WITH DEPENDENT PROBABILITY MEASURES

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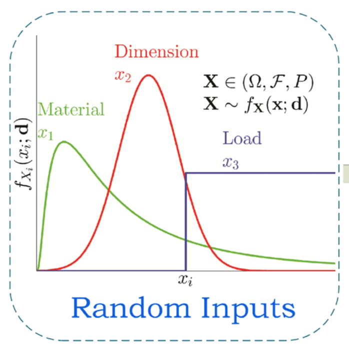
INTRODUCTION



- Moments

$\mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X})] := \int_{\mathbb{R}^N} y_l(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x}$

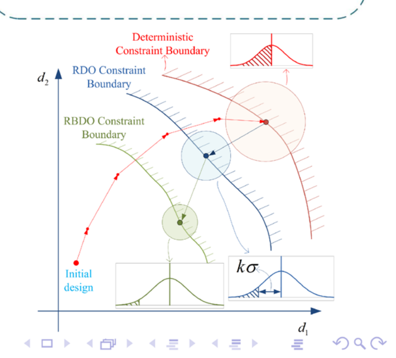
$\mathrm{var}_{\mathbf{d}}[y_l(\mathbf{X})] := \int_{\mathbb{R}^N} (y_l(\mathbf{x}) - \mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X})])^2 f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x}$

- Probability of Failure

$P_{\mathbf{d}}(\mathbf{X} \in \Omega_{F,l}) = \int_{\mathbb{R}^N} I_{\Omega_{F,l}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x}$

Design Under Uncertainty

- Robust Design Optimization (RDO): find an optimal design with reduced variability of the system performance, leading to an insensitive design.
- Reliability-Based Design Optimization (RBDO): find an optimal design with low probabilities of failure.



INTRODUCTION

- **RDO**

$$\begin{aligned} \min_{\mathbf{d}\in\mathcal{D}\subseteq\mathbb{R}^M} \quad & c_0(\mathbf{d}) := w_1 \mathbb{E}_{\mathbf{d}}[y_0(\mathbf{X})]/\mu_0^* + w_2\sqrt{\mathrm{var}_{\mathbf{d}}[y_0(\mathbf{X})]}/\sigma_0^*, \\ \text{subject to} \quad & c_l(\mathbf{d}) := \alpha_l\sqrt{\mathrm{var}_{\mathbf{d}}[y_l(\mathbf{X})]} - \mathbb{E}_{\mathbf{d}}[y_l(\mathbf{X})] \le 0, \\ & l = 1,\cdots,K, \\ & d_{k,L} \le d_k \le d_{k,U},\ k = 1,\cdots,M \end{aligned}$$

Needs: moments, design sensitivity of moments, optimization method

- **RBDO**

$$\begin{aligned} \min_{\mathbf{d}\in\mathcal{D}\subseteq\mathbb{R}^M} \quad & c_0(\mathbf{d}), \\ \text{subject to} \quad & c_l(\mathbf{d}) := P_{\mathbf{d}}[\mathbf{X} \in \Omega_{F,l}(\mathbf{d})] - p_l \le 0,\ l = 1,\cdots,K, \\ & d_{k,L} \le d_k \le d_{k,U},\ k = 1,\cdots,M, \end{aligned}$$

Needs: reliability, design sensitivity of reliability, optimization method

INTRODUCTION

Project Goal

Create new theoretical foundations, accompanied by robust numerical algorithms, for design optimization of high-dimensional complex systems subject to random input following an arbitrary dependent probability measure

Project Objectives

- Develop generalized ADD and PDD approximations for high-dimensional stochastic response functions (Year 1)
- Generate new PDD-based formulae and scalable algorithms for moments, reliability, and design sensitivities (Year 2)
- Develop fast and reliable gradient-based algorithms for RDO and RBDO (Year 3)

(Project duration: August 15, 2015 - July 31, 2018)

STOCHASTIC DESIGN OPTIMIZATION

Input $\mathbf{X} \in \mathbb{R}^N \rightarrow$ **COMPLEX SYSTEM** $\rightarrow$ Output $y(\mathbf{X};\mathbf{d}) \in \mathcal{L}_2(\mathbb{R})$

$\mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x};\mathbf{d})$; arbitrary but with grid-closed support

- **Generalized ANOVA Dim. Decomposition (ADD)**

$$\begin{aligned} y(\mathbf{X};\mathbf{d}) &= \sum_{u\subseteq\{1,\cdots,N\}} y_{u,G}(\mathbf{X}_u) \\ y_{\emptyset,G} &= \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x}, \\ y_{u,G}(\mathbf{x}_u) &= \int_{\mathbb{R}^{N-|u|}} y(\mathbf{x}) f_{\mathbf{X}_{-u}}(\mathbf{x}_{-u};\mathbf{d}) d\mathbf{x}_{-u} - \sum_{v\subset u} y_{v,G}(\mathbf{x}_v) - \\ & \sum_{\substack{\emptyset\ne v\subseteq\{1,\cdots,N\} \\ v\cap u\ne\emptyset,\, v\not\subseteq u}} \int_{\mathbb{R}^{|v\cap -u|}} y_{v,G}(\mathbf{x}_v) f_{\mathbf{X}_{v\cap -u}}(\mathbf{x}_{v\cap -u};\mathbf{d}) d\mathbf{x}_{v\cap -u} \end{aligned}$$

Generalized ADD component functions have zero means and are hierarchically orthogonal

STOCHASTIC DESIGN OPTIMIZATION

$\{\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u),\ \mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}\} \rightarrow$ Mult. Ortho. Poly. System (MOPS)

- **Gen. Polynomial Dim. Decomposition (PDD)**

$$y(\mathbf{X}) = y_{\emptyset,G}(\mathbf{d}) + \sum_{\emptyset\ne u\subseteq\{1,\cdots,N\}} \sum_{\mathbf{j}_{|u|}\in\mathbb{N}^{|u|}} C_{u\mathbf{j}_{|u|}}(\mathbf{d}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})$$

- **S-variate, mth-order PDD Approximation**

$$\tilde{y}_{S,m}(\mathbf{X}) = y_{\emptyset,G}(\mathbf{d}) + \sum_{\substack{\emptyset\ne u\subseteq\{1,\cdots,N\} \\ 1\le|u|\le S}} \sum_{k=|u|}^{m} \sum_{\substack{\|\mathbf{j}_{|u|}\|=k \\ \prod_{i=1}^{|u|} j_i\ne 0}} C_{u\mathbf{j}_{|u|}}(\mathbf{d}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

The coefficients $C_{u\mathbf{j}_{|u|}}(\mathbf{d})$ can be estimated by dimension-reduction integration and solution of linear equations

STOCHASTIC DESIGN OPTIMIZATION

- **Two Important Properties of MOPS**

$$\mathbb{E}_{\mathbf{d}}\left[\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\right] = 0$$

$$\mathbb{E}_{\mathbf{d}}\left[\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\psi_{v\mathbf{j}_{|v|}}(\mathbf{X}_v;\mathbf{d})\right] = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } v \subset u. \end{cases}$$

- **Second-Moment Statistics**

$$\mathbb{E}_{\mathbf{d}}[\tilde{y}_{S,m}(\mathbf{X})] = y_{\emptyset,G}(\mathbf{d})$$

$$\mathrm{var}_{\mathbf{d}}[\tilde{y}_{S,m}(\mathbf{X})] = \sum_{\substack{\emptyset\ne u\subseteq\{1,\cdots,N\} \\ 1\le|u|\le S}} \mathbb{E}_{\mathbf{d}}[y_u^2(\mathbf{X}_u)] + \sum_{\substack{\emptyset\ne u,v\subseteq\{1,\cdots,N\} \\ 1\le|u|\le S,\ u\not\subseteq v\not\subseteq u}} \mathbb{E}_{\mathbf{d}}[y_u(\mathbf{X}_u)y_v(\mathbf{X}_v)]$$

When $S \rightarrow N$, $m \rightarrow \infty$, $\tilde{y}_{S,m} \rightarrow y$ in the *m.s.* sense

STOCHASTIC DESIGN OPTIMIZATION

Given a failure domain $\Omega_F \subset \mathbb{R}^N$ and indicator function I_{Ω_F}, the failure probability

$$P_F(\mathbf{d}) := P_{\mathbf{d}}[\mathbf{X} \in \Omega_F] = \mathbb{E}_{\mathbf{d}}[I_{\Omega_F}(\mathbf{X})].$$

- **Reliability Analysis**

$$\tilde{P}_{F,S,m}(\mathbf{d}) = \mathbb{E}_{\mathbf{d}}\left[I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X})\right] = \lim_{L\to\infty} \frac{1}{L}\sum_{l=1}^{L} I_{\tilde{\Omega}_{F,S,m}}(\mathbf{x}^{(l)})$$

$$I_{\tilde{\Omega}_{F,S,m}}(\mathbf{x}^{(l)}) = \begin{cases} 1 & \mathbf{x}^{(l)} \in \tilde{\Omega}_{F,S,m}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{\Omega}_{F,S,m} := \begin{cases} \{\mathbf{x} : \tilde{y}_{S,m}(\mathbf{x}) < 0\}, & \text{component}, \\ \{\mathbf{x} : \cup_i \tilde{y}_{i,S,m}(\mathbf{x}) < 0\}, & \text{series system}, \\ \{\mathbf{x} : \cap_i \tilde{y}_{i,S,m}(\mathbf{x}) < 0\}, & \text{parallel system}. \end{cases}$$

The sample size (L) can be arbitrarily large in the PDD method

STOCHASTIC DESIGN OPTIMIZATION

- **Score Function**

$$\begin{aligned} \frac{\partial \mathbb{E}_{\mathbf{d}}[y^r(\mathbf{X})]}{\partial d_k} &= \int_{\mathbb{R}^N} y^r(\mathbf{x}) \overbrace{\frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\mathbf{d})}{\partial d_k}}^{=: s_k(\mathbf{x};\mathbf{d})} f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) d\mathbf{x} \\ &=: \mathbb{E}_{\mathbf{d}}[y^r(\mathbf{X}) s_k(\mathbf{X};\mathbf{d})] \end{aligned}$$

- **Design Sensitivity of rth-order Moment**

$$\frac{\partial \mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}^r(\mathbf{X})\right]}{\partial d_k} = \mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}^r(\mathbf{X}) s_k(\mathbf{X};\mathbf{d})\right]$$

- **Design Sensitivity of Failure Probability**

$$\frac{\partial \tilde{P}_{F,S,m}(\mathbf{d})}{\partial d_k} = \mathbb{E}_{\mathbf{d}}\left[I_{\tilde{\Omega}_{F,S,m}}(\mathbf{X}) s_k(\mathbf{X};\mathbf{d})\right]$$

No additional response evaluations are required for sensitivities

STOCHASTIC DESIGN OPTIMIZATION

- **Computational Complexity**

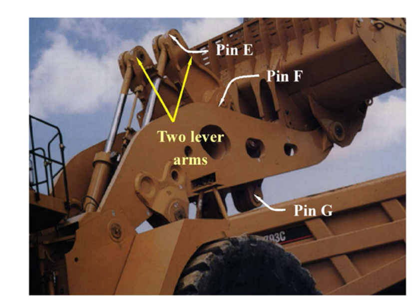
N = no. of random variables; m = max. order of polynomials; S = max. degree of interaction

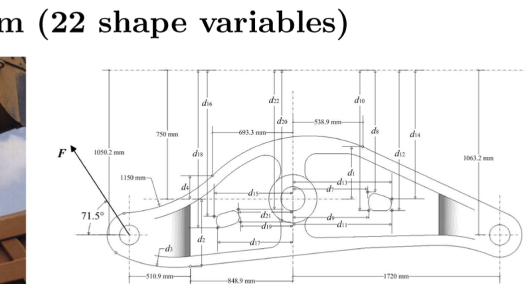
Approximation	No. of Func. Eval.	Cost Scaling
Univariate ($S = 1$)	$N(m+1)+1$	Linear
Bivariate ($S = 2$)	$N(N-1)(m+1)^2/2 + N(m+1)+1$	Quadratic
...	...	...
S-variate	$\sum_{k=0}^{S} \binom{N}{S-k}(m+1)^{S-k}$	Sth degree

- Cost increases polynomially, not exponentially; deflates curse of dimensionality to some extent
- Adaptive PDD further reduces the computational effort

RESULTS

- **RDO of a Lever Arm (22 shape variables)**





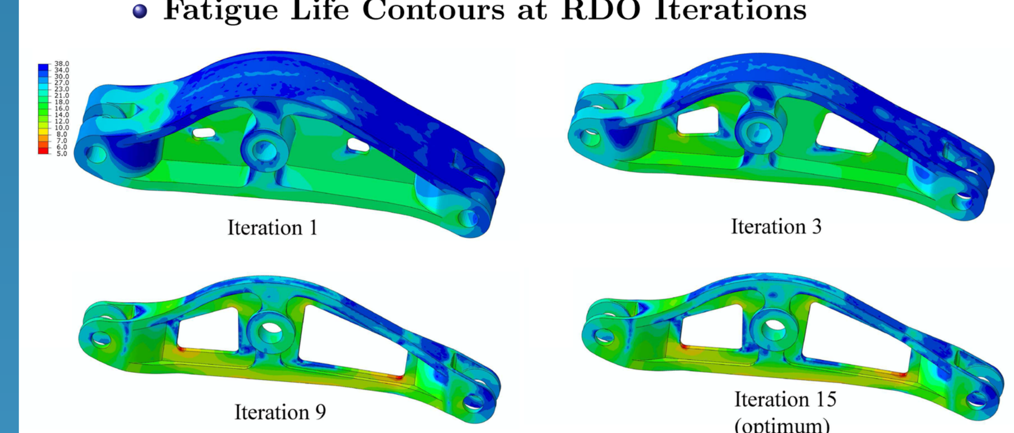
$$\begin{aligned} \min_{\mathbf{d}\in\mathcal{D}} \quad & c_0(\mathbf{d}) = 0.5\frac{\mathbb{E}_{\mathbf{d}}[y_0(\mathbf{X})]}{\mathbb{E}_{\mathbf{d}_0}[y_0(\mathbf{X})]} + 0.5\frac{\sqrt{\mathrm{var}_{\mathbf{d}}[y_0(\mathbf{X})]}}{\sqrt{\mathrm{var}_{\mathbf{d}_0}[y_0(\mathbf{X})]}}, \\ \text{subject to} \quad & c_1(\mathbf{d}) = 3\sqrt{\mathrm{var}_{\mathbf{d}}[y_1(\mathbf{X})]} - \mathbb{E}_{\mathbf{d}}[y_1(\mathbf{X})] \le 0 \end{aligned}$$

$$y_0(\mathbf{X}) = \rho\int_{\mathcal{D}'(\mathbf{X})} d\mathcal{D}';\ y_1(\mathbf{X}) = N_{\min}(\mathbf{X}) - N_c$$

$X_i \sim$ truncated Gaussian variables; $d_i = \mathbb{E}[X_i]$; $i = 1,\cdots,22$

RESULTS

- **Fatigue Life Contours at RDO Iterations**

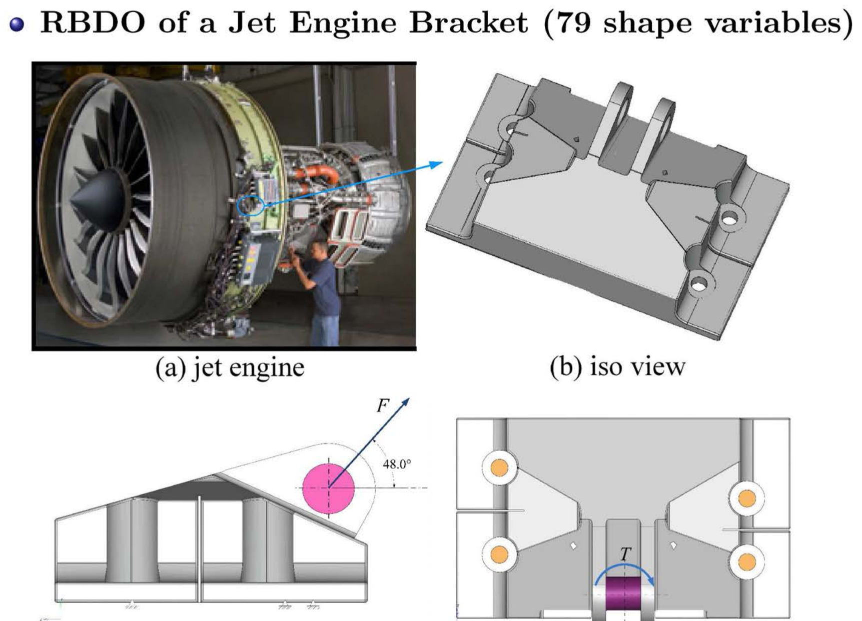


Summary

- Optimal mass: 1263 kg (79% reduction of initial mass)
- Required 15 iterations and 675 FEA

RESULTS

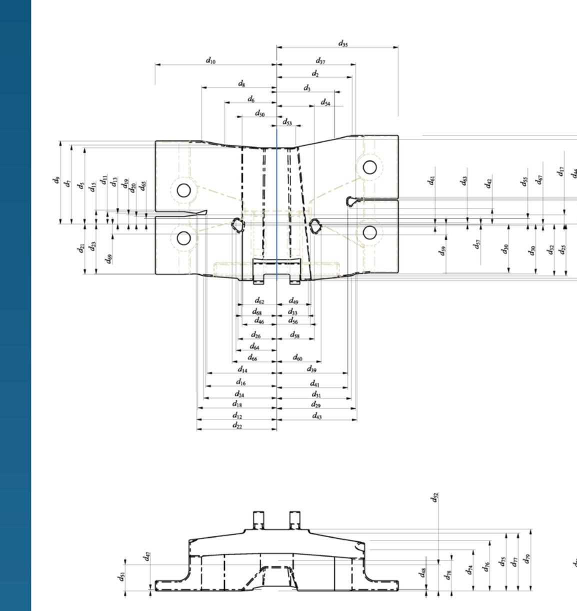
- **RBDO of a Jet Engine Bracket (79 shape variables)**



(a) jet engine (b) iso view (c) side view (d) top view

RESULTS

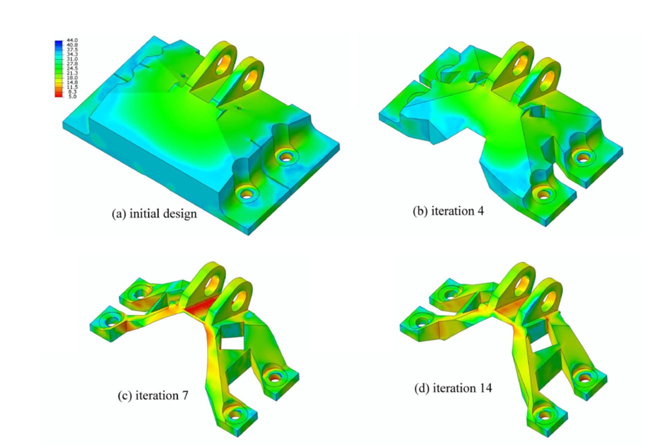
- **Design Parametrization**



$$\begin{aligned} \min_{\mathbf{d}\in\mathcal{D}} \quad & c_0(\mathbf{d}) = \rho\int_{\mathcal{D}'(\mathbf{d})} d\mathcal{D}', \\ \text{subject to} \quad & c_1(\mathbf{d}) = P_{\mathbf{d}}[y_1(\mathbf{X}) < 0] \\ & \quad -\Phi(-3) \le 0, \\ & d_{i,L} \le d_i \le d_{i,U}, \\ & i = 1,\dots,79 \\ & y_1(\mathbf{X}) = N_{\min}(\mathbf{X}) - N_c \end{aligned}$$

RESULTS

- **Fatigue Life Contours at RBDO Iterations**



Summary

- Optimal mass: 0.4904 kg (84% reduction of initial mass)
- Required 14 iterations and 2808 FEA

BROADER IMPACT

Fundamental Aspects

- Novel computational methods for complex systems design subject to uncertainties
- Many stochastic problems in basic & applied sciences will be solved

Industrial Relevance

- Improved design of civil, automotive, and aerospace infrastructures
- Applications: durability, noise-vibration-harshness, creep, and crashworthiness

Knowledge Transfer

- Symposia on stochastic design optimization
- Peer-reviewed journal publications & presentations at major conferences
- Collaboration with industry (Caterpillar)

Educational Impact

- One Ph.D. student
- Software tools in upgrading CAE & stochastic-mechanics courses
- Publication of courseware on reliability and robustness analyses and design

PUBLICATIONS

- Ren, X. and Rahman, S., "Stochastic Design Optimization Accounting for Structural and Distributional Design Variables," submitted to *AIAA Journal*, 2015.
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