Improving Virtual Product Development Through Isogeometric Analysis Based on Rational Triangular Bézier Splines

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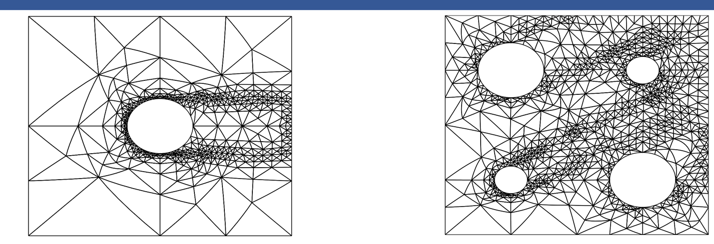
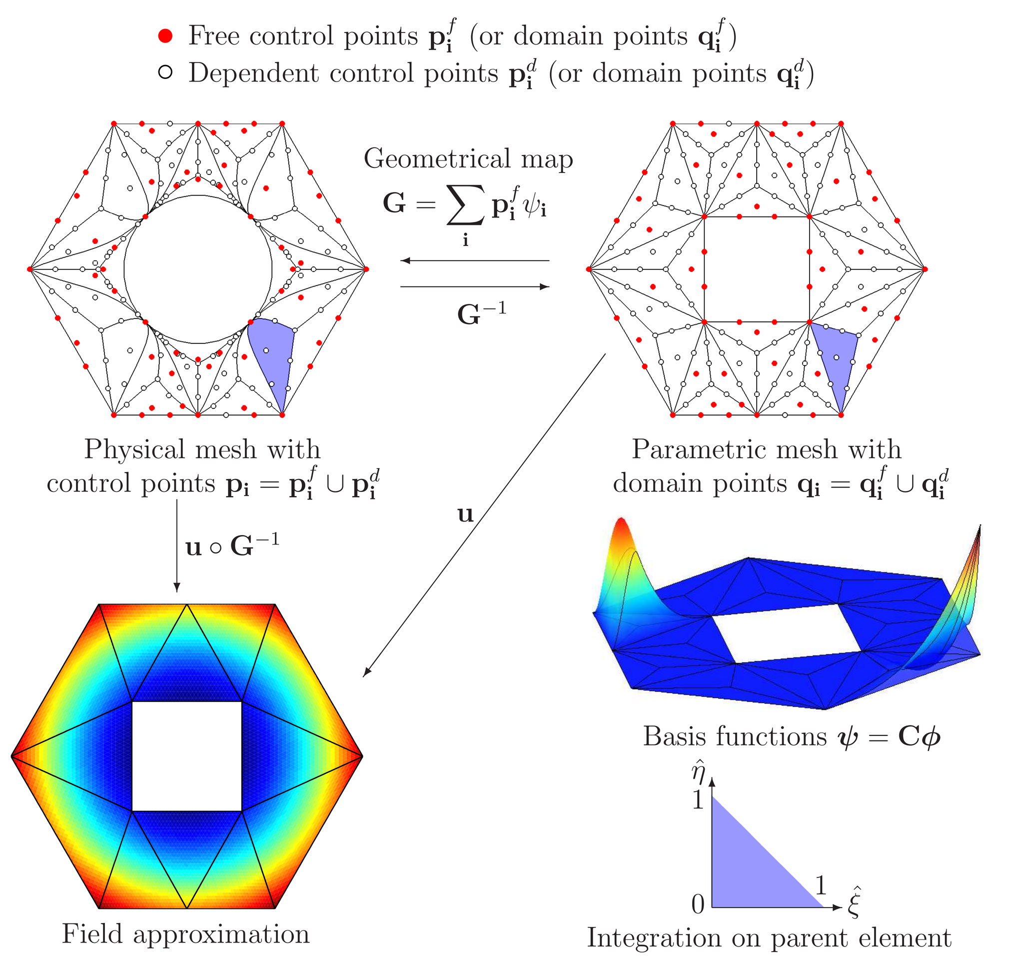
Motivation: integrating CAD and FEA

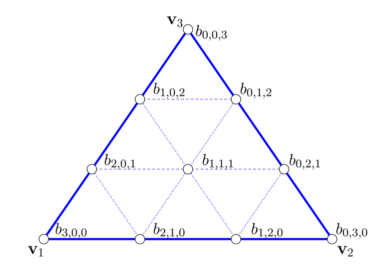
- Exact representation of CAD geometry
- Applicable to complex topologies
- High order of smoothness
- Automatic parametrization
- Ease of local refinement



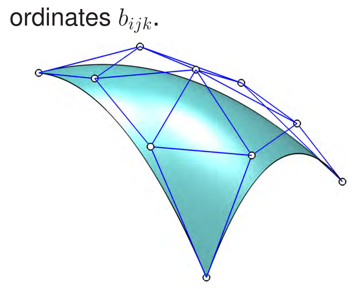
Basic approach: IGA on triangulations in 2D



- Bivariate basis function
$$B_{ijk,d}(\boldsymbol{\xi}) = \frac{d!}{i!j!k!} u^i v^j w^k$$
- Triangular Bézier patch:
$$\mathbf{b}(\boldsymbol{\xi}) = \sum_{i+j+k=d} \mathbf{p}_{ijk} B_{ijk,d}(\boldsymbol{\xi})$$
- Triangular Bézier Spline (TBS)
$$f(\boldsymbol{\xi}) = \sum_{i+j+k=d} b_{ijk} B_{ijk,d}(\boldsymbol{\xi})$$



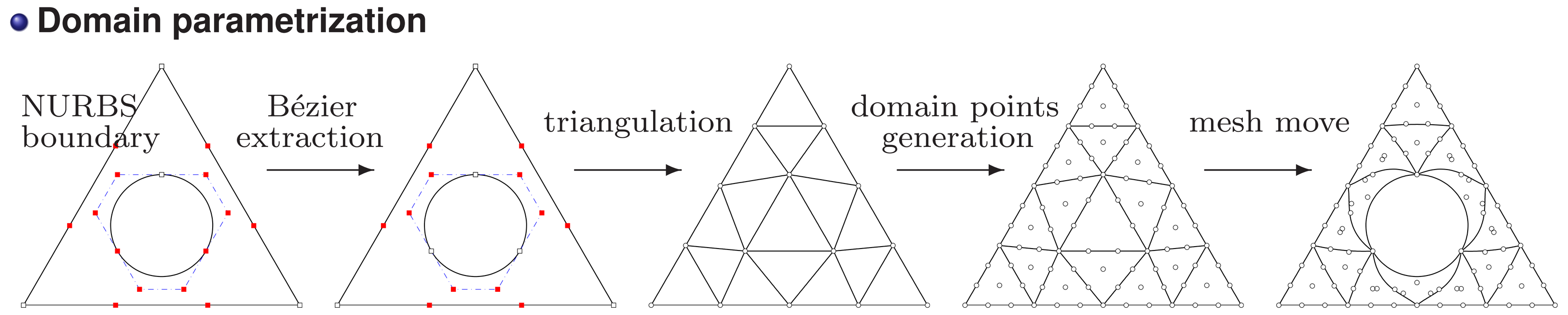
(a) Domain points of the Bézier ordinates b_{ijk}.



(b) Triangular Bézier patch $\mathbf{b}(\boldsymbol{\xi})$.

Construction of C^r basis

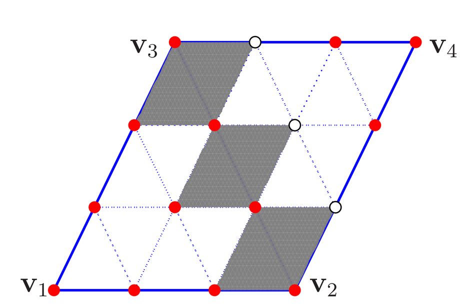
- **Domain parametrization**



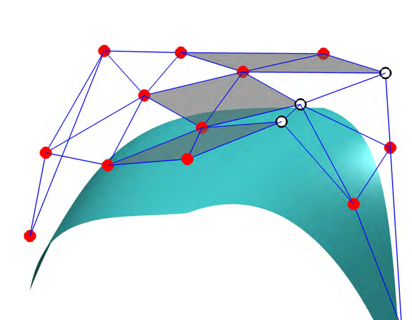
- **Smoothness conditions**

Let $\boldsymbol{\xi}$ be the barycentric coordinates of $\mathbf{v}_4$ w.r.t. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then two patches will be C^r continuous iff
$$\tilde{b}_{\rho,j,k} = \sum_{\mu+\nu+\kappa=\rho} b_{\mu,k+\nu,j+\kappa} B^{\rho}_{\mu\nu\kappa}(\mathbf{v}_4)$$
for all $\rho \le r$ and $\rho + j + k = d$.

- **Macro-element spaces**
  - Polynomial macro-element spaces: $d \ge 4r+1$, e.g., $\mathcal{S}_5^1(T), \mathcal{S}_5^{1,2}(T), \mathcal{S}_9^{2,4}(T)$
  - PS macro-element spaces: $\begin{cases} d \ge \frac{9r-1}{4}, & \text{for } r \text{ odd, e.g., } \mathcal{S}_2^1(T_{ps}) \\ d \ge \frac{9r+4}{4}, & \text{for } r \text{ even, e.g., } \mathcal{S}_5^{2,3}(T_{ps}) \end{cases}$
  - CT macro-element spaces: $\begin{cases} d \ge 3r, & \text{for } r \text{ odd, e.g., } \mathcal{S}_3^1(T_{ct}) \\ d \ge 3r+1, & \text{for } r \text{ even, e.g., } \mathcal{S}_7^{2,3}(T_{ct}) \end{cases}$
- **Construction of C^r basis**
  - Direct construction in macro-element spaces
    1. Directly select a minimum number of free nodes $\mathbf{b}^f$ according to the connectivity of the elements
    2. Determine the values of other dependent nodes by the continuity condition $\mathbf{b}_{\mathcal{D}} = \mathbf{C}^T \mathbf{b}^f$
  - Gaussian elimination
    1. Apply the continuity condition on all adjoining element pairs $\mathbf{A}\mathbf{b}_{\mathcal{D}} = 0$
    2. Get rid of redundant continuity constraints by Gaussian elimination $\mathbf{b}_{\mathcal{D}} = \mathbf{C}^T \mathbf{b}^f$



(a) Two domain triangles with C^1 constraints.

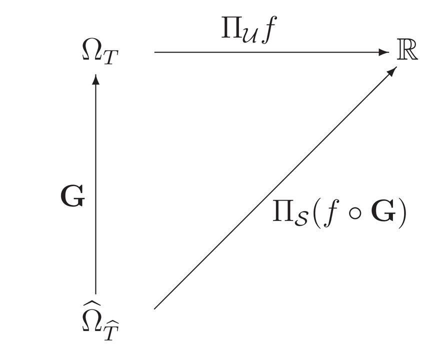


(b) Two Bézier patches with C^1 continuity.

Conditions for optimal convergence

- **Approximation in rTBS space**
  - Parametric domain

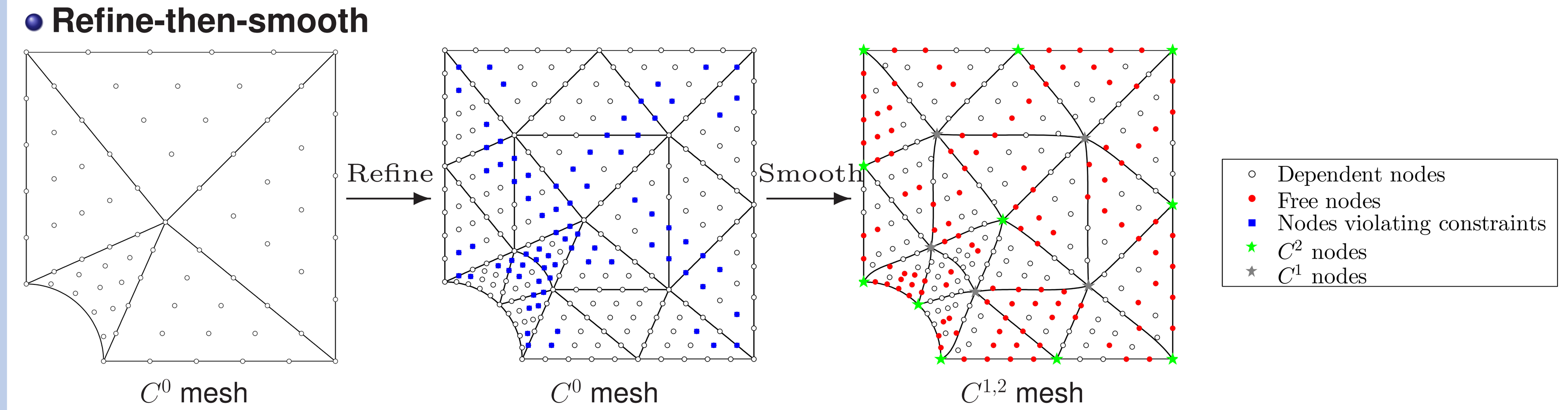
    If there exists a space $\mathcal{S}_d^r(\widehat{T})$ with a set of stable local basis, then for every $f \in W^{k,d+1}$, there exists a spline $s \in \mathcal{S}_d^r(\widehat{T})$ such that
$$|f-s|_{W^{k,d+1}(\widehat{\Omega})} \le C h_{\widehat{T}}^{d+1-k} |f|_{W^{d+1,d+1}(\widehat{\Omega})}, \qquad 0 \le k \le d. \qquad \text{[Lai, M.J., et al., 2007]}$$
  - Physical domain
$$|f - \Pi_{\mathcal{U}} f|_{H^k(T)} \le C_w h_T^{d+1-k} \sum_{i=0}^{d+1} \|\nabla \mathbf{G}\|^{i-d-1}_{L^\infty(\mathbf{G}^{-1})} |f|_{H^i(T)}, \ \forall f \in H^{d+1}(\Omega)$$



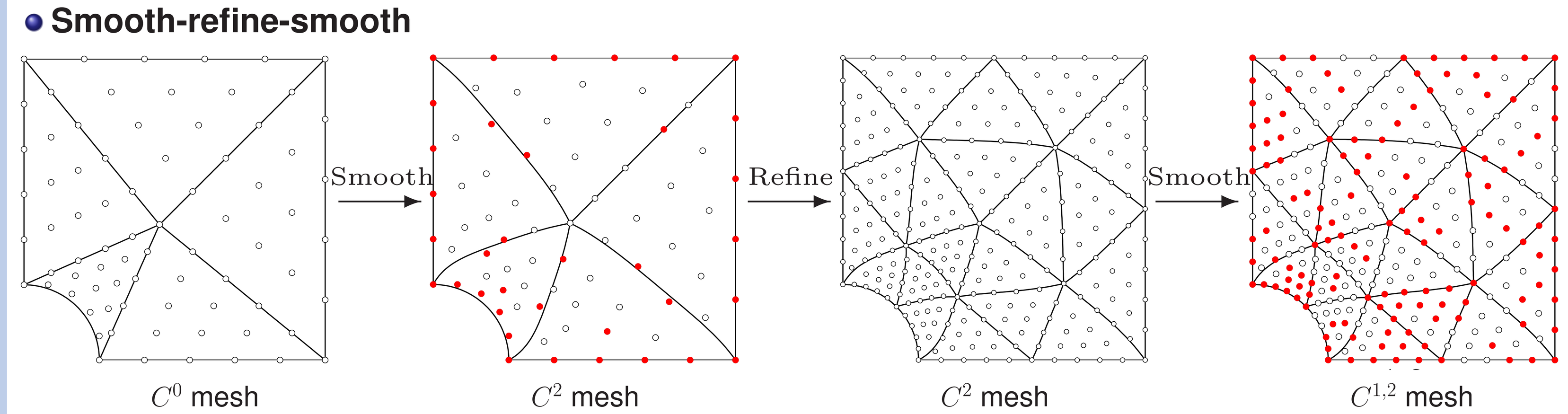
- **Conditions for optimal convergence**
  - Spaces with stable local basis: all C^0 spaces and $C^r, C^{r,\rho}$ macro-element spaces
  - Same geometric map $\mathbf{G}$ during refinement $\mathbf{G}(\boldsymbol{\xi}) = \sum_i^m \mathbf{p}_i^f \psi_i(\boldsymbol{\xi}) = \sum_j^n \mathbf{p}_j \phi_j(\boldsymbol{\xi})$

Refinement procedures

- **Refine-then-smooth**



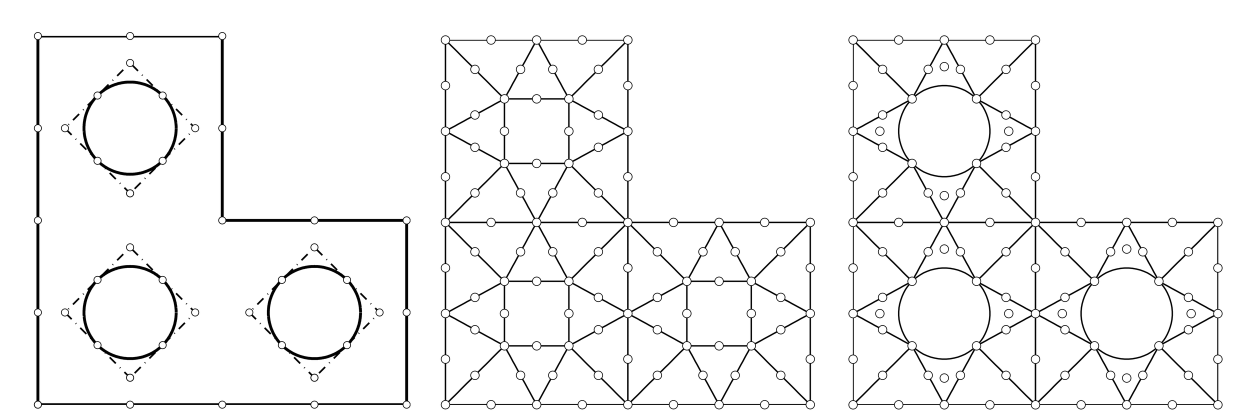
  - Initial mesh does not have enough smoothness.
  - Some control points (blue squares) need to be relocated after refinement to satisfy the continuity constraints.
  - Relocation of the control points leads to a change of the geometric map.
- **Smooth-refine-smooth**



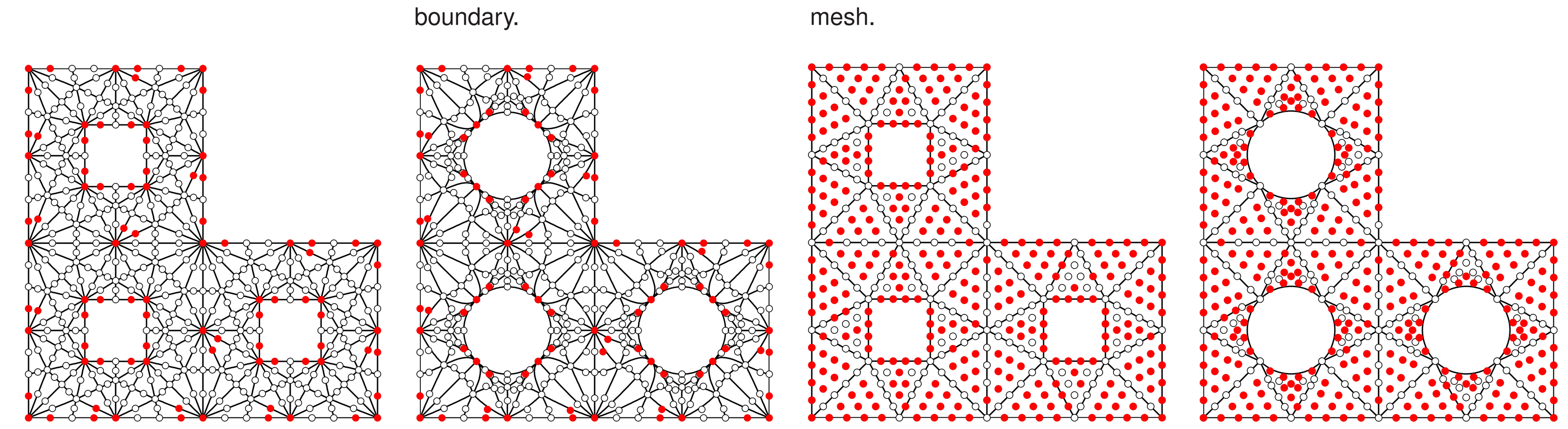
  - Construct a mesh with enough smoothness before refinement.
  - Geometric map stays the same during refinement.

Example: Poisson problem

- Problem definition
$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega, \\ u = \bar{u} & \text{on } \Gamma, \end{cases}$$
where
$$f(x,y) = 2\sin(x)\sin(y),$$
$$\bar{u}(x,y) = \sin(x)\sin(y).$$

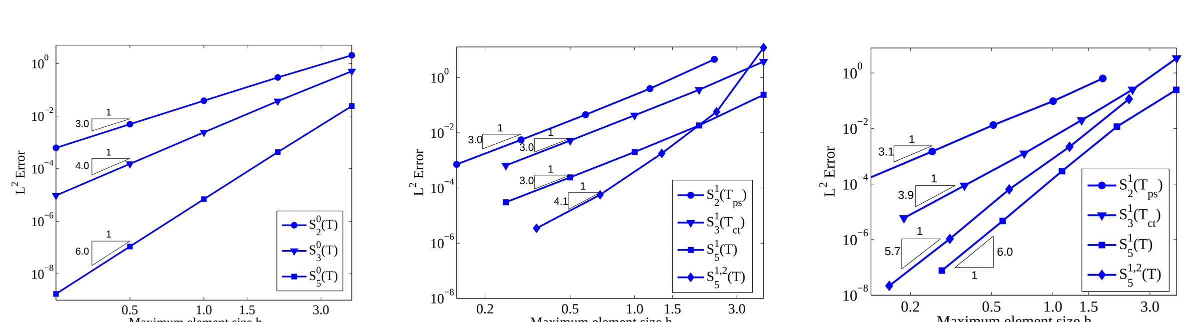


(a) Input domain with NURBS boundary. (b) Initial quadratic parametric mesh. (c) Initial quadratic physical mesh.



(d) Parametric mesh in $\mathcal{S}_2^1(\widehat{T}_{ps})$. (e) Physical mesh in $\mathcal{S}_2^1(\widehat{T}_{ps})$. (f) Parametric mesh in $\mathcal{S}_5^1(\widehat{T})$. (g) Physical mesh in $\mathcal{S}_5^1(\widehat{T})$.

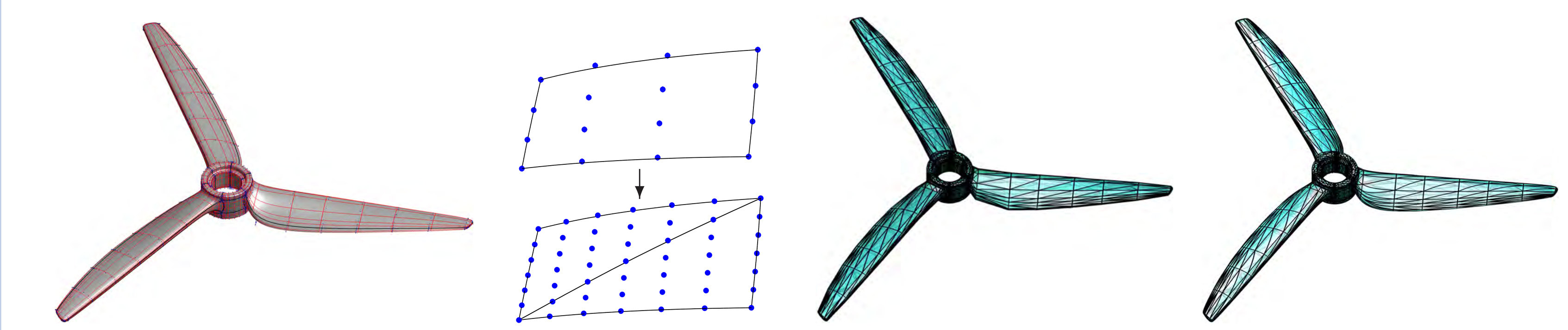
- Optimal convergence rates are obtained in all C^0 spaces.
- Optimal convergence rates are obtained in C^1 spaces only when the smooth-refine-smooth scheme is used.



(h) Optimal convergence rates in C^0 spaces. (i) Decreased convergence rates in C^1 spaces with refine-then-smooth scheme. (j) Optimal convergence rates in C^1 spaces with smooth-refine-smooth scheme.

IGA on triangulations in 3D

- **Parametrization**

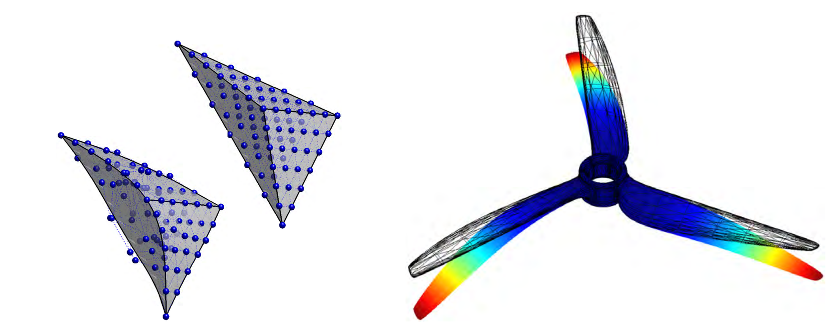


(a) Input: bicubic NURBS boundary patches (b) Convert bicubic Bézier patch to sextic Bézier triangles (c) Lienarize the Bézier triangles and tetrahedralize the surface triangulation (d) Replace boundary and smooth the interior control points

- **Example: elasticity on a propeller**

A wind loading is simulated by
  - fixing the interior cylindrical surface
  - setting traction $\boldsymbol{t} = [0, 0, -10 n_z]$ if $n_z > 0$ and zero otherwise.
  - setting Young's modulus $E = 10^5$, Poisson's ratio $\nu = 0.3$.

Mesh data
  - 6,431 elements and 798,366 dofs



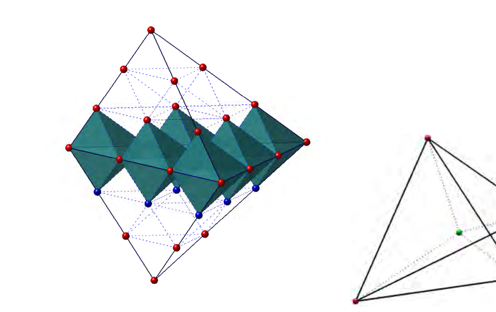
(a) Element samples (b) Deformed shape

Optimal convergence in C^r space in 3D

- **Smoothness conditions**

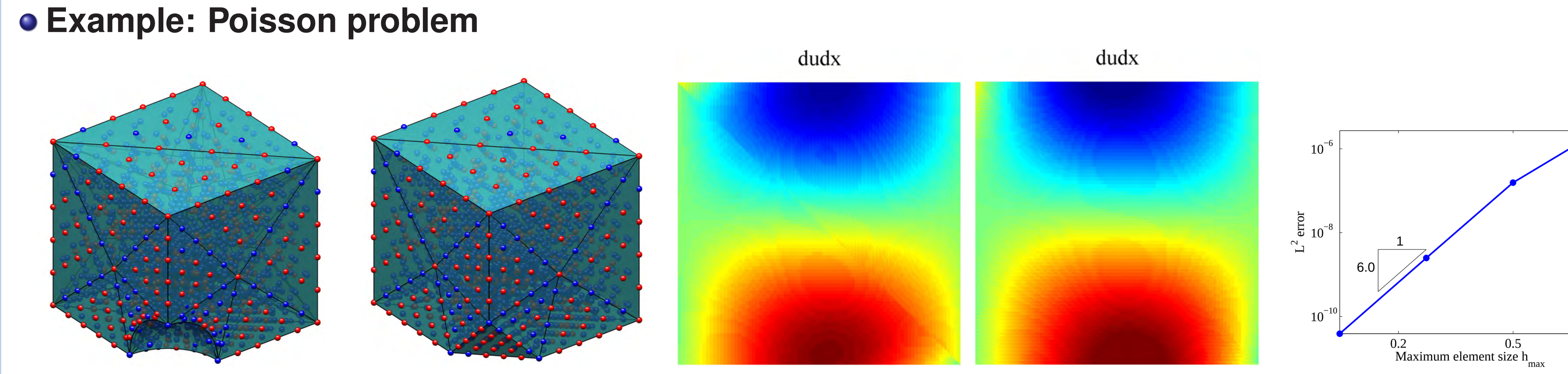
Suppose $T_1 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $T_2 := \{\mathbf{v}_5, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_3\}$ are two tetrahedra sharing the face $F := \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ of $\triangle$. Two polynomials f_1 and f_2 of degree d on T_1 and T_2 join together with C^r continuity across the face F if and only if for $m = 0, \cdots, r$,
$$\tilde{b}_{mijk} = \sum_{\nu+\mu+\kappa+\delta=m} b_{\nu,i+\mu,k+\kappa,j+\delta} B^m_{\nu\mu\kappa\delta}(\mathbf{v}_5), \quad \forall\, i+j+k = d-m.$$

- **C^1 macro-element space with Alfeld split**
$$\mathcal{S}_1(\triangle_A) := \{s \in \mathcal{S}_5^1(\triangle_A) : s \in C^2(v), \forall v \in \mathcal{V}, s \in C^4(v_T), \forall T \in \triangle\}$$



(a) C^1 constraints (b) Alfeld split

- **Example: Poisson problem**



(a) $\mathcal{S}_1(\triangle_A)$ physical mesh (b) $\mathcal{S}_1(\triangle_A)$ parametric mesh (c) Non-smooth gradient across the diagonal (element edge) with C^0 elements. (d) Smooth gradient across elements with C^1 elements. (e) Optimal convergence in C^1 space.

Conclusion

- A smooth-refine-smooth scheme is developed. It is the only scheme that has demonstrated optimal convergence rates for C^r elements involving extraordinary nodes.
- The parametrization can be fully automated for complex domain. Local refinement can also be easily implemented.
- A prototype software of isogeometric analysis based on rational triangular Bézier splines (rTBS) is developed. Any form of C^r Bézier elements can be used.
- The prototype software has been successfully applied for both 2D and 3D problems.

Publications

1. Xia, S., Wang, X., and Qian, X., Continuity and convergence in rational triangular Bézier spline based isogeometric analysis, *Computer Methods in Applied Mechanics and Engineering* 297 (2015) 292-324.
2. Jaxon, N. and Qian, X., Isogeometric analysis on triangulations, *Computer-Aided Design* 46 (2014) 45-57.

Patent pending

1. Qian, X. and Xia, S., Isogeometric analysis with Bézier triangles, Wisconsin Alumni Research Foundation, filing date: 03/16/2015.

Special issue pending

- Co-editing a special issue, Isogeometric Design and Analysis, in journal *Computer-Aided Design*.