ECE-409

Linear Control Systems

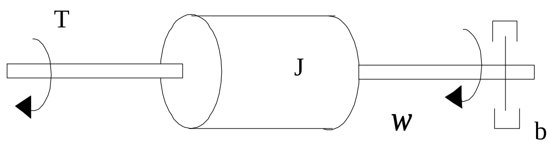
Prof. Darren M. Dawson

Notes Prepared by:

Siddharth Nagarkatti
Aman Behal
Pradeep Setlur

Example of a System that needs Control

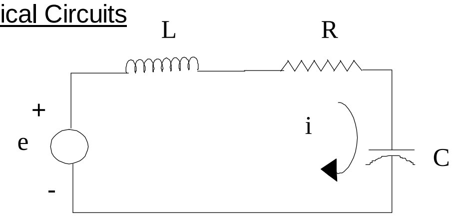
Motor Turning a Mechanical Load



T : torque input
X : output position
J : inertia
B : friction

$$J\ddot{X} + b\dot{X} = T \quad \}$$ Newton's Law

Electrical Circuits



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e \quad \Big\}$$ Kirchoff's Voltage Law

$$i = \frac{dq}{dt}$$ q charge

e : input voltage
q : output charge
i : current

R : resistance
L : inductance
C : capacitance

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = e \quad \Big\}$$ after some algebraic manipulation

Note : Two different systems are governed by similar-looking Differential Equations

Control is the "shaping" of the output of the Differential Equation by changing the input to the system via feedback

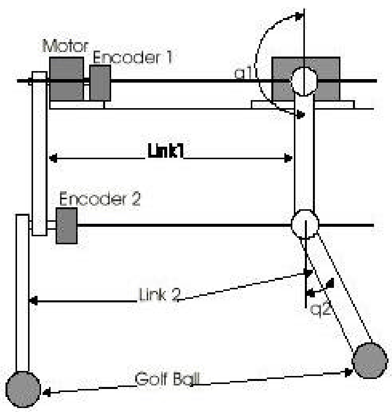
The Pendubot

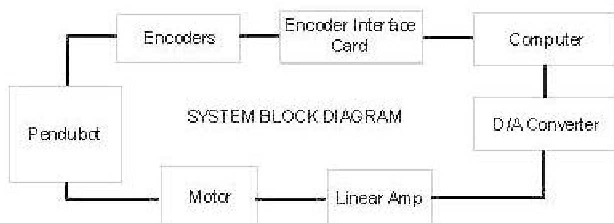
An Example of a System that needs Control

The Pendubot, short for PENDUlum RoBOT, is an under-actuated two-link mechanism. The system consists in two rigid links, the first of them actuated by a DC-motor and the second link is not actuated. The Pendubot is similar in spirit to the classical inverted pendulum on a cart or a rotational inverted pendulum.

The Pendubot presents some unique features and challenges for control research. One can use the Pendubot to investigate linear control, nonlinear control, system identification, intelligent control, hybrid and switching control, and gain scheduling.

Control Objective: To balance the pendubot and make the pendubot perform a specific acrobatic maneuver (i.e. the pendubot starts at the stationary vertical down position and must swing up to the inverted position)





Block Diagram of the Pendubot Setup

Movie (.avi) of a working Pendubots
[ECE 496 Senior Design Project Spring 1998]

Differential Equations

Definition

A differential equation is an equation which involves differentials or derivatives

For example: $y(t) = m \cdot \frac{d^2}{dt^2} x(t)$ is Newton's law of motion

In this class, we will be concerned with ordinary differential equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y^{(1)} + a_0 y = b_{n-1} u^{(n-1)} + ... + b_1 u^{(1)} + b_0 u$$

where u(t) and y(t) are functions of time. (Note : $y^{(n)}$ means the nth derivative)

a_i and b_i are constants

Time Invariant

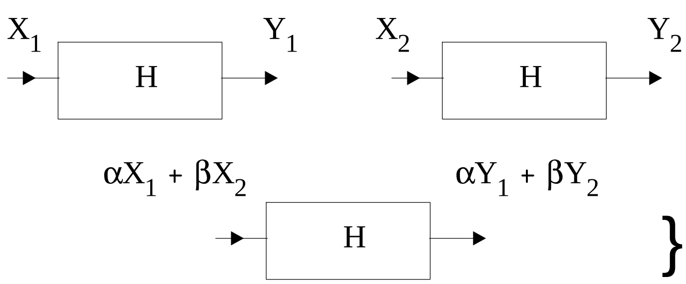
The differential equations will be assumed to be time-invariant.

That is, the coefficients, denoted by a_i and b_i do not depend explicitly on time.

Linearity

The differential equations are also assumed to be linear.

What is a linear system ?



} Given a system with two differential inputs

α and β are constants

} we obtain an output which conforms to standard linear operations (add and subtract by a constant OR multiply and divide by a constant)

Causality

A system is called causal if the output depends only on the present and past values of the input. That is, if y(t) is the output, then x(t) depends only on the input $x(\tau)$ for all values of τ less than or equal to t. A car, a light switch, a TV are all causal

The "D" Notation for Analyzing DE's

Examine the differential equation

$$\frac{d^n}{dt^n}y + a_{n-1}\cdot\frac{d^{n-1}}{dt^{n-1}}y + \text{.........} + a_1\cdot\frac{d}{dt}y + a_0\cdot y = u$$

Define $D = \frac{d}{dt}(\blacksquare)$ and $D^n = \frac{d^n}{dt^n}(\blacksquare)$

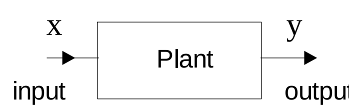
So the above equation can be written as

$$D^n\,y + a_{n-1}\,D^{n-1}\cdot y + \text{.......} + a_1\,D\cdot y + a_0\,y = u$$

$$\left(D^n + a_{n-1}\,D^{n-1} + \text{.......} + a_1\,D + a_0\right)\cdot y = u$$

$$\Delta(D) = \left(D^n + a_{n-1}\,D^{n-1} + \text{......} + a_1\,D + a_0\right) = 0$$

$\Delta(D)$ is called the **characteristic equation**



Example Find $\Delta(D)$ for $y'' + 3y' + 2y = x$

$$\Delta(D) = \left(D^2 + 3\cdot D + 2\right) = 0$$

$$\Delta(D) = (D+1)\cdot(D+2) = 0$$

so $D = -1$ or $D = -2$

Determine the "shape" of y(t)

Roots of $\Delta(D)$ are sometimes referred to as the **eigen values or poles**

It is now obvious that this class involves the study of Differential Equations

Where are we headed ?

1. Given x(t), it would be advantageous to have a general procedure for finding y(t)
2. It would also be advantageous to develop analysis techniques for examining systems without knowing the input

} Problems

3. Develop the Laplace Transform Tool

} Solutions

Time for a Movie

Click on image to start movie

Laplace Transforms

Why ? Because it is easy to work with an Algebraic Equation as opposed to a Differential Equation

Laplace Transformations are used to change a differential equation into an algebraic equation

$$\mathcal{L}(f(t)) = F(s) = \int_{0^p}^{\infty} f(t)\cdot e^{-st}\,dt \quad \Big\}$$

Definition of the Laplace Transform (one - sided)

where $F(s)$ is the Laplace transform of $f(t)$, $s = \sigma + j\omega$ and $j = \sqrt{-1}$

$\mathcal{L}$ denotes the Laplace Transform Operation

0^p denotes just to the right-hand side of zero (sometimes called zero-plus)

Definition **When can you take a Laplace Transform ?**

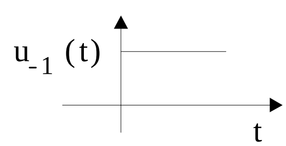
If $f(t)$ is single valued for $t>0$ and $F(s)$ is convergent for some σ_0 , that is

Apply the vertical line test

$$\int_{0^p}^{\infty} (|f(t)|)\cdot e^{-\sigma_0 t}\,dt < \infty$$

then $f(t)$ is Laplace transformable for Re(s) $>\sigma_0$

Example Checking the region of convergence



$$f(t) = e^{-t}\cdot u_{-1}(t)$$

$$\int_{0^p}^{\infty} \left(\left|e^{-t}\right|\right)\cdot e^{-\sigma_0 t}\,dt = \int_{0^p}^{\infty} e^{-\left(1+\sigma_0\right) t}\,dt = \frac{-1}{1+\sigma_0}\cdot e^{-\left(1+\sigma_0\right)\cdot t}\Bigg|_{0^p}^{\infty} = \frac{1}{1+\sigma_0} < \infty$$

if $1 + \sigma_0 > 0$ or $\boxed{\sigma_0 > -1}$ This is sometimes called the **Region of Convergence**

Note: We never worry about the ROC in Control Theory since we only use the one-sided Laplace Transform and all of our systems are causal

Example Find the Laplace Transform of e^{-t}

$$\mathcal{L}\left(e^{-t}\right) = \int_{0^p}^{\infty} e^{-t}\cdot e^{-s\,t}\,dt = \frac{-1}{s+1}\, e^{-(s+1)\cdot t}\Bigg|_{0^p}^{\infty} = \frac{1}{s+1}$$

Note: We hardly ever use this method of calculating the Laplace Transform in practice; rather, we use the tables of Laplace Transforms and algebraic manipulation to find the Laplace Transform of a function

Some Properties of Laplace Transforms

$\mathcal{L}(f(t)) = F(s)$

1. $\mathcal{L}\left(\frac{d}{dt}f(t)\right) = sF(s) - f(0^p)$ } Often used for solving DEs

2. $\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}$

3. Initial Value Theorem: $f(0^p) = \lim_{t \to 0^p} f(t) = \lim_{s \to \infty} sF(s)$

4. Final Value Theorem: $f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

} often used for setting performance specs for control design

5. $\mathcal{L}\left(f\left(\frac{t}{a}\right)\right) = |a| \cdot F(a \cdot s)$ } scaling ("a" is a constant)

6. $\mathcal{L}(f(t - T)) = e^{-sT}\,F(s)$ } delay ("T" is a constant)

7. $\mathcal{L}\left(e^{-at} \cdot f(t)\right) = F(s + a)$ } frequency scaling ("a" is a constant)

8. $\mathcal{L}\left[f_1(t)\right] * \left[f_2(t)\right] = F_1(s) \cdot F_2(s)$ } convolution

where $\left[f_1(t)\right] * \left[f_2(t)\right] = \int_0^t f_1(\tau) \cdot f_2(t - \tau)\,d\tau$ } convolution equation is avoided by multiplication in the frequency domain

What are the above properties good for ?

These properties can be used to manipulate a function into a form in which the transition, between the time domain and the frequency domain is eased (e.g. taking the Laplace Transform or the Inverse Laplace Transform)

Solution of Differential Equations by Laplace Transform

1. Take the Laplace Transform of both sides of the equation to work in the frequency domain
2. Solve the algebraic equation for the unknown variable
3. Take the Inverse Laplace Transform to find the time domain expression

Example Find y(t)

$$\frac{d}{dt}y + 4\cdot y = 2\,e^{-3\,t} \qquad y(0^p) = 3$$

$$sY(s) - y(0^p) + 4\,Y(s) = \frac{2}{s+3}$$ } Turning a DE into an AE

$$(s+4)\,Y(s) = \frac{2}{s+3} + y(0^p)$$

$$Y(s) = \frac{2}{(s+3)\cdot(s+4)} + \frac{y(0^p)}{s+4}$$ } Solve the AE

Partial Fraction Expansion

$$Y(s) = \frac{2}{s+3} + \frac{-2}{s+4} + \frac{3}{s+4}$$

$$Y(s) = \frac{2}{s+3} + \frac{1}{s+4}$$

} Preparing the expression for using the tables

Inverse Laplace Transform

$$y(t) = \left(2\,e^{-3\,t} + e^{-4\,t}\right)$$ } The answer

Check if the initial condition is satisfied

$$y(0) = 3$$ } A quick good way to catch errors

Plug in the solution and check if the original equation is recovered

$$y(t) = 2\,e^{-3\,t} + e^{-4\,t}$$

$$\frac{d}{dt}y + 4\cdot y = 2\cdot e^{-3\cdot t}$$

$$\frac{d}{dt}\left(2\,e^{-3\,t} + e^{-4\,t}\right) + 4\left(2\,e^{-3\,t} + e^{-4\,t}\right) = 2\,e^{-3\,t}$$

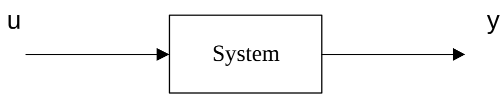
$$\left(-6\,e^{-3\,t} - 4\,e^{-4\,t}\right) + \left(8\,e^{-3\,t} + 4\,e^{-4\,t}\right) = 2\,e^{-3\,t}$$

} The answer is correct

$$2\,e^{-3\,t} = 2\,e^{-3\,t}$$

Finding the Output from a Linear Differential Equation

Given a system and an input, how do we find the output ?



We use an nth order differential equation to describe the system as follows

$$y^{n} + a_{n-1}\,y^{(n-1)} + \,.....\, + a_1\,y^{(1)} + a_0\,y = b_{n-1}\,u^{(n-1)} + \,.....\, + b_1\,u^{(1)} + b_0\,u$$

If we assume that all initial conditions are zero, then we can take the Laplace Transform

$$\left(s^{n} + a_{n-1}\cdot s^{n-1} + \,......\, + a_1\cdot s + a_0\right)\cdot Y(s) = \left(b_{n-1}\cdot s^{n-1} + \,.....\, + b_1\cdot s + b_0\right)\cdot U(s)$$

so

$$Y(s) = \left[\frac{\left(b_{n-1}\cdot s^{n-1} + \,.....\, + b_1\cdot s + b_0\right)}{\left(s^{n} + a_{n-1}\cdot s^{n-1} + \,......\, + a_1\cdot s + a_0\right)}\right]\cdot U(s)$$

We could now substitute for $U(s)$ and then use the Inverse Laplace Transform to find $y(t)$

Side Note: Transfer Function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\left(b_{n-1}\cdot s^{n-1} + \,.....\, + b_1\cdot s + b_0\right)}{\left(s^{n} + a_{n-1}\cdot s^{n-1} + \,......\, + a_1\cdot s + a_0\right)}$$

} Definition of Transfer Function

A transfer function is an Input/Output Relation

Remember : Set the initial conditions to zero when calculating the transfer function

More Math Tools

The **Partial Fraction** tool is important for taking the Inverse Laplace Transform

Example for distinct roots

$$H(s) = \frac{\left(s^4 + 8 s^3 + 25 s^2 + 31 s + 15\right)}{\left(s^3 + 6 s^2 + 11 s + 6\right)}$$

} Problem : Find h(t)

Step 1: Make the transfer function *strictly proper,i.e,*, order of (denominator) > order of (numerator)

Long Division

$$\begin{array}{r|l} & s + 2 \\ s^3 + 6 s^2 + 11 s + 6 & s^4 + 8 s^3 + 25 s^2 + 31 s + 15 \\ & -\left(s^4 + 6 \cdot s^3 + 11 \cdot s^2 + 6 \cdot s\right) \\ \hline & 2 s^3 + 14 s^2 + 25 s + 15 \\ & -\left(2 s^3 + 12 s^2 + 22 s + 12\right) \\ \hline & 2 s^2 + 3 s + 3 \end{array}$$

Don't let yourself get burned on the test

Stop when the order of the numerator is one less than the denominator

$$H(s) = s + 2 + H_1(s)$$

$$H_1(s) = \frac{2 s^2 + 3 s + 3}{s^3 + 6 s^2 + 11 s + 6}$$

} Now work with $H_1(s)$ via the PFE tool.

Step 2: Find $h_1(t)$ by using the PFE tool and tables

$$H_1(s) = \frac{2 s^2 + 3 s + 3}{s^3 + 6 s^2 + 11 s + 6} = \frac{2 s^2 + 3 s + 3}{(s + 1) \cdot (s + 2) \cdot (s + 3)}$$

} There is no easy way to factor the denominator.

$$H_1(s) = \frac{K_1}{s + 1} + \frac{K_2}{s + 2} + \frac{K_3}{s + 3}$$

Now Find K_1 K_2 and K_3

Residue Formula: $K_i = (s + s_i)\cdot H(s) \Big|_{s = s_i}$ } This can be used as long as the root at s_i is not repeated

$$K_1 = (s+1)\ H_1(s) \Big|_{s=-1} = \frac{2s^2 + 3s + 3}{(s+2)\cdot(s+3)} \Big|_{s=-1} = \frac{2 - 3 + 3}{(1)\cdot(2)} = 1$$

$$K_2 = (s+2)\ H_1(s) \Big|_{s=-2} = \frac{2s^2 + 3s + 3}{(s+1)\cdot(s+3)} \Big|_{s=-2} = \frac{8 - 6 + 3}{(-1)\cdot(1)} = -5$$

$$K_3 = (s+3)\ H_1(s) \Big|_{s=-3} = \frac{2s^2 + 3s + 3}{(s+1)\cdot(s+2)} \Big|_{s=-3} = \frac{18 - 9 + 3}{(-1)\cdot(-2)} = 6$$

$$H_1(s) = \frac{6}{(s+3)} - \frac{5}{(s+2)} + \frac{1}{(s+1)}$$

$$= \frac{2s^2 + 3s + 3}{(s+1)\cdot(s+2)\cdot(s+3)}$$

How do you check this ?
Get a common denominator

After taking the Inverse Laplace Transform (see the table), we have

$$h_1(t) = 6\cdot\exp(-3\cdot t) - 5\cdot\exp(-2\cdot t) + \exp(-t)$$ } answer

So $h(t) = \mathcal{L}^{-1}(s) + 2\,\delta(t) + h_1(t)$

The whole point of the PFE is to get the expression in the frequency domain into a form we can recognize in the tables

<u>PFE for Repeated Roots</u>

$$H(s) = \frac{\left(b_{n-1}\cdot s^{n-1} + \ldots\ldots + b_1\cdot s + b_0\right)}{\left(s^n + a_{n-1}\cdot s^{n-1} + \ldots\ldots + a_1\cdot s + a_0\right)}$$

} s_n .. repeated root multiplicity "q"
s_1 .... s_r - distinct roots

$$H(s) = \frac{K_1}{s + s_1} + \ldots + \frac{K_r}{s + s_r} + \frac{A_1}{(s + s_n)} + \frac{A_2}{(s + s_n)^2} + \ldots\ldots + \frac{A_q}{(s + s_n)^q}$$

Find K_1 ..... K_r with the residue formula

Find A_1 ..... A_q with the formula below

$$A_q = (s + s_n)^q \cdot H(s) \Big|_{s = s_n}$$

$$A_{q-1} = \frac{1}{1!}\left[\frac{d}{ds}\left[(s + s_n)^q \cdot H(s)\right]\right] \Bigg|_{s = s_n}$$

$$A_{q-2} = \frac{1}{2!}\left[\frac{d^2}{ds^2}\left[(s + s_n)^q \cdot H(s)\right]\right] \Bigg|_{s = s_n}$$ and so on

Example for repeated roots

Find h(t) for $H(s) = \dfrac{11 s^2 + 23 s + 1}{s(s + 1)^2} = \dfrac{K_1}{s} + \dfrac{A_1}{s + 1} + \dfrac{A_2}{(s + 1)^2}$ } one distinct root (s=0) one repeated root (s=-1) multiplicity two (q=2)

$$K_1 = (s \cdot H(s)) \Big|_{s = 0} = \frac{1}{1} = 1$$

$$A_2 = \left[(s + 1)^2 \cdot H(s)\right] \Big|_{s = -1} = \frac{11 - 23 + 1}{-1} = 11$$

$$A_1 = \left[\frac{d}{ds}\left[(s + 1)^2 \cdot H(s)\right]\right] \Bigg|_{s = -1} = \frac{d}{ds}\left(\frac{11 s^2 + 23 s + 1}{s}\right) \Bigg|_{s = -1}$$

$$= \frac{s(22 s + 23) - 11 s^2 - 23 s - 1}{s^2} \Bigg|_{s = -1} = 10$$

$$H(s) = \frac{1}{s} + \frac{10}{s + 1} + \frac{11}{(s + 1)^2}$$ } Take the Inverse Laplace Transform with the tables

$$h(t) = 1 + 10 \cdot \exp(-t) + 11 \cdot t \cdot \exp(-t)$$ } answer

More Math Tools

$$\mathcal{L}(x(t)) = X(s)$$

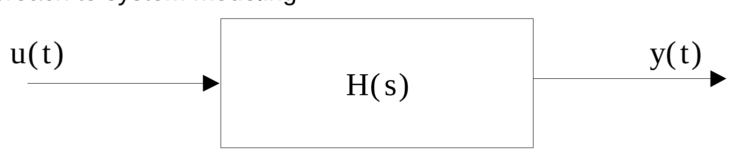
Property 9. $\frac{d^n}{dt^n} x(t) = s^n X(s) - s^{n-1} x\left(0^p\right) - s^{(n-2)} x^{(1)}\left(0^p\right) - .... - x^{(n-1)}\left(0^p\right)$

Property 10. $\int_{0^p}^{\infty} \delta(t) \cdot e^{-st} dt = 1$ note $\mathcal{L}(\delta(t)) = 1$ } $\delta(t)$ is defined this way
"The 0^p skeleton"

$\int_{-\infty}^{\infty} f(t) \cdot \delta(t - T)\, dt = f(T)$ } Definition of delta function

Transfer Function

Input / Output approach to system modeling



$\frac{Y(s)}{U(s)} = H(s)$ Transfer function is calculated by assuming all initial conditions are zero

$$y^n + a_{n-1} y^{(n-1)} + ... + a_1 y^{(1)} + a_0 y = b_{n-1} u^{(n-1)} + ... + b_1 u^{(1)} + b_0 u$$

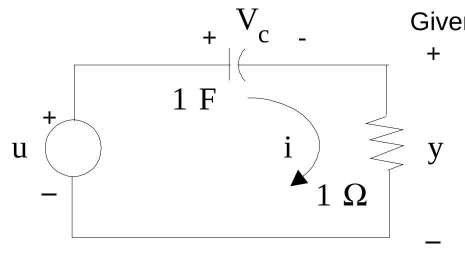
$$\frac{Y(s)}{U(s)} = H(s) = \frac{\left(b_{n-1} \cdot s^{n-1} + ... + b_1 \cdot s + b_0\right)}{\left(s^n + a_{n-1} \cdot s^{n-1} + ... + a_1 \cdot s + a_0\right)}$$

Roots of the numerator - zeros of H(s)

Roots of the denominator - poles of H(s)

Stability **H(s) is stable if all the poles have negative real parts**

Example



Given V_c voltage across the capacitor

u : input voltage
y : output voltage
i : current

$u = 2\,e^{-t}$ Find y(t)

$V_c\left(0^p\right) = 1$

Voltage loop $\{\quad u = V_c + y$

$y = i \qquad i = \frac{d}{dt}V_c \quad \Rightarrow \quad y = \frac{d}{dt}V_c$

Circuit Analysis

Take Derivative $\{\quad \frac{d}{dt}u = \frac{d}{dt}V_c + \frac{d}{dt}y$

Substitute for $\frac{d}{dt}V_c$ $\{\quad \frac{d}{dt}u = y + \frac{d}{dt}y$

Take Laplace Transform of DE

$$\mathcal{L}(u(t)) = \frac{2}{s+1}$$

$$sU(s) - u\left(0^p\right) = sY(s) - y\left(0^p\right) + Y(s)$$

$$y\left(0^p\right) = u\left(0^p\right) - V_c\left(0^p\right) = 1$$

} Calculate $y\left(0^p\right)$ from problem statement (Note $u\left(0^p\right) = 2$)

$$sU(s) - 2 = sY(s) - 1 + Y(s)$$

} Substitute initial conditions

$$Y(s) = \frac{2\,s}{(s+1)^2} - \frac{2}{s+1} + \frac{1}{s+1}$$

} Solve for Y(s)

$$Y(s) = \frac{2\,s}{(s+1)^2} - \frac{1}{s+1}$$

} Simplify

Use Partial Fraction Tool

$$Y(s) = \frac{-2}{(s+1)^2} + \frac{B}{s+1} - \frac{1}{s+1}$$

} Apply Residue Formula for Distinct Roots

$$B = \frac{d}{ds}\left[(s+1)^2 \cdot \frac{2\,s}{(s+1)^2}\right]\Bigg|_{s=-1} = 2$$

} Use Repeated Root Formula

Inverse Laplace Transform from Table

$$y(t) = -2\,te^{-t} + 2\,e^{-t} - 1\,e^{-t} = -2 \cdot t \cdot e^{-t} + e^{-t}$$

} Answer

How does a Mathematician check this answer ?
How does an Electrical Engineer check this answer ?

More Formulas

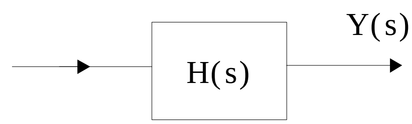
New Laplace Transformation Table Entries

$$\mathcal{L}\left(e^{-at}\sin\left(\omega_0 t\right)\right) = \frac{\omega_0}{(s+a)^2 + \left(\omega_0\right)^2}$$

ω_0 and a are positive constants

$$\mathcal{L}\left(e^{-at}\cos\left(\omega_0 t\right)\right) = \frac{s+a}{(s+a)^2 + \left(\omega_0\right)^2}$$

Side note on the Impulse Response



$h(t)$ is the Impulse Response for $u(t) = \delta(t)$

for $u(t) = \delta(t)$ $\mathcal{L}(u(t)) = 1$

$Y(s) = H(s)\cdot U(s) = H(s)\cdot 1$ note $\mathcal{L}(u(t)) = 1$

hence $y(t) = h(t)$ when $u(t) = \delta(t)$

Example: Calculating an Output

$$\frac{d^2}{dt^2}y + 2\cdot\frac{d}{dt}y + 5\cdot y = u(t) \qquad y(0) = \frac{d}{dt}y\,(0) = 0$$

$$s^2\,Y(s) + 2\,s\,Y(s) + 5\,Y(s) = U(s)$$

$$Y(s) = \frac{1}{s^2 + 2\,s + 5}\,U(s)$$

$$\frac{Y(s)}{U(s)} = H(s) = \frac{1}{s^2 + 2\,s + 5} \qquad h(t) = \mathcal{L}^{-1}(H(s))$$

***Example* - Continued**

Now let $u(t) = 3$ find $y(t)$

From the tables $U(s) = \frac{3}{s}$

poles always come in conjugate pairs if the coefficients are real

$$Y(s) = H(s)\cdot U(s) = \frac{3}{s\left(s^2 + 2s + 5\right)}$$

} Use quadratic formula for 2nd order roots

The roots of the polynomial in paranthesis are given by $s = -1 + j2$ and $s = -1 - j2$

A partial fraction expansion is given by

$$Y(s) = \frac{A_0}{s} + \frac{B_0}{s + 1 + j2} + \frac{C_0}{s + 1 - j2}$$

There are no complex numbers in the tables, so we use a different PFE method

Another partial fraction expansion is given by

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

} Find A,B and C

1. Since there are 3 roots, you need 3 weighting coefficients
2. The degree of the numerator polynomial must be one less than that of the denominator

} PFE Rules

$$Y(s) = \frac{3}{5}\cdot\frac{1}{s} + \frac{Bs + C}{s^2 + 2s + 5} = \frac{3}{s\left(s^2 + 2s + 5\right)}$$

} "A" was calculated with the Residue Formula

$$\left(\frac{3}{5}s^2 + \frac{6}{5}s + 3\right) + Bs^2 + Cs = 3$$

} "B" and "C" are found with the common denominator method

After equating the coefficients in the above equation

$B = \frac{-3}{5}$ and $C = \frac{-6}{5}$

$$Y(s) = \frac{3}{5}\cdot\frac{1}{s} - \left(\frac{3}{5}\right)\frac{s + 2}{s^2 + 2s + 5}$$

} still does not look quite like the tables

$$\frac{s + 2}{s^2 + 2s + 5} = \frac{s + 2}{(s + 1)^2 + 2^2} = \frac{s + 1 + 1}{(s + 1)^2 + 2^2} = \frac{s + 1}{(s + 1)^2 + 2^2} + \frac{1}{2}\left[\frac{2}{(s + 1)^2 + 2^2}\right]$$

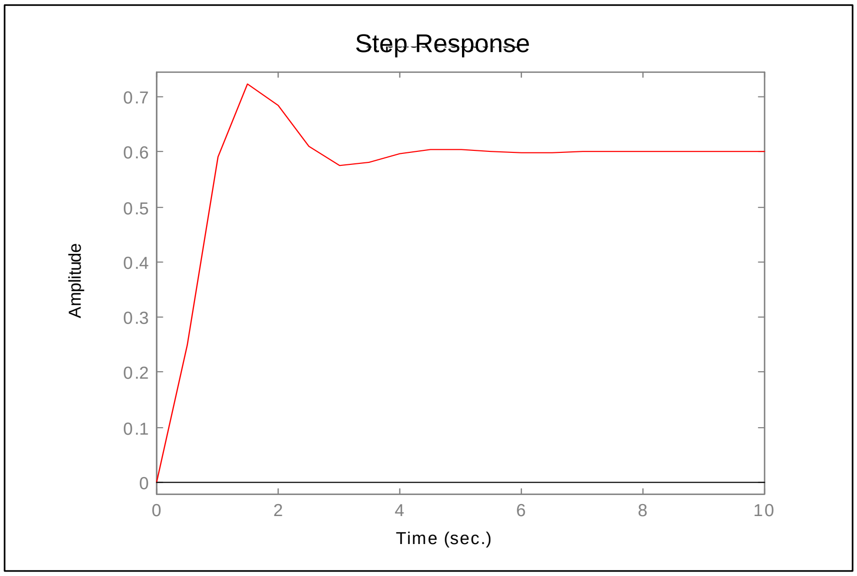
complete the squares

which looks like

From the tables......... $\frac{s + a}{(s + a)^2 + \left(\omega_0\right)^2}$ $\frac{\omega_0}{(s + a)^2 + \left(\omega_0\right)^2}$

$$Y(s) = \frac{3}{5}\cdot\frac{1}{s} - \left[\frac{3}{5}\,\frac{s+1}{(s+1)^2+2^2}\right] - \left(\frac{3}{10}\right)\frac{2}{(s+1)^2+2^2}$$ } Final Form

$$y(t) = \frac{3}{5} - \frac{3}{5}\cdot\exp(-t)\cdot\cos(2\cdot t) - \frac{3}{10}\cdot\exp(-t)\cdot\sin(2\cdot t)$$ } Take the inverse Laplace Transform to obtain answer



} Plot of y(t)

The overshoot and slight ripple is cause by the complex poles which also give rise to cos(t) and sin(t) terms

How do you check the answer ?

1. Check $y(0) = \frac{d}{dt}y(0) = 0$

2. Plug the solution into $\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 5\cdot y = u(t)$ } $u(t) = 3$

Checking for Stability Without Factoring the Denominator

$$H(s) = \frac{N(s)}{D(s)} \qquad D(s) = s^n + a_{n-1} s^{n-1} + \ldots\ldots + a_1 s + a_0$$ } Denominator determines stability

$$D_1(s) = s^n + a_{n-2} \cdot s^{n-2} + \ldots\ldots$$

$$D_2(s) = a_{n-1} \cdot s^{n-1} + a_{n-3} \cdot s^{n-3} + \ldots\ldots$$ } Formulate $D_1(s)$ and $D_2(s)$

$$\frac{D_1(s)}{D_2(s)} = h_1 s + \cfrac{1}{h_2 s + \cfrac{1}{h_3 s + \cfrac{1}{h_4 s + \cfrac{1}{\ldots h_n s}}}}$$ } Form the continued fraction expansion

If all $h_1 \ldots\ldots h_n$ are positive then all of the roots of $D(s)$ are in the OLHP (stable)

(Note OLHP - open left half plane-poles have negative real parts)

Example: Continued Fraction Expansion

Is $D(s)$ **stable ?**

$$D(s) = s^3 + 6 s^2 + 12 s + 8$$ } Denominator determines stability

$$D_1(s) = s^3 + 12 s \qquad D_2(s) = 6 s^2 + 8$$ } Formulate $D_1(s)$ and $D_2(s)$

$$\frac{D_1(s)}{D_2(s)} = \frac{s^3 + 12 s}{6 s^2 + 8} \quad =>$$

$$\begin{array}{r|l} & \frac{1}{6} s \\ \hline 6 s^2 + 8 & s^3 + 12 s \\ & -\left(s^3 + \frac{8}{6} s\right) \\ \hline & \frac{32}{3} s \end{array}$$

$$\frac{D_1(s)}{D_2(s)} = \frac{1}{6} s + \frac{1}{\dfrac{6 s^2 + 8}{\dfrac{32}{3} \cdot s}}$$ } Continue Dividing

$$\frac{D_1(s)}{D_2(s)} = \frac{1}{6}s + \frac{1}{\frac{6s^2+8}{\frac{32}{3}s}}$$

} Continued Fraction Expansion is completed

$$\Rightarrow \quad \frac{32}{3}s \,\overline{\big)\, 6s^2 + 8} \quad \text{quotient: } \frac{9}{16}s$$

$$\begin{array}{l} 6s^2 + 8 \\ -\left(6s^2\right) \\ \hline \, 8 \end{array}$$

$$\frac{D_1(s)}{D_2(s)} = \frac{1}{6}s + \frac{1}{\frac{9}{16}s + \frac{1}{\frac{4}{3}s}}$$

} Algebraic Simplication

$h_1 = \frac{1}{6}$ $h_2 = \frac{9}{16}$ and $h_3 = \frac{4}{3}$

So all the roots of $D(s)$ have all negative real parts (i.e., it is stable)

Example: Continued Fraction Expansion

Find the values of K such that the system described by the following equation is stable(Note K>0)

$$\Delta(s) = s^3 + 14s^2 + 56s + K$$

$$\frac{D_1(s)}{D_2(s)} = \frac{s^3 + 56s}{14s^2 + K}$$

Long Division

$$\begin{array}{r|l} & \frac{1}{14}s \\ \hline 14s^2 + K & s^3 + 56s \\ & -\left(s^3 + \frac{K}{14}\cdot s\right) \\ \hline & \left(56 - \frac{K}{14}\right)s \end{array}$$

$$\frac{D_1(s)}{D_2(s)} = \frac{1}{14}s + \frac{\left(56 - \frac{K}{14}\right)s}{14s^2 + K} = \frac{s}{14} + \frac{1}{\dfrac{14s^2 + K}{\left(56 - \frac{K}{14}\right)s}}$$

Performing long division again

$$\begin{array}{r|l} & s\cdot\left(\frac{14}{56 - \frac{K}{14}}\right) \\ \hline s\left(56 - \frac{K}{14}\right) & 14s^2 + K \\ & -14s^2 \\ \hline & K \end{array}$$

$$\frac{D_1(s)}{D_2(s)} = \frac{s}{14} + \frac{1}{s\left(\dfrac{14}{56 - \frac{K}{14}}\right) + \dfrac{1}{\dfrac{s\left(56 - \frac{K}{14}\right)}{K}}}$$

$$h_1 = \frac{1}{14} \qquad h_2 = \frac{14}{56 - \frac{K}{14}} \qquad h_3 = \frac{1}{\left(\dfrac{56 - \frac{K}{14}}{K}\right)}$$

For each $h_i > 0$ we must have

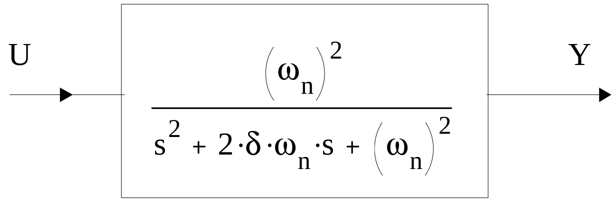
$$\frac{14}{56 - \frac{K}{14}} > 0 \quad \text{therefore} \quad 56 - \frac{K}{14} > 0 \quad \Rightarrow \quad K < 784$$

For stability K should satisfy $0 < K < 784$

For $K > 784$ => there is at least one unstable pole

Describing a Second Order System

$$\frac{d^2}{dt^2}y + 2\,\delta\cdot\omega_n\cdot\frac{d}{dt}y + \left(\omega_n\right)^2\cdot y = \left(\omega_n\right)^2 u \qquad \frac{Y(s)}{U(s)} = H(s)$$



$$\frac{\left(\omega_n\right)^2}{s^2 + 2\cdot\delta\cdot\omega_n\cdot s + \left(\omega_n\right)^2}$$

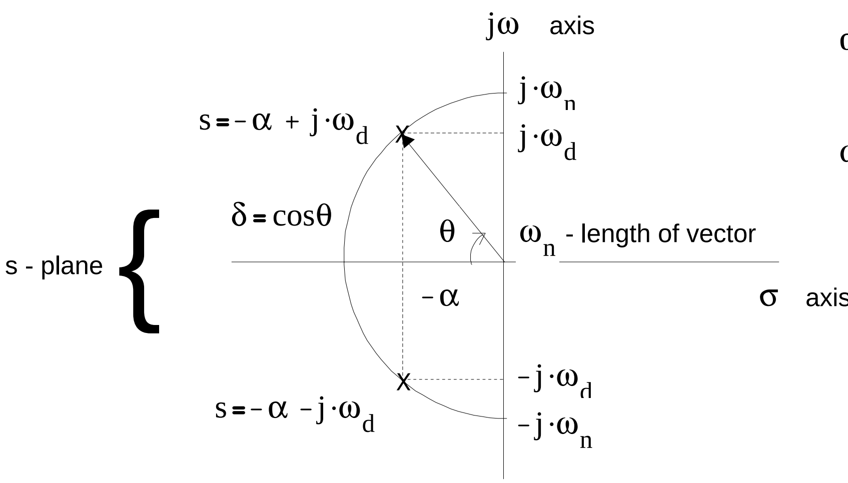
ω_n undamped natural frequency

δ damping ratio

The poles of $H(s)$ are $s = -\delta\cdot\omega_n + j\cdot\omega_n\cdot\sqrt{1-\delta^2}$ and $s = -\delta\cdot\omega_n - j\cdot\omega_n\cdot\sqrt{1-\delta^2}$

which is also written as $s = -\alpha - j\cdot\omega_d$ and $s = -\alpha + j\cdot\omega_d$

$\frac{1}{\alpha}$ is called the time constant $\qquad \omega_d$ is the damped natural frequency



$\alpha = \delta\cdot\omega_n$

$\omega_d = \omega_n\cdot\sqrt{1-\delta^2}$

If $\delta > 1$ poles are negative and real

If $\delta = 1$ poles are real and equal

If $0 < \delta < 1$ poles are complex conjugate pairs $s = -\delta\cdot\omega_n + j\cdot\omega_n\cdot\sqrt{1-\delta^2}$ and $s = -\delta\cdot\omega_n - j\cdot\omega_n\cdot\sqrt{1-\delta^2}$

If $\delta = 0$ poles are imaginary $s = j\cdot\omega_n$ and $s = -j\omega_n$

δ determines how much cosine and sine action an output might have

More Laplace Transforms

Second Order System

If $\delta = 1$ $\quad \mathcal{L}\left(t \cdot e^{-\omega_n \cdot t}\right) = \left[\dfrac{1}{\left(s + \omega_n\right)^2}\right]$ } Poles are real and repeated

If $\delta = 0$ $\quad \mathcal{L}\left(\sin\left(\omega_n \cdot t\right)\right) = \dfrac{\omega_n}{s^2 + \left(\omega_n\right)^2}$ } Poles are purely imaginary

If $0 < \delta < 1$ $\quad \mathcal{L}\left(\dfrac{\omega_n}{\sqrt{1 - \delta^2}} \cdot e^{-\delta \cdot \omega_n \cdot t} \cdot \sin\left(\sqrt{1 - \delta^2} \cdot t\right)\right) = \dfrac{\left(\omega_n\right)^2}{s^2 + 2 \cdot \delta \cdot \omega_n \cdot s + \left(\omega_n\right)^2}$

If $\delta > 1$ roots are distinct and real; hence standard PFE/Tables can be applied

Stability: Time Domain and Frequency Domain Relationships

Second Definition of Stability

A system is stable if its impulse response approaches zero as time goes to infinity, that is

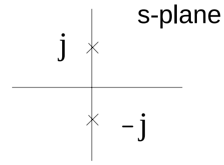
$$\lim_{t \to \infty} h(t) = \mathcal{L}^{-1}(H(s)) = 0$$

which means that all poles are in the OLHP

(Note OLHP - open left hand plane - poles have negative real parts)

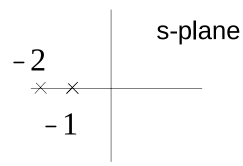
Examples

Transform Pairs
$$\begin{cases} H(s) = \dfrac{1}{s^2 + 1} \\ h(t) = \sin(t) \end{cases}$$



Not Stable

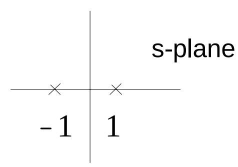
Transform Pairs
$$\begin{cases} H(s) = \dfrac{1}{(s+1)\cdot(s+2)} \\ h(t) = \exp(-t) - \exp(-2\cdot t) \end{cases}$$



Stable

} Pole Plots

Transform Pairs
$$\begin{cases} H(s) = \dfrac{1}{(s+1)\cdot(s-1)} \\ h(t) = \dfrac{-1}{2}\cdot\exp(-t) + \dfrac{1}{2}\cdot\exp(t) \end{cases}$$



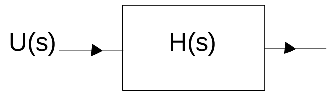
Not Stable

So stability is only dependent on the poles of the transfer function

Time for a Movie

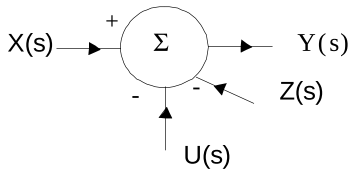
Click on image to start movie

Block Diagram Simplification



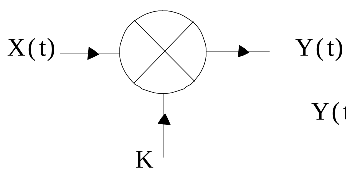
} Goal of Block Diagram Simplification

Summer

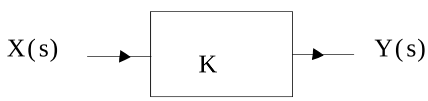


$Y(s) = X(s) - U(s) - Z(s)$ } Mathematical Expression

multiplier

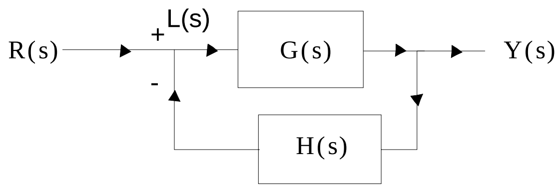


$Y(t) = K \cdot X(t)$ } Mathematical Expression



$Y(s) = K \cdot X(s)$ } Mathematical Expression

Canonical Form



} Most common Block Diagram

G(s) Forward Transfer Function

H(s) Feedback Transfer Function

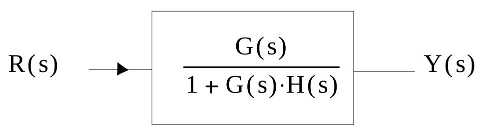
$L(s) = -H(s) \cdot Y(s) + R(s)$ } from Block Diagram

$Y(s) = L(s) \cdot G(s)$ } from Block Diagram

$Y(s) = (-H(s) \cdot Y(s) + R(s))\ G(s)$ } substitute for L(s)

$Y(s)\ (1 + H(s)\ G(s)) = R(s) \cdot G(s)$ } algebraic manipulation

$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s) \cdot H(s)}$ } answer

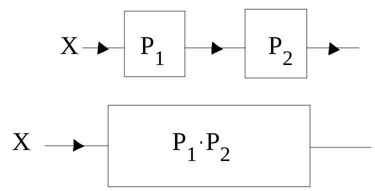


} Block Diagram has been simplified

The need for simplification of Block Diagrams spawned the creation of the rules and procedures

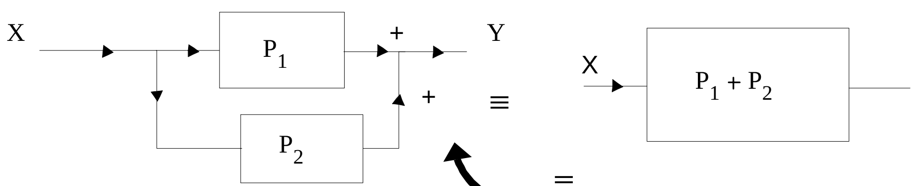
Block Diagram Simplification Rules

Cascade $Y = P_1 \cdot P_2 \cdot X$

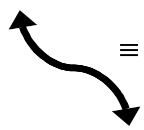


} simplified

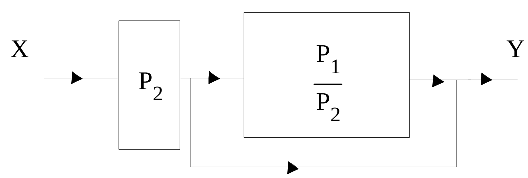
Parallel $Y = \left(P_1 + P_2\right) \cdot X$



} simplified



Remove a Block from a Path

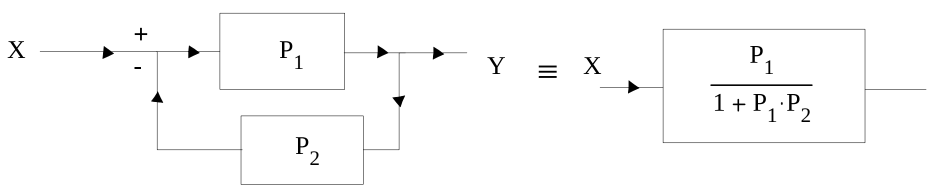


} equivalent to above block.

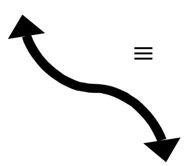
$$Y = P_2 \cdot \left(\frac{P_1}{P_2} + 1 \right) X$$

Feedback loop $Y = P_1 \left(X - P_2 \cdot Y\right)$

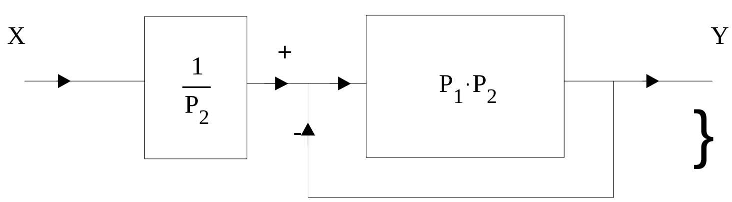
$$Y = \left(P_1 + P_2\right) \cdot X$$



} simplified



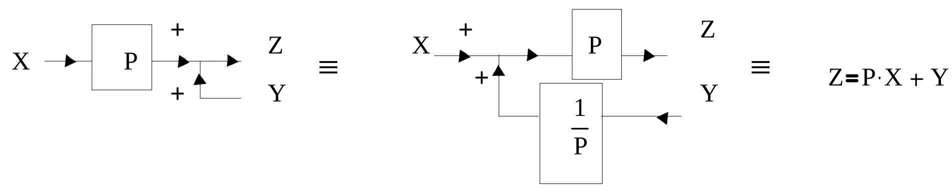
Remove a Block from a Path

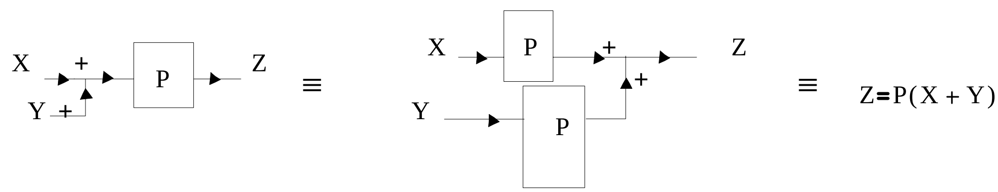


$$Y = \frac{1}{P_2}\left(\frac{P_1 P_2}{1 + P_1 P_2}\right) X$$

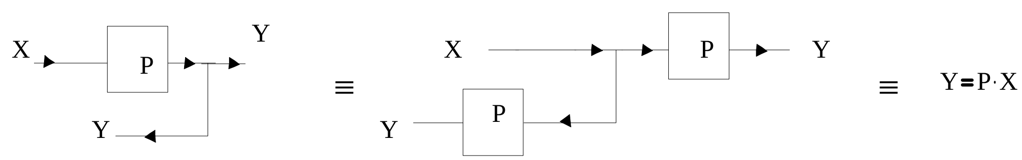
} equivalent to above block.

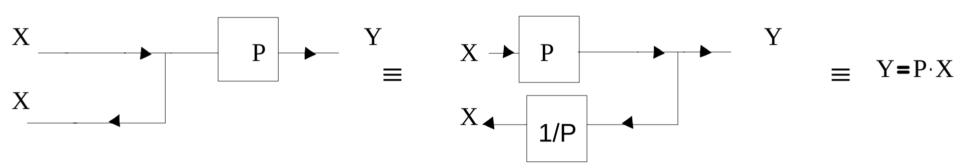
Moving a Summing Point ahead of a Block or beyond





Moving a Take-off point





Bottom Line : If you move wires or blocks, you must ensure the mathematical expression remains equal to the same thing that you started with

Multiple Inputs (Some problems have multiple inputs)

When we have multiple inputs, each is treated independently of the other inputs.

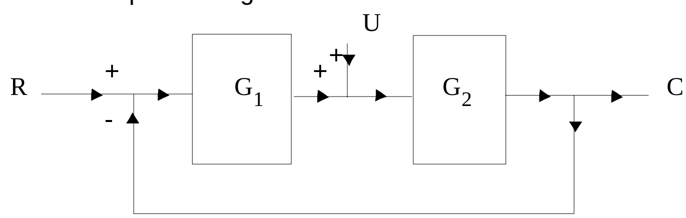
Step 1: Set all inputs except one input equal to zero

Step 2: Calculate the response due to the non-zero input.

Step 3: Repeat step 1 and step 2 as needed

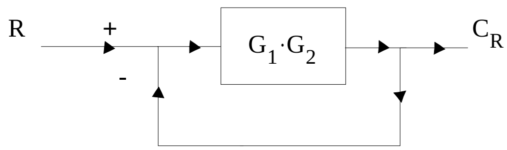
Step 4: Add all the responses together.

Example



} Find C(s)

Set U = 0

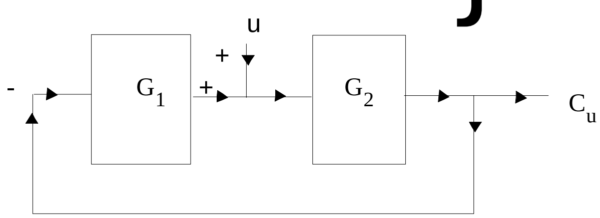


} Response of C(s) due to R(s) only

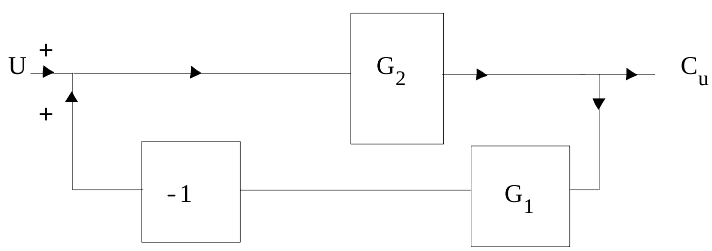
$$C_R = \frac{G_1 \cdot G_2}{1 + G_1 \cdot G_2} \cdot R$$

} Block Diagram Simplification Rules

Set R = 0



} Response of C(s) due to U(s) only



} Rearrange Block Diagram

$$C_u = \frac{G_2}{1 + G_1 \cdot G_2} U$$

} Block Diagram Simplification Rules

The total response is given by

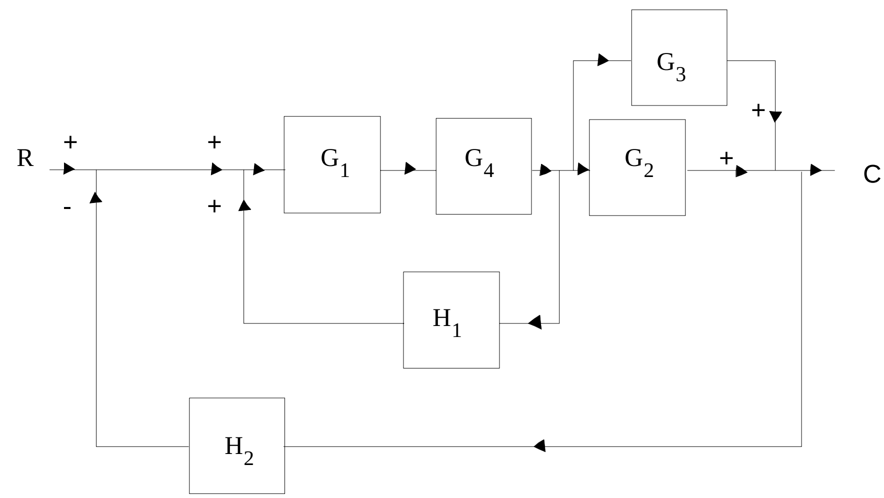
$$C = C_u + C_R = \frac{G_1 \cdot G_2}{1 + G_1 \cdot G_2} \cdot R + \frac{G_2}{1 + G_1 \cdot G_2} U$$

Rule-Based Reduction of Complicated Diagrams (Method 1)

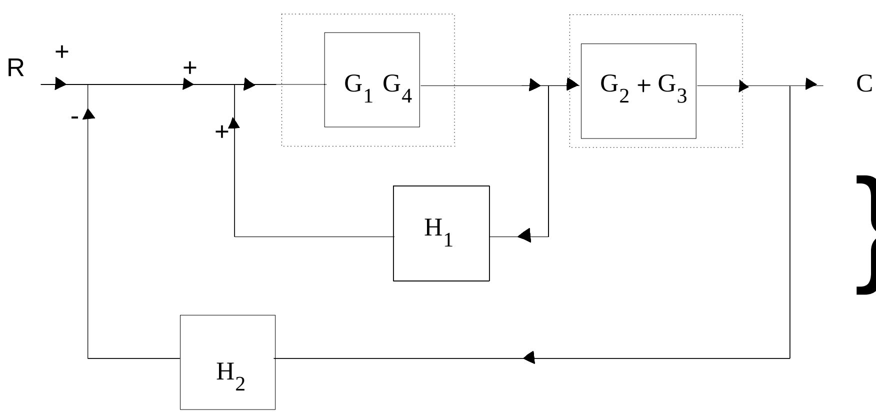
1. Combine Cascade and Parallel Blocks

2. Eliminate Feedback Loops.

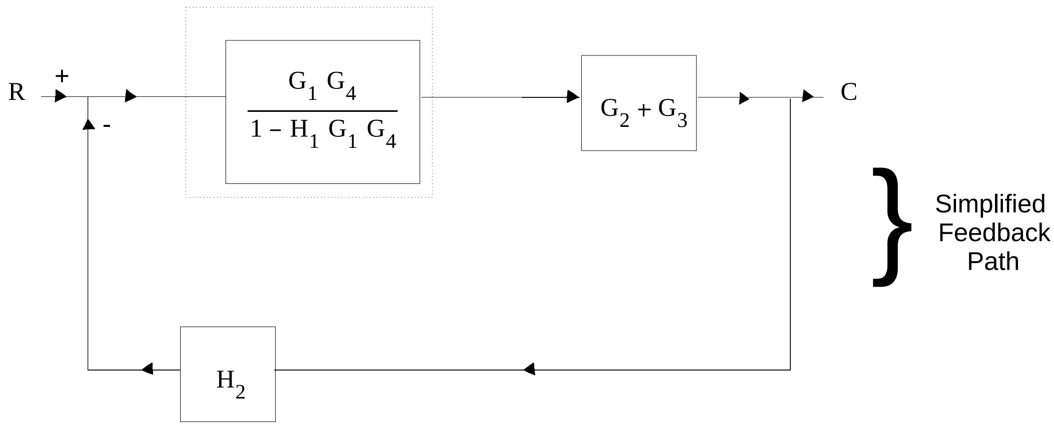
Example 1

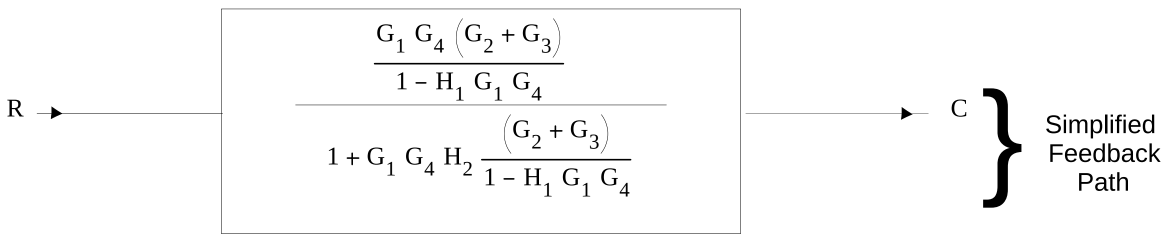


Find $\dfrac{C(s)}{R(s)}$

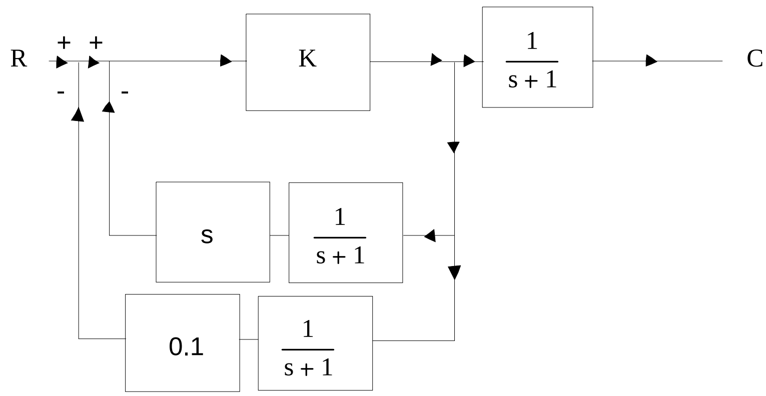


Combined Cascade and Parallel Blocks

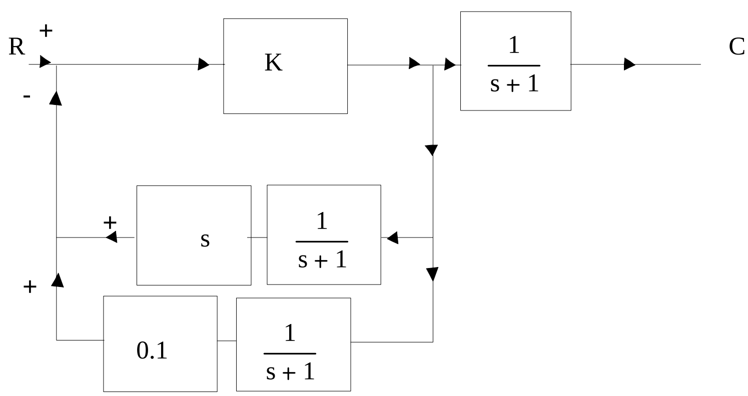

R
+
-
$\frac{G_1 G_4}{1 - H_1 G_1 G_4}$
$G_2 + G_3$
C
H_2
Simplified Feedback Path


R
$\dfrac{\dfrac{G_1 G_4 (G_2 + G_3)}{1 - H_1 G_1 G_4}}{1 + G_1 G_4 H_2 \dfrac{(G_2 + G_3)}{1 - H_1 G_1 G_4}}$
C
Simplified Feedback Path

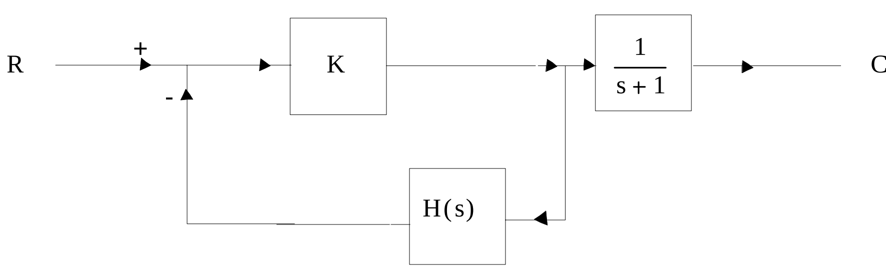
Example 2



Find $\frac{C(s)}{R(s)}$



identify parallel paths



add parallel paths

$$H(s) = \left(\frac{s}{s+1}\right) + \frac{.1}{s+1} = \frac{(s+.1)}{(s+1)}$$

add parallel paths

$$\frac{C(s)}{R(s)} = \left(\frac{K}{1+KH(s)}\right)\cdot\left(\frac{1}{s+1}\right)$$

close feedback loop

$$\frac{C(s)}{R(s)} = \left(\frac{K}{1+\frac{K(s+0.1)}{s+1}}\right)\cdot\left(\frac{1}{s+1}\right)$$

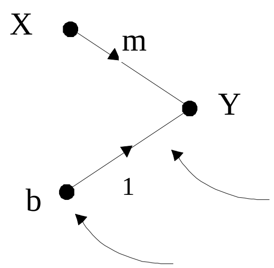
answer

Time for a Movie

Click on image to start movie

Signal Flow Graphs (Method 2)

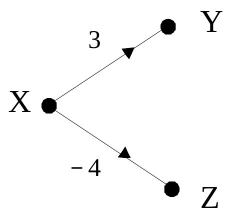
Nomenclature



$Y = m \cdot x + b$ } mathematical expression

branch - which way you go

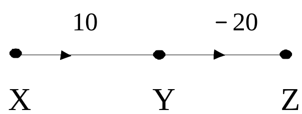
Node - variables are represented by nodes

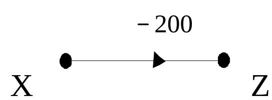


$Y = 3 \cdot X$

$Z = -(4 \cdot X)$

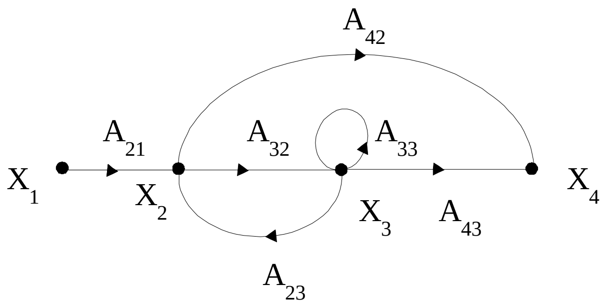
} mathematical expression





$Y = 10 \cdot X$ $\quad Z = -(20 \cdot Y)$ $\quad Z = -(200 \cdot X)$

Example and Definitions



} Signal Flow Graph that is used to explain Mason's terminology

Definition - A path is a continuous unidirectional succession of branches along which no node is passed more than once.

X_1 to X_2 to X_3 to X_4

X_2 to X_3 to X_4 X_2 to X_3 to X_2

} Examples of Paths

Definition - an input node is a node with only outgoing branches Example X_1

Definition - an output node is a node with only incoming branches Example X_4

Definition - a forward path is a path from input node to output node Example X_1, X_2, X_4

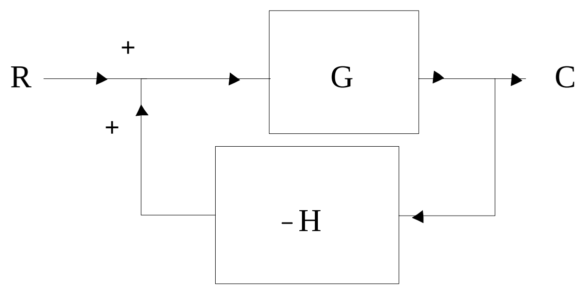
Definition - a path gain is the product of the branch gains encountered

Example -- $X_1 \cdot X_2 \cdot X_4 = A_{21}\ A_{42}$ } path gain for a forward path

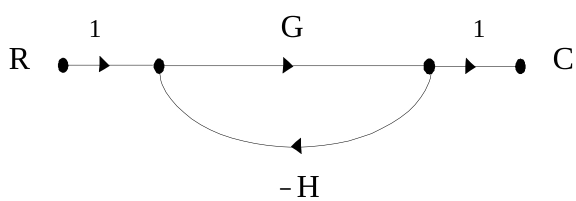
Definition - A loop gain is the product of the branch gains of the loop

Example -- $X_2 \cdot X_3 \cdot X_2 = A_{32}\ A_{23}$ } loop gain for a loop

Construction of signal flow graphs

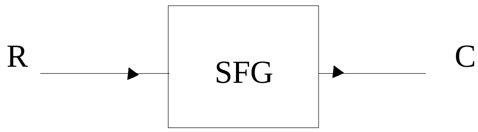


Block Diagram



Signal Flow Graph

Simplification of Signal Flow Graphs



} First draw the signal flow graph

$$\frac{C}{R} = H = \sum_i \frac{P_i \Delta_i}{\Delta}$$ } Mason's Rule

P_i = ith forward path

P_{jk} = jth possible gain product of K non-touching loops

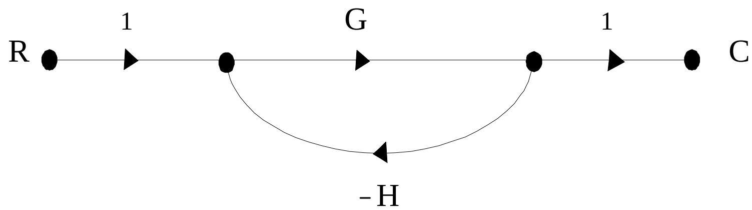
$$\Delta = 1 - (-1)^{K+1} \sum_K \sum_J P_{JK}$$ **}** Original Formula

Δ = 1 - (sum of all loop gains) + (sum of gain-products of 2 non-touching loops) - (sum of gain-products of 3 non-touching loops) + (....... 4.........) - ...... **}** Alternate Formula

Δ_i = Δ evaluated with all loops touching P_i being eliminated

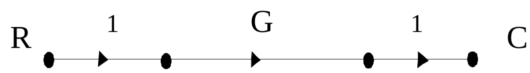
Definitions - Two loops, paths, or a loop and a path are non-touching if they have no nodes in common

Example - Simplification of an SFG



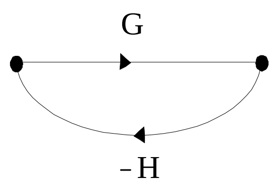
1) Find forward paths

$P_1 = G$ } One forward path



2) Find loops

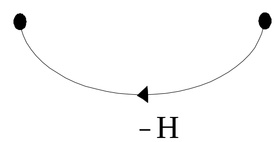
$P_{11} = -GH$ } One loop



3) Find Δ and Δ_i's

$\Delta = 1 + GH$ } Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

$\Delta_1 = 1 - 0 = 1$ } recalculate Δ with P_1 removed

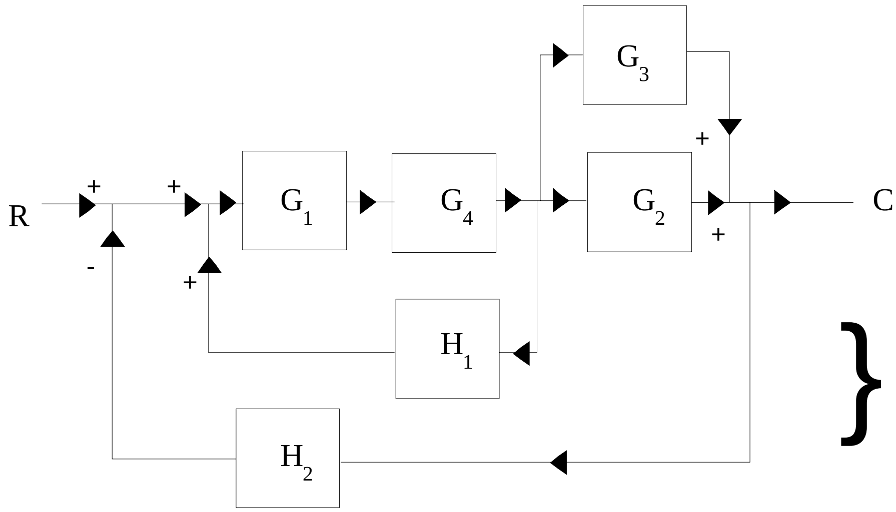


4) Calculate H(s)

$$H(s) = \frac{P_1 \Delta_1}{\Delta} = \frac{G}{1 + GH}$$

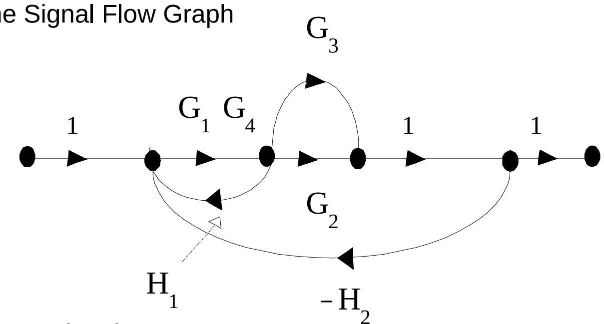
} Does this agree with what you know the answer is supposed to be ?

Example - Simplify Block Diagram



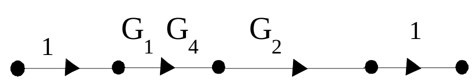
worked this example before with standard block diagram simplification rules

1) First, draw the Signal Flow Graph

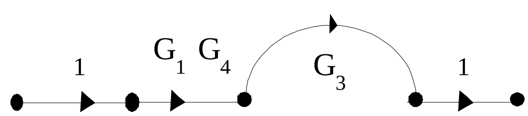


2) Find the forward paths

$$P_1 = G_1 \; G_2 \; G_4$$

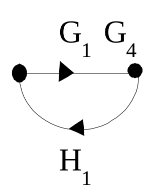


$$P_2 = G_1 \; G_3 \; G_4$$

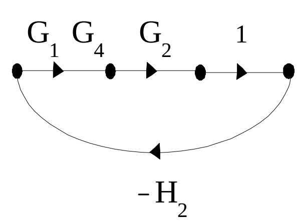


3) Find the loops

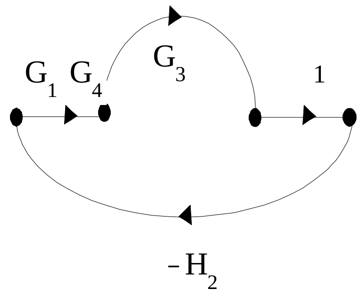
$$P_{11} = G_1 \; G_4 \; H_1$$



$$P_{12} = -G_1 \; G_2 \; G_4 \; H_2$$



$P_{21} = -G_1 G_3 G_4 H_2$

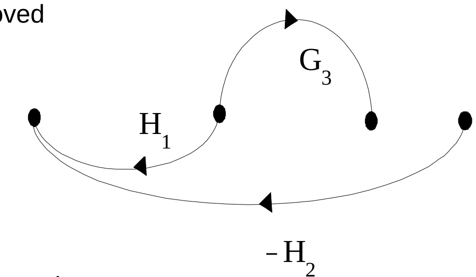


4) Find Δ and Δ_i's

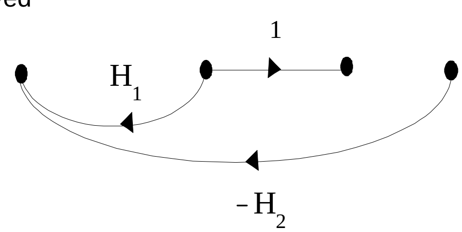
$\Delta = 1 - \left(P_{11} + P_{12} + P_{21}\right) + 0$ } Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

Δ_i = Δ evaluated with all loops touching P_i being eliminated

$\Delta_1 = 1$ } Recalculate Δ with P_1 removed



$\Delta_2 = 1$ } Recalculate Δ with P_2 removed

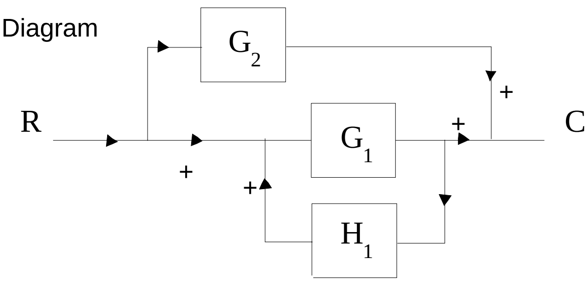


5) Find H(s)

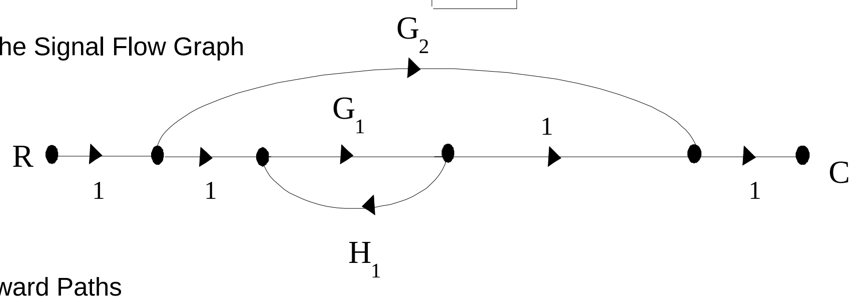
$$H = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$H = \frac{G_1 G_4 G_2 (1) + G_1 G_3 G_4 (1)}{1 - G_1 G_4 H_1 + G_1 G_2 G_4 H_2 + G_1 G_3 G_4 H_2}$$ } same as before but with different technique

Example -Simplify Block Diagram

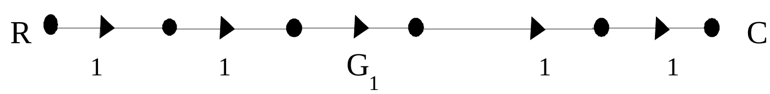


1) First, draw the Signal Flow Graph

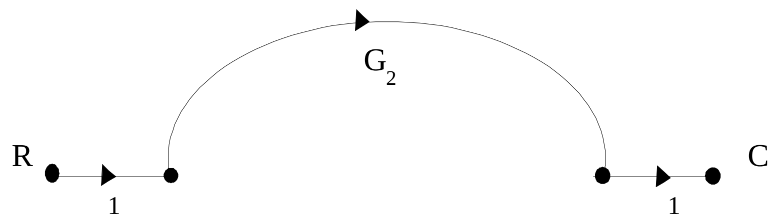


2) Find the forward Paths

$P_1 = G_1$

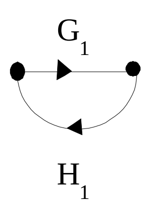


$P_2 = G_2$



3) Find the loops

$P_{11} = G_1\ H_1$

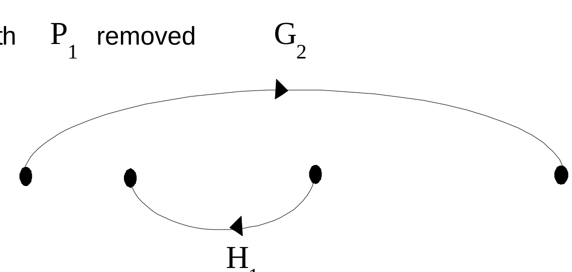


4) Find Δ and Δ_i's

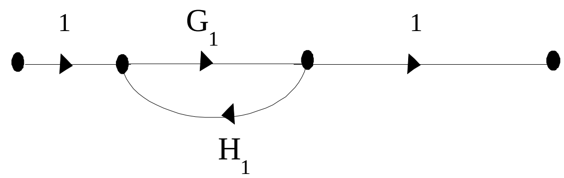
$\Delta = 1 - G_1 H_1$ }

Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

$\Delta_1 = 1$ } Recalculate Δ with P_1 removed



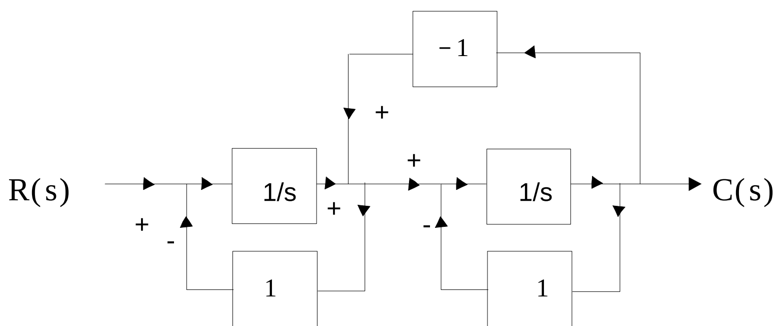
$\Delta_2 = 1 - G_1 H_1$ } Recalculate Δ with P_2 removed



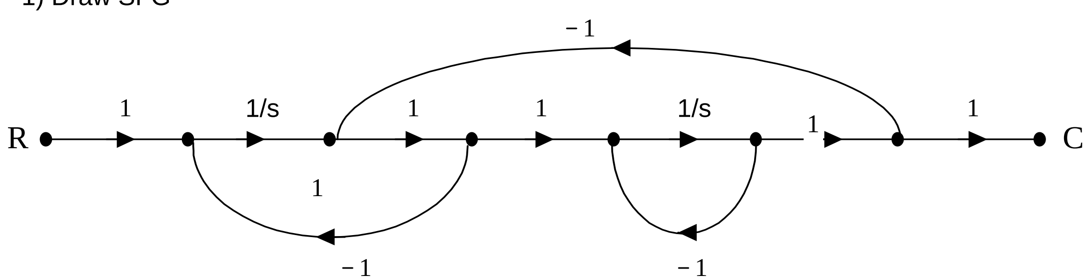
5) $H(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1 + G_2(1 - G_1 H_1)}{1 - G_1 H_1}$

__Example - Putting it all together__

If R(s) = 1/s   Find c(t)

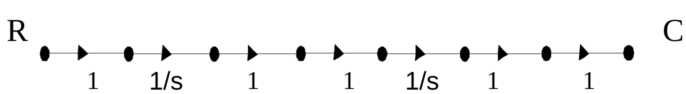


1) Draw SFG



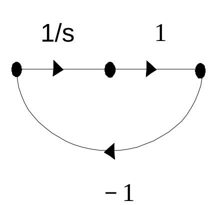
2) Find the forward loops

$P_1 = \frac{1}{s^2}$

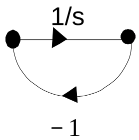


3) Find the loops

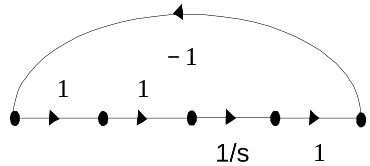
$A = \frac{-1}{s}$



$B = \frac{-1}{s}$



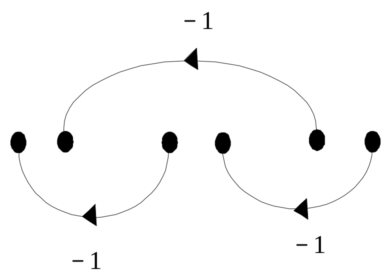
$C = \frac{-1}{s}$



4) Find Δ and Δ_i's

$\Delta = 1 - (A + B + C) + AB$ } Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

$\Delta_1 = 1$ } Recalculate Δ with P_1 removed



5) Find $\frac{C(s)}{R(s)} = H(s)$

$$\frac{C}{R} = \frac{P_1 \Delta_1}{\Delta}$$

$$\frac{C}{R} = \frac{\frac{1}{s^2}}{1 - (A + B + C) + AB} = \frac{\frac{1}{s^2}}{1 + \frac{3}{s} + \frac{1}{s^2}} = \frac{1}{s^2 + 3s + 1}$$

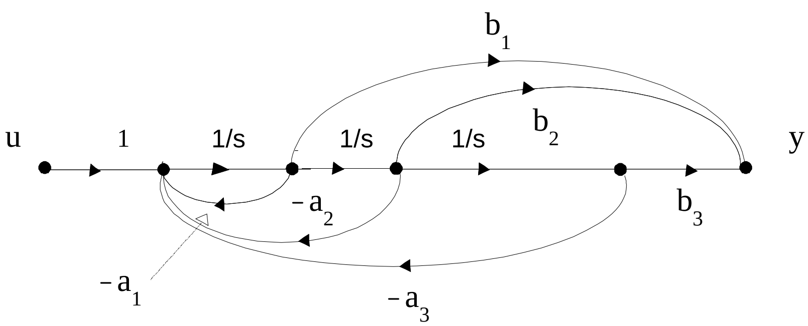
$$C(s) = \frac{1}{s \cdot (s^2 + 3s + 1)} = \frac{1}{s(s+a)(s+b)}$$ } $R(s) = \frac{1}{s}$

$$a = -\left(\frac{-3 + \sqrt{5}}{2}\right) \qquad b = -\left(\frac{-3 - \sqrt{5}}{2}\right)$$

$$C(s) = \frac{\frac{1}{ab}}{s} + \frac{\frac{1}{-a(-a+b)}}{s+a} + \frac{\frac{1}{-b(-b+a)}}{s+b}$$ } PFE

$$c(t) = \frac{1}{ab} + \frac{-1}{(b-a)a} e^{-at} + \frac{-1}{(a-b)b} e^{-bt}$$ } Inverse Laplace Transform

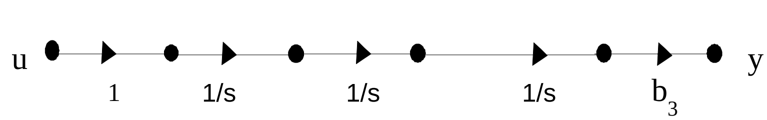
Example Simplify SFG



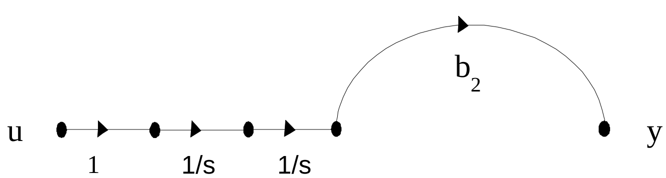
} Reachable Canonical Form

1) Find the forward paths

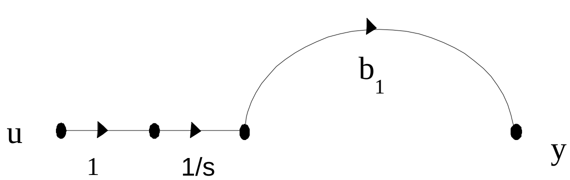
$$P_1 = \frac{b_3}{s^3}$$



$$P_2 = \frac{b_2}{s^2}$$

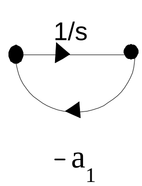


$$P_3 = \frac{b_1}{s}$$

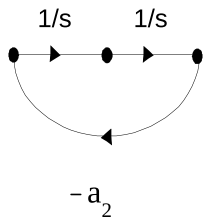


2) Find the Loops

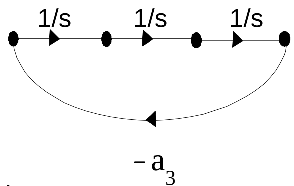
$$A = \frac{-a_1}{s}$$



$$B = \frac{-a_2}{s^2}$$



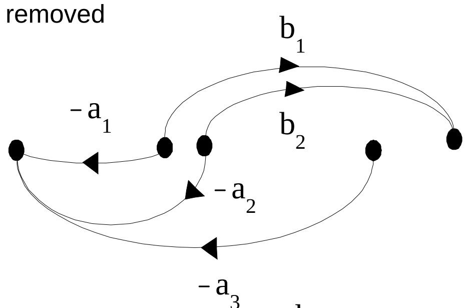
$$C = \frac{-a_3}{s^3}$$



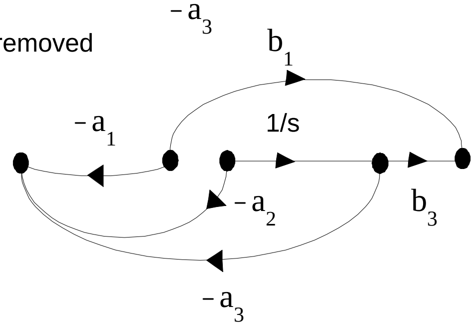
3) Find Δ and Δ_i's

$\Delta = 1 - (A + B + C)$ } Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

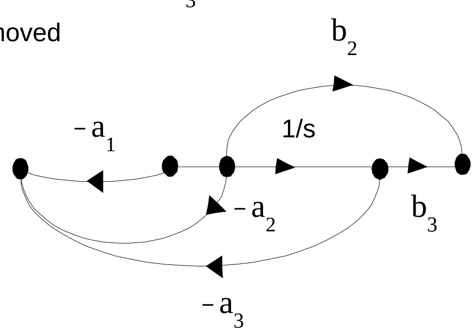
$\Delta_1 = 1$ } Recalculate Δ with P_1 removed



$\Delta_2 = 1$ } Recalculate Δ with P_2 removed



$\Delta_3 = 1$ } Recalculate Δ with P_3 removed



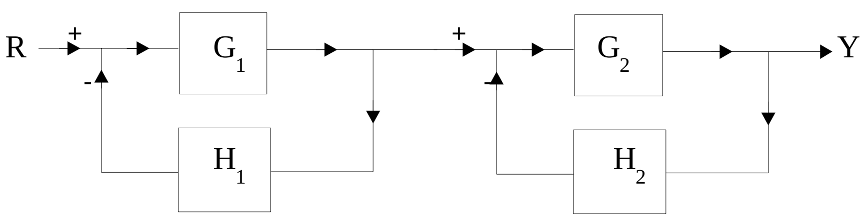
4) Find H(s)

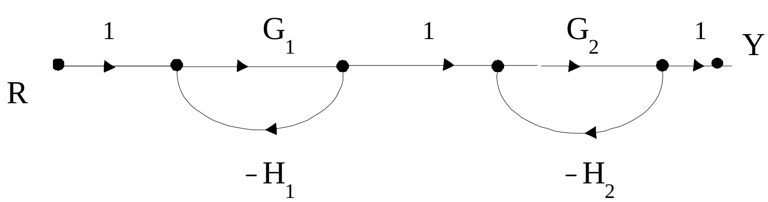
$$\frac{Y(s)}{U(s)} = \sum \frac{P_i \Delta_i}{\Delta} = \frac{P_1 + P_2 + P_3}{1 - (A + B + C)}$$

$$= \frac{\frac{b_3}{s^3} + \frac{b_2}{s^2} + \frac{b_1}{s}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}}$$

$$= \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$ } General Form of a 3rd order Transfer Function

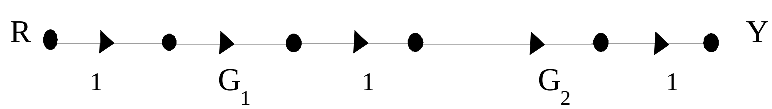
Example





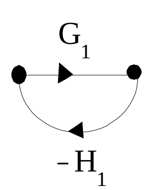
1) Find the forward paths

$P_1 = G_1 G_2$

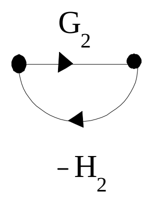


2) Find the loops

$P_{11} = -G_1 H_1$



$P_{12} = -G_2 H_2$

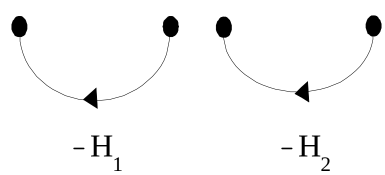


3) Find Δ and Δ_i's

$$\Delta = 1 - (P_{11} + P_{12}) + (P_{11} P_{12})$$

} Δ = 1 - (sum of all loop gains)
+ (sum of gain-products of 2 non-touching loops)
- (sum of gain-products of 3 non-touching loops)
+ (....... 4.........) - ......

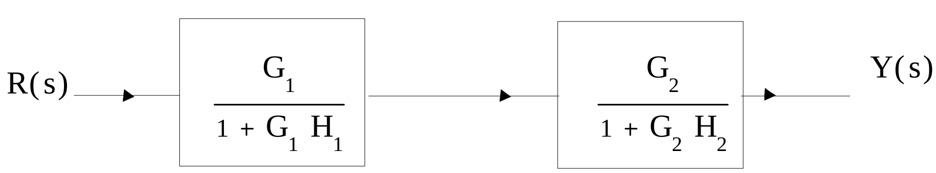
$\Delta_1 = 1$ } Recalculate Δ with P_1 removed



4) Find H(s)

$$\frac{Y(s)}{R(s)} = \frac{P_1 \Delta_1}{\Delta} = \frac{G_1 G_2}{1 + G_1 H_1 + G_2 H_2 + G_1 G_2 H_1 H_2}$$

Alternative method to solve



} closed both loops

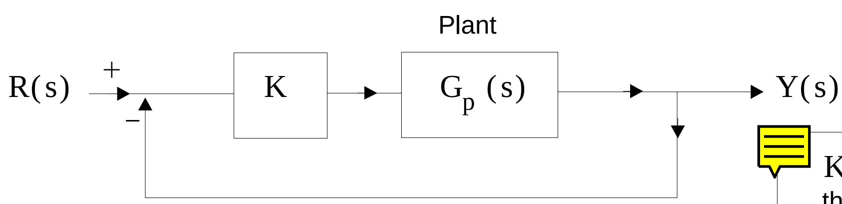
$$\frac{Y(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 H_1 + G_2 H_2 + G_1 G_2 H_1 H_2}$$

} multiply cascade blocks

Time for a Movie

Click on image to start movie

Root Locus: Construction and Design



K is a positive parameter that can be changed

$$\frac{Y(s)}{R(s)} = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \Big\} \text{ Closed Loop Transfer Function}$$

$$G_p(s) = \frac{s^m + a_{m-1} \cdot s^{m-1} + \ldots\ldots + a_0}{s^n + b_{n-1} \cdot s^{n-1} + \ldots\ldots\ldots + b_0} = \frac{N(s)}{D(s)} \Big\} \text{ General Form for the Transfer Function}$$

$N(s)$ and $D(s)$ are polynomials where $m \le n$

$$\frac{Y(s)}{R(s)} = \frac{K \cdot \frac{N(s)}{D(s)}}{1 + K \cdot \frac{N(s)}{D(s)}} = \frac{K \cdot N(s)}{D(s) + K \cdot N(s)}$$

The closed loop poles are the roots of the characteristic equation

$$\Delta(s) = D(s) + K \cdot N(s) = 0$$

The location of the roots of $\Delta(s)$ in the s-plane change as K is varied from 0 to ∞

A "Locus" of these roots plotted in the s-plane as a function of K is called the **<u>Root Locus</u>**

Different ways to write the same thing

$$\Delta(s) = 1 + K \cdot \frac{N(s)}{D(s)} = D(s) + K \cdot N(s) = 0$$

Gives the roots of the closed loop system

Root Locus Rules

1. Starting points (K = 0) — The root loci start at the open-loop poles.

2. Termination points — The root loci terminate at the open-loop zeros. The open loop zeros include those at infinity.

3. Number of distinct root loci — There will be as many root loci as the largest number of finite open loop poles or zeros. For the majority of systems, the number of finite open-loop poles will be greater than the number of finite open-loop zeros.

4. Symmetry of root loci — The root loci are symmetrical with respect to the real axis.

5. Asymptote intersection — The asymptotes intersect the real axis at a point given by

centroid formula {

$$\sigma_i = \frac{\sum open\ loop\ poles - \sum open\ loop\ zeros}{n-m}$$

6. Root Locus locations on the real axis — The root loci may be found on portions of the real axis to the left of an odd number of open-loop poles and zeros.

7. Rool locus asymptotes — The root loci are asymptotic to straight lines, for large values of s, with angles given by

$$\theta = \frac{(1+2k)\pi}{n-m}$$

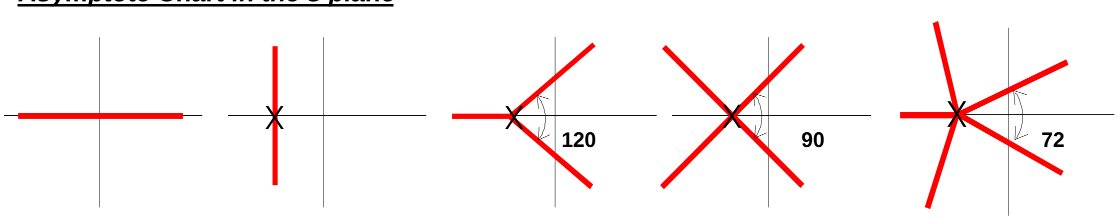
$$k = 0,1,\ldots,n-m-1$$

Let RD (relative degree) = n - m

$$n = \text{no. of finite open loop poles}$$

$$m = \text{no. of finite open loop zeros}$$

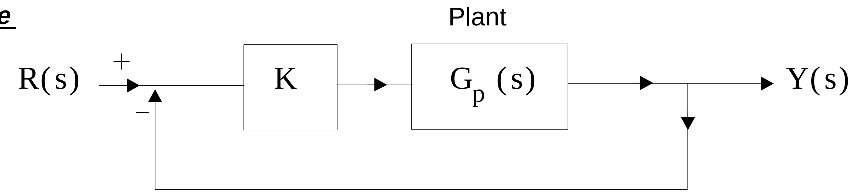
Asymptote Chart in the s plane



RD = 1 RD = 2 RD = 3 RD = 4 RD = 5

The centroids are marked **X** in the asymptote charts above

Example

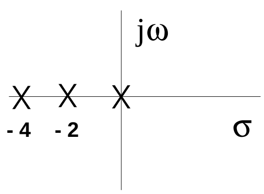


$$G_p(s) = \frac{1}{s\cdot(s+2)\cdot(s+4)}$$ } Open loop poles are at $s = 0, -2, -4$

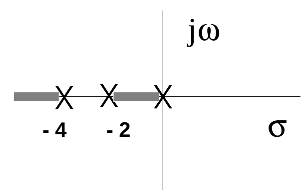
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{s\cdot(s+2)\cdot(s+4)}}{1 + K\cdot\frac{1}{s\cdot(s+2)\cdot(s+4)}}$$ } Closed Loop Transfer Function

$$\Delta(s) = 1 + K\cdot\left[\frac{1}{s\cdot(s+2)\cdot(s+4)}\right]$$ } Root Locus Form for the Denominator

Step 1. Pole Zero Plot



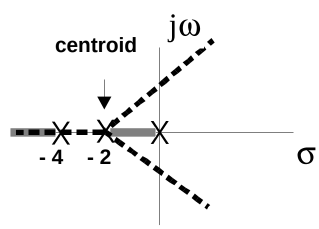
Step 2. Real Axis



Step 3. Centroid / Asymptotes

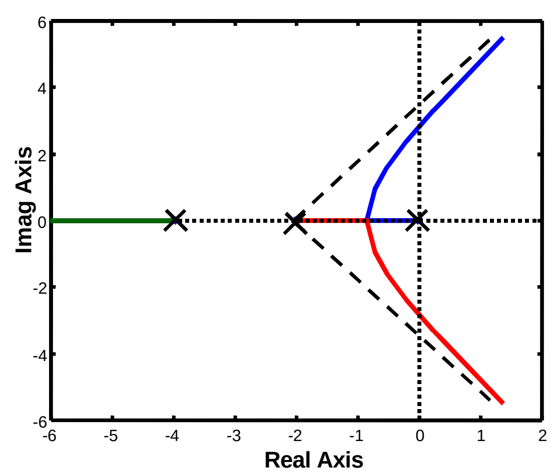
$$\text{centroid} = \frac{0 - 2 - 4 - 0}{3} = -2$$

$$RD = 3$$



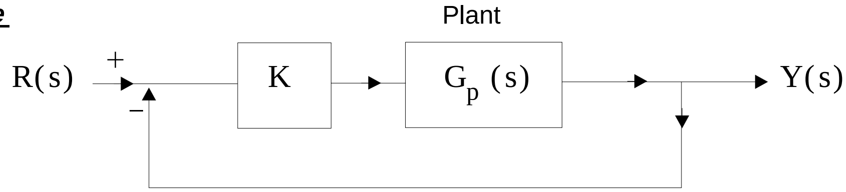
Step 4. Draw the Root Locus

a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 3
c. Locus starts at open loop poles and goes to the open loop zeros



Note: Locus is symmetric about the real axis because complex poles always come in conjugate pairs

Example



$$G_p(s) = \frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}$$

} Open loop zero $s = -2$
Open loop poles $s = -1, -3 \pm j3$

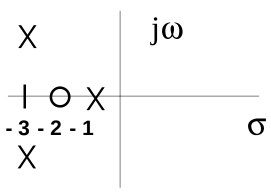
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}}{1+K\cdot\frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}}$$

} Closed Loop Transfer Function

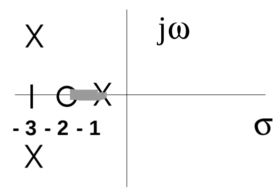
$$\Delta(s) = 1 + K\cdot\left[\frac{s+2}{(s+1)\cdot(s+3+3j)\cdot(s+3-3j)}\right]$$

} Root Locus Form for the Denominator

<u>Step 1</u>. Pole Zero Plot



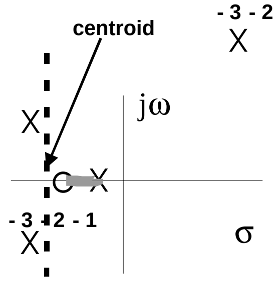
<u>Step 2</u>. Real Axis



<u>Step 3</u>. Centroid / Asymptotes

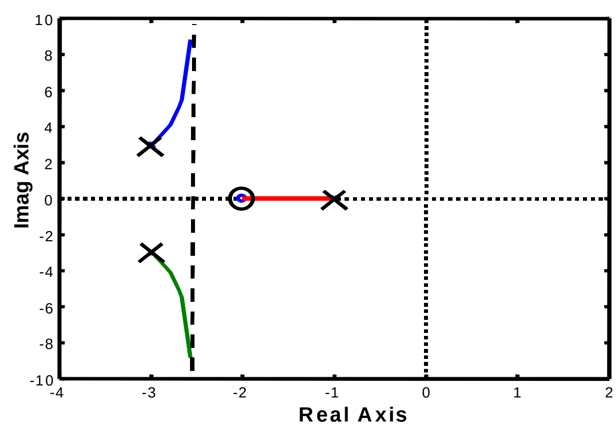
$$\text{Centroid} = \frac{-3-3-1-(-2)}{2} = \frac{-5}{2}$$

$$RD = 2$$

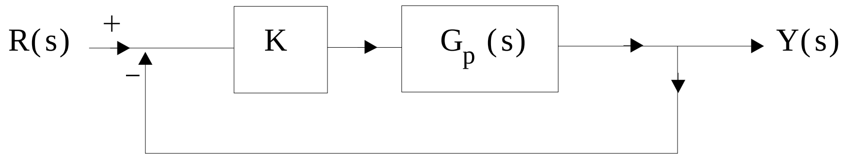


<u>Step 4</u>. Draw the Root Locus

a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 2
c. Locus starts at open loop poles and
goes to the open loop zeros



Example

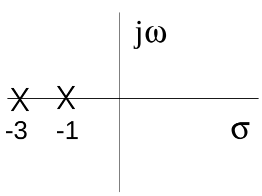


$$G_p(s) = \frac{1}{(s+1)\cdot(s+3)}$$ } Open loop poles $s = -1, -3$

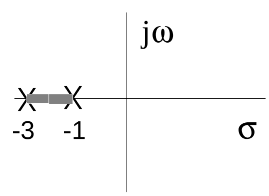
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{(s+1)\cdot(s+3)}}{1 + K\cdot\frac{1}{(s+1)\cdot(s+3)}}$$ } Closed Loop Transfer Function

$$\Delta(s) = 1 + K\cdot\left[\frac{1}{(s+1)\cdot(s+3)}\right]$$ } Root Locus Form for the Denominator

Step 1. Pole Zero Plot



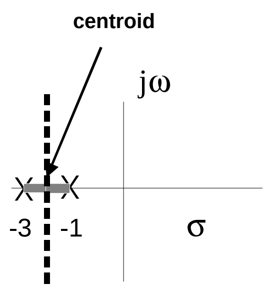
Step 2. Real Axis



Step 3. Centroid / Asymptotes

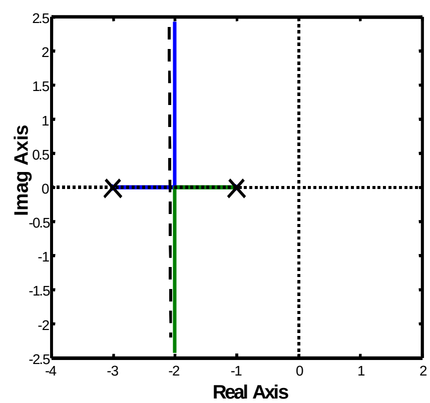
$$\text{Centroid} = \frac{-3-1}{2} = -2$$

$$RD = 2$$



Step 4. Draw the Root Locus

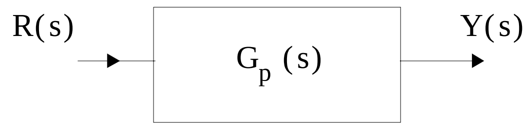
a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 2
c. Locus starts at open loop poles and
   goes to the open loop zeros



Example - Continued

Open Loop System

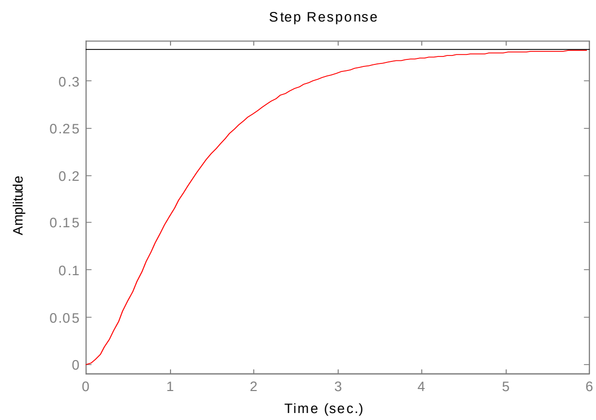
$$G_p(s) = \frac{1}{(s+1)\cdot(s+3)}$$



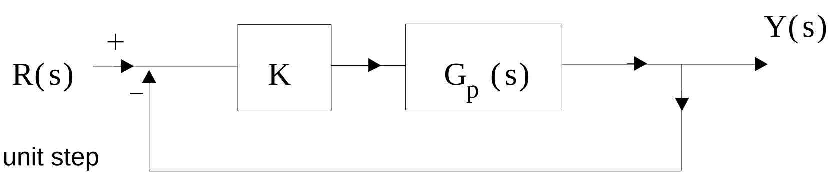
Let R(s) = Unit step

$$Y(s) = \frac{\left(\frac{1}{3}\right)}{s} + \frac{\left(\frac{-1}{2}\right)}{s+1} + \frac{\left(\frac{1}{6}\right)}{s+3}$$

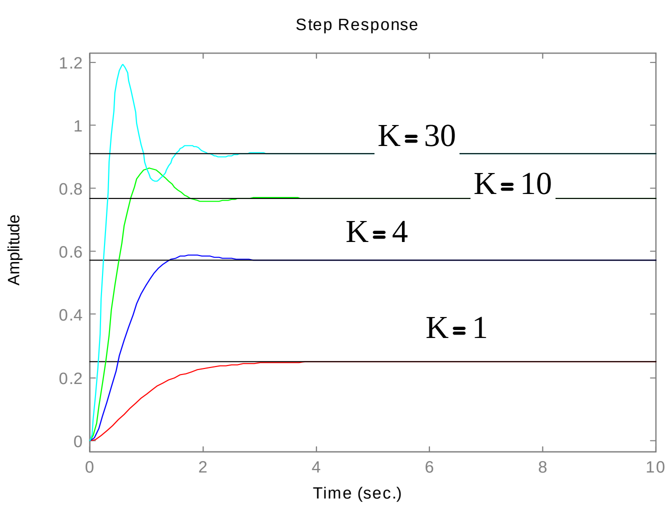
$$y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$



Closed Loop System



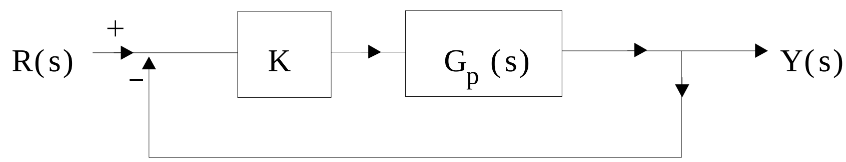
Let R(s) = unit step



} as K changes it is clear that the transient response can be "shaped"

Note: Higher values of K speed up the closed-loop response when compared to the open-loop response

Example



$$G_p(s) = \frac{1}{s\cdot(s+3)\cdot(s+3-j\cdot 7.4)\cdot(s+3+j\cdot 7.4)}$$

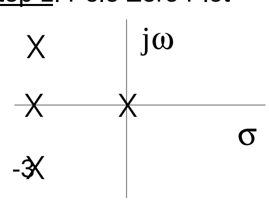
$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{s\cdot(s+3)\cdot(s+3-j\cdot 7.4)\cdot(s+3+j\cdot 7.4)}}{1+K\cdot\frac{1}{s\cdot(s+3)\cdot(s+3-j\cdot 7.4)\cdot(s+3+j\cdot 7.4)}}$$

} Closed-Loop Transfer Function

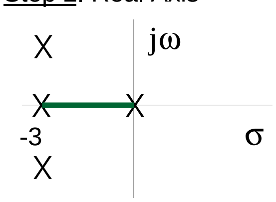
$$\Delta(s) = 1 + K\cdot\left[\frac{1}{s\cdot(s+3)\cdot(s+3-j\cdot 7.4)\cdot(s+3+j\cdot 7.4)}\right]$$

} Root Locus Form for the Denominator

Step 1. Pole Zero Plot



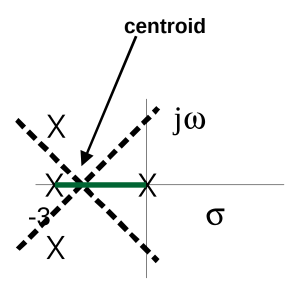
Step 2. Real Axis



Step 3. Centroid Asymptotes

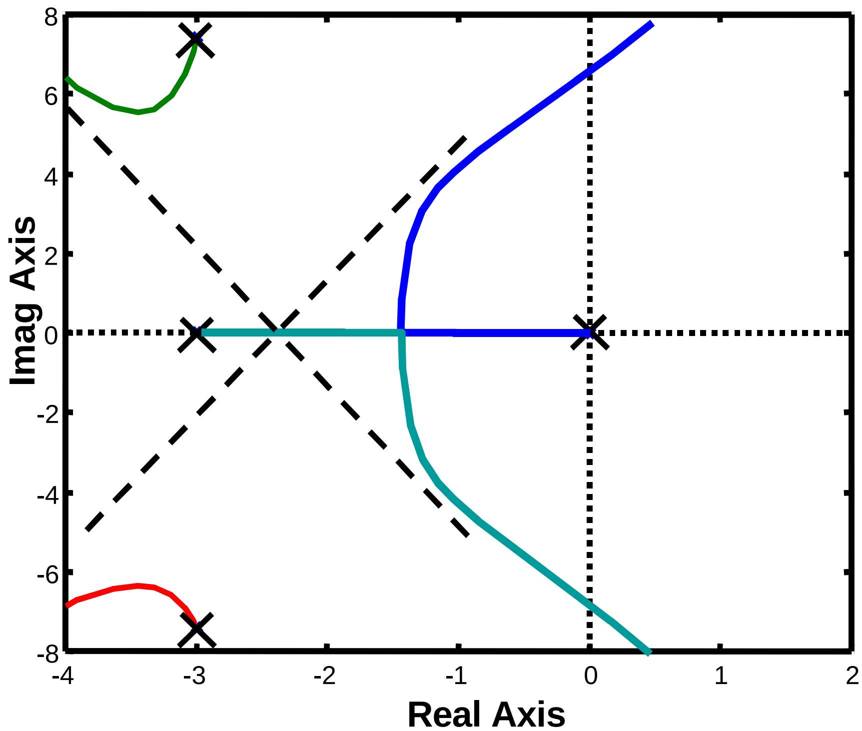
$$\text{Centroid} = \frac{-3-3-3-0}{4} = \frac{-9}{4} = -2.25$$

$$RD = 4$$



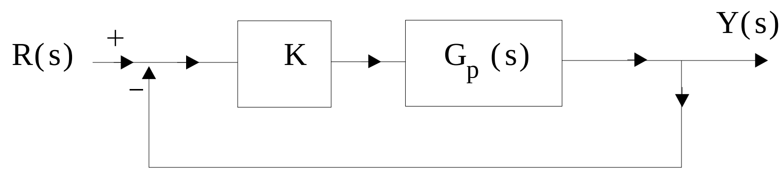
Step 4. Draw the Root Locus

a. Locus must be symmetric about real axis
b. Open loop zeros at infinity = 4
c. Locus starts at open loop poles and
   goes to the open loop zeros



Magnitude and Angle Conditions

$$\frac{Y(s)}{R(s)} = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)}$$



Poles are determined by

$$\Delta(s) = 1 + K \cdot G_p(s)$$ } characteristic equation

$$\Delta(s) \Big|_{s = clp} = 0$$ } clp denotes closed-loop pole

Magnitude Condition

$$\Delta(s) = 1 + K \cdot G_p(s) \Big|_{s = clp} = 0 \quad \Rightarrow \quad K \cdot G_p(s) \Big|_{s = clp} = -1$$ } Expression for finding clps

$$\left| K \cdot G_p(s) \right| \Big|_{s = clp} = \left| -1 \right|$$ } Take the magnitude of both sides

(Note K is a real number)

$$\boxed{K \cdot \left| G_p(s) \right| \Big|_{s = clp} = 1}$$ } Magnitude Condition

Angle Condition

$$K \cdot G_p(s) \Big|_{s = clp} = -1$$ } Expression for finding clps

$$\text{Angle}\left[K \cdot G_p(s) \right] \Big|_{s = clp} = \text{Angle}(-1)$$ } Take the angle of both sides

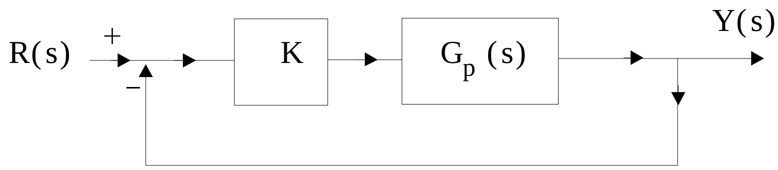
$$\text{Angle}\left[G_p(s) \right] \Big|_{s = clp} = -180$$ } Use -180 degrees as opposed to 180 degrees by convention (only here)

(Note K is a real number)

$$\boxed{\text{Angle}\left[G_p(s) \right] \Big|_{s = clp} = -180}$$ } **Angle Condition**

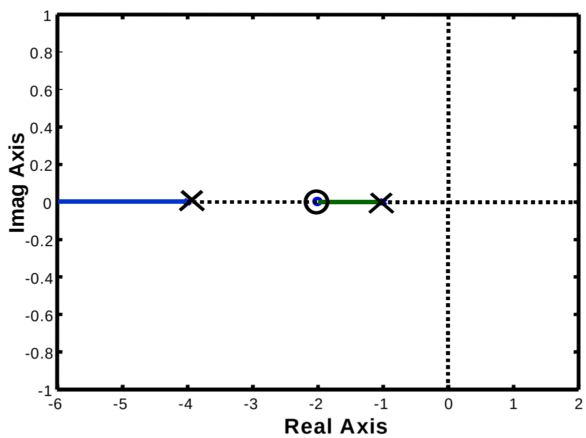
Example Find the value of K which places a closed loop pole at -5

$$G_p(s) = \frac{s+2}{(s+1)\cdot(s+4)}$$



Find the transfer function

$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{s+2}{(s+1)\cdot(s+4)}}{\left[1+K\cdot\frac{s+2}{(s+1)\cdot(s+4)}\right]}$$ } closed loop TF



} Draw the Root Locus

Note that -5 lies on the root locus

Magnitude Condition

$$K\cdot\left|G_p(s)\right|\Big|_{s=-5} = K\cdot\left|\frac{s+2}{(s+1)\cdot(s+4)}\right|\Big|_{s=-5} = 1$$

$$= K\cdot\left|\frac{-3}{(-4)\cdot(-1)}\right| = 1 \quad => \quad K = \frac{4}{3}$$

Angle Condition

$$\text{Angle}\left[G_p(s)\right]\Big|_{s=-5} = \text{Angle}(s+2) - \text{Angle}(s+4) - \text{Angle}(s+1)\Big|_{s=-5} = -180$$

$$= \text{Angle}(-3) - (\text{Angle}(-1)) - \text{Angle}(-4) = 180 - (180+180) = -180$$ } Angle Condition is satisfied

Design Example: DC Motor

The differential equation for a dc motor is given by

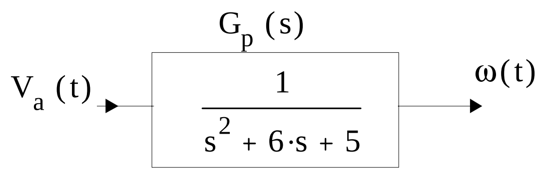
$$\frac{d^2}{dt^2}\omega(t) + \left(\frac{R_a}{L_a} + \frac{B}{J}\right)\cdot\frac{d}{dt}\omega(t) + \left[\frac{R_a}{L_a}\cdot\frac{B}{J} + \frac{(G_a\cdot I_f)^2}{L_a\cdot J}\right]\cdot\omega(t) = \frac{G_{af}\cdot I_f}{J\cdot L_a}\cdot V_a(t)$$

where $V_a(t)$ is the input voltage and $\omega(t)$ is the motor speed

After testing the motor and determining the parameters, we obtained

$$\frac{d^2}{dt^2}\omega(t) + 6\cdot\frac{d}{dt}\omega(t) + 5\cdot\omega(t) = V_a(t)$$ } Plant Model

$$\frac{\omega(s)}{V_a(s)} = \frac{1}{s^2 + 6\cdot s + 5} = G_p(s)$$



Open Loop system that we want to control

Poles of the plant are given by

$$\Delta(s) = s^2 + 6\cdot s + 5 = (s + 1)\cdot(s + 5) = 0$$

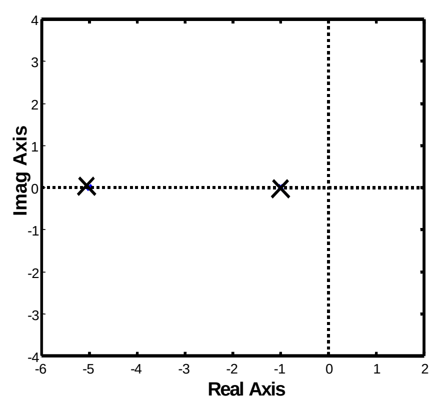
Poles are at $s = -1$ and $s = -5$

Let $V_a(t) = \delta(t)$ then $\omega(t) = \mathcal{L}^{-1}\left[G_p(s)\cdot V_a(s)\right]$

$$\omega(s) = \frac{0.25}{s + 1} + \frac{-0.25}{s + 5}$$ } PFE Tool

$$\omega(t) = \frac{-1}{4}\cdot\exp(-5\cdot t) + \frac{1}{4}\cdot\exp(-t)$$ } Inverse Laplace Transform

Pole Plot



Definition

Dominant Poles are the closest poles to the $j\omega$ axis (Assume that the system is stable)

The dominant pole corresponds to the slowest mode

The open loop system's output response will be dominated by e^{-t} **Why ?**

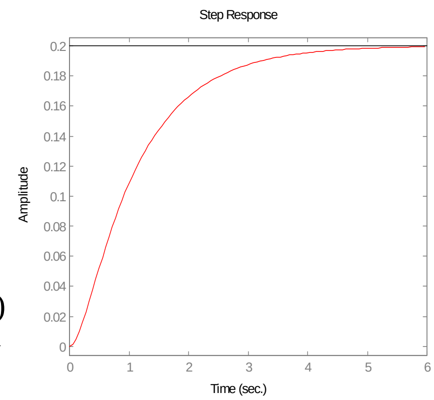
Design Example - Continued

Let us examine the response to a step input

$$V_a(s) = \frac{1}{s} \qquad \text{hence} \qquad \omega(s) = G_p(s) \cdot \frac{1}{s}$$

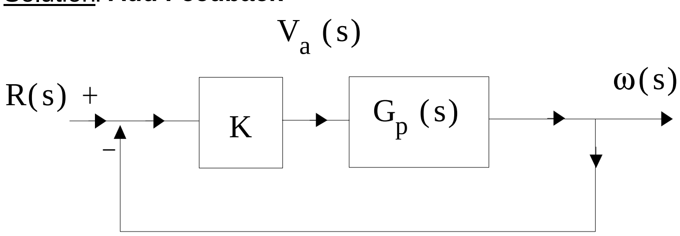
$$\omega(s) = \frac{1}{s(s+1)\cdot(s+5)} = \frac{1}{5}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{1}{(s+1)} + \frac{1}{20}\cdot\frac{1}{s+5} \Bigg\} \quad \text{PFE Tool}$$

$$\omega(t) = \frac{1}{5} - \frac{1}{4}\cdot\exp(-t) + \frac{1}{20}\cdot\exp(-5\cdot t) \Big\} \quad \text{Inverse Laplace Transform}$$



<u>Problem</u>: The response of the system maybe too slow to suit our specifications

<u>Solution</u>: **Add Feedback**

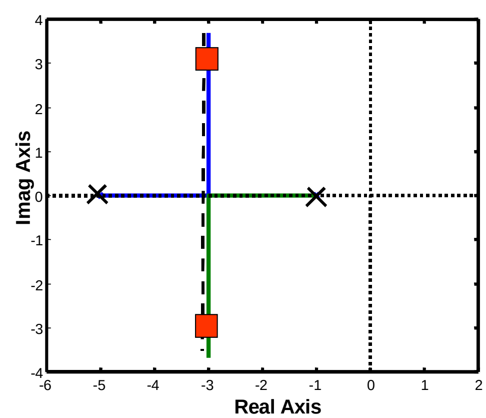


$$\frac{\omega(s)}{R(s)} = \frac{K\cdot G_p(s)}{1 + K\cdot G_p(s)} \Big\} \quad \text{Closed Loop Transfer Function}$$

Find K so that the closed loop poles are at

$$s = -3 - j\cdot 3 \quad \text{and} \quad s = -3 + j\cdot 3$$

Draw the Root Locus



<u>From Magnitude Condition</u>

$$K\cdot\left|\frac{1}{(s+1)\cdot(s+5)}\right| \Bigg|_{s=-3+j\cdot 3} = 1$$

$$K\cdot\left|\frac{1}{(-2+j\cdot 3)\cdot(2+j\cdot 3)}\right| = 1$$

$K = 13$ places the closed loop poles at

$s = -3 + j\cdot 3$ and $s = -3 - j\cdot 3$

Design Example - Continued

With K = 13 and R(s) = 1/s , find $\omega(t)$

$$\frac{\omega(s)}{R(s)} = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} = \frac{\frac{13}{(s+1)\cdot(s+5)}}{1 + \frac{13}{(s+1)\cdot(s+5)}}$$ } Closed Loop Transfer Function

$$\frac{\omega(s)}{R(s)} = \frac{13}{s^2 + 6 \cdot s + 18}$$ } $R(s) = \frac{1}{s}$

$$\omega(s) = \frac{13}{s \cdot \left(s^2 + 6\ s + 18\right)}$$ } Use PFE Tool

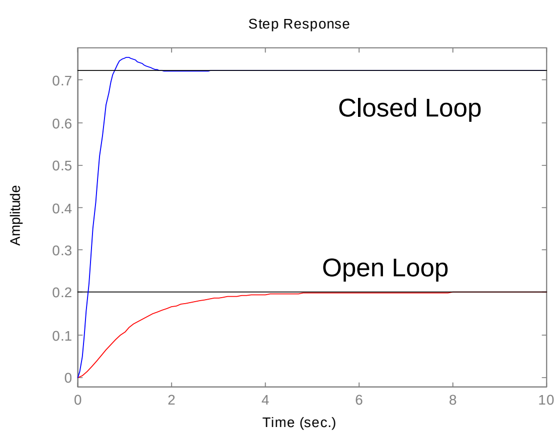
$$\omega(s) = 13 \cdot \left[\frac{1}{18} \cdot \frac{1}{s} - \frac{\frac{1}{18}s + \frac{1}{3}}{(s+3)^2 + 3^2}\right] = 13 \cdot \left[\frac{1}{18} \cdot \frac{1}{s} - \frac{1}{18} \cdot \left[\frac{s+3+3}{(s+3)^2 + 3^2}\right]\right]$$ } Make it look like the tables

$$\omega(s) = 13 \cdot \left[\frac{1}{18} \cdot \frac{1}{s} - \frac{1}{18} \cdot \left[\frac{s+3}{(s+3)^2 + 3^2} + \frac{3}{\left[(s+3)^2 + 3^2\right]}\right]\right]$$

$$\omega(t) = \frac{13}{18} - \frac{13}{18} \cdot \exp(-3 \cdot t) \cdot \sin(3 \cdot t) - \frac{13}{18} \cdot \exp(-3 \cdot t) \cdot \cos(3 \cdot t)$$ } Inverse Laplace Transform

Closed Loop $e^{-3\ t}$ — Open Loop e^{-t}

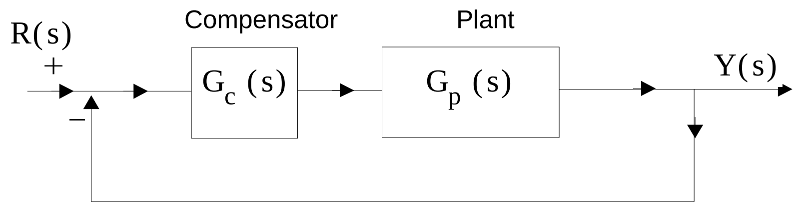
Faster Dominant Pole — Slower Dominant Pole



} Note it takes e^{-t} 5 seconds to die out while it takes $e^{-5\ t}$ one second to die out

Note that the closed loop response is faster than the open loop response

Compensation



$$G_p(s) = \frac{1}{s\cdot(s+1)}$$

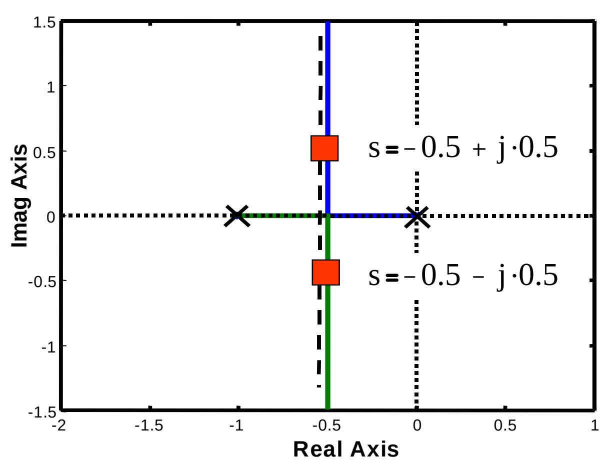
Let $G_c(s) = K$

Find K so the closed loop poles are at

$s = -0.5 + j\cdot 0.5$ and $s = -0.5 - j\cdot 0.5$

$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K}{s\cdot(s+1)}}{1 + \dfrac{K}{s\cdot(s+1)}} = \frac{K}{s^2 + s + K}$$ } Closed Loop Transfer Function

Root Locus tells us that desired closed loop poles are possible with $G_c(s) = K$



} Draw the Root Locus

Method 1: Coefficient Matching

$\Delta(s) = s^2 + s + K$ } Actual denominator

$\Delta_d(s) = (s + 0.5 + j\cdot 0.5)\cdot(s + 0.5 - j\cdot 0.5) = s^2 + s + 0.5$ } Desired denominator

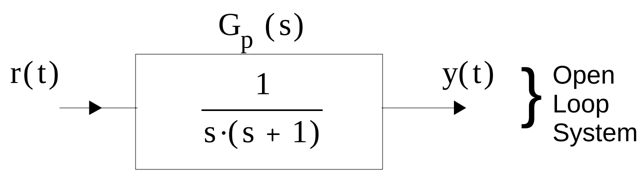
After equating coefficients, we obtain $K = 0.5$

Method 2: Magnitude Condition

$$\Delta(s) = 1 + \frac{K}{s\cdot(s+1)} \Bigg|_{s=-0.5+j\cdot 0.5} = 0$$

$$K = |s|\cdot|s+1| \Bigg|_{s=-0.5+j\cdot 0.5} \quad \text{hence} \quad K = \left(\sqrt{\frac{1}{4}+\frac{1}{4}}\cdot\sqrt{\frac{1}{4}+\frac{1}{4}}\right) = \frac{1}{2}$$

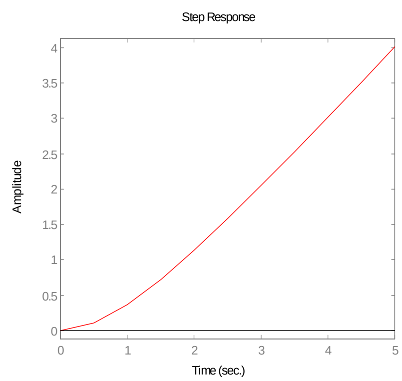
Applications of FVT and IVT for open loop systems



$$R(s) = \frac{1}{s} \qquad Y(s) = G_p(s)\cdot R(s)$$

$$Y(s) = \frac{1}{s^2\cdot(s+1)}$$

$$y(t) = A + Bt + C\cdot e^{-t}$$ } Use PFE tool



Initial Value Theorem

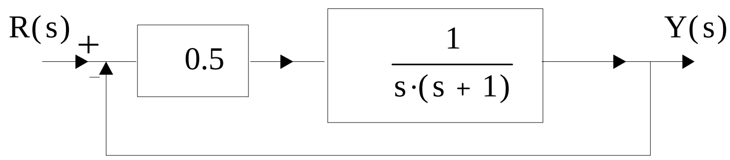
$$f(0^p) = \lim_{s\to\infty} s\cdot F(s) \qquad y(0^p) = \lim_{s\to\infty} \frac{s}{s^2\cdot(s+1)} = 0$$ } Checks with graph

Final Value Theorem

$$f(\infty) = \lim_{s\to 0} s\cdot F(s) \qquad y(\infty) = \lim_{s\to 0} \frac{s}{s^2\cdot(s+1)} = \infty$$ } Checks with graph

Application of FVT and IVT for Closed Loop System

$$R(s) = \frac{1}{s}$$



} Closed Loop System

$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{2}\cdot\frac{1}{s\cdot(s+1)}}{1 + \frac{1}{2}\cdot\frac{1}{s\cdot(s+1)}} = \frac{0.5}{s^2 + s + 0.5}$$ } closed loop TF

$$Y(s) = \frac{0.5}{s\cdot\left(s^2 + s + 0.5\right)} = \frac{0.5}{s\cdot\left[(s+0.5)^2 + (0.5)^2\right]}$$ } Use PFE tool

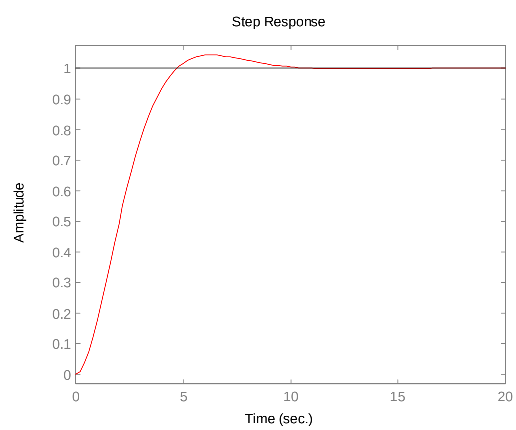
$$y(t) = A + Be^{-0.5\cdot t}\cdot\cos(0.5\cdot t) + C\cdot e^{-0.5\cdot t}\cdot\sin(0.5\cdot t)$$ } Inverse Laplace Transform

FVT :

$$y(\infty) = \lim_{s \to 0} \frac{0.5\cdot s}{s\cdot\left(s^2 + s + 0.5\right)} = 1$$

IVT :

$$y(0) = \lim_{s \to \infty} \frac{0.5\cdot s}{s\cdot\left(s^2 + s + 0.5\right)} = 0$$



} $y(t)$

FVT - tells us A = 1

IVT - tells us B = -1

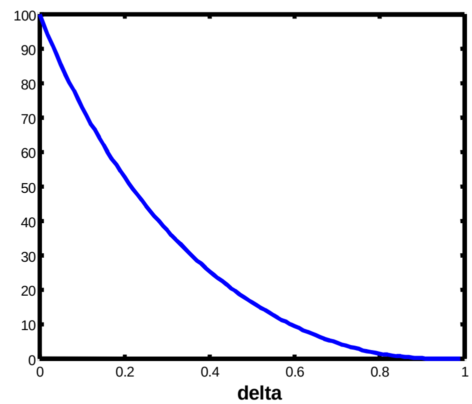
2nd Order Time Domain Transient Specifications

$$H(s) = \frac{(\omega_n)^2}{s^2 + 2\,\delta\cdot\omega_n\cdot s + (\omega_n)^2}$$ } standard form for 2nd order Transfer Function

1. **Percent Overshoot** (PO)

$$PO = \frac{\text{max value of output - final value of output}}{\text{final value of output}}$$ } time domain calculation

$$PO = 100\cdot\exp\left(\frac{-\delta\cdot\pi}{\sqrt{1-\delta^2}}\right)$$



} Graphical Solution

2. **Settling Time** - Time required for the system output to remain within b% of the final value

Typical values: $b = 2\ \%$ $\quad t_s = \frac{4}{\delta\cdot\omega_n}$ and $b = 5\ \%$ $\quad t_s = \frac{3}{\delta\cdot\omega_n}$

3. **Rise Time** - Time for the system output to go from 10% to 90% of the final value.

$$t_r = \frac{(\pi - \theta_{rad})}{\omega_n\cdot\sqrt{1-\delta^2}}$$ where θ_{rad} is in radians

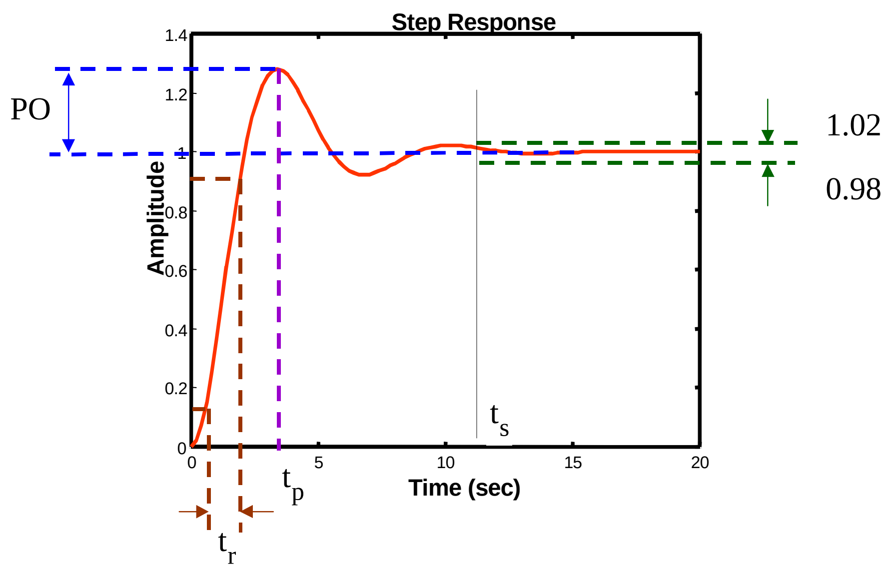
$$\delta = \cos(\theta)$$

4. **Peak Time** - Time required for system output to reach first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_n\cdot\sqrt{1-\delta^2}}$$

Above relationships are used to convert time domain specification to pole locations.

Typical Output to a Step Input



$$t_r = \frac{(\pi - \theta_{rad})}{\omega_n \cdot \sqrt{1 - \delta^2}}$$

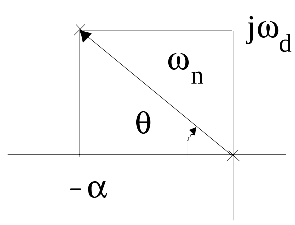
$$t_s = \frac{4}{\delta \cdot \omega_n}$$

} Conversion formulas

$$t_p = \frac{\pi}{\omega_n \cdot \sqrt{1 - \delta^2}}$$

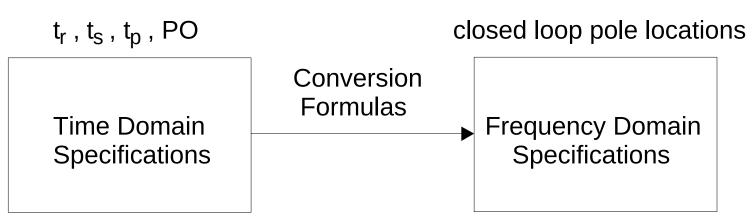
$$PO = 100 \cdot \exp\left(\frac{-\delta \cdot \pi}{\sqrt{1 - \delta^2}}\right)$$

s-plane {

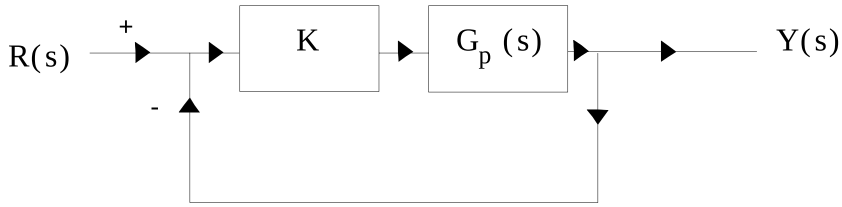


$$\delta = \cos\theta$$

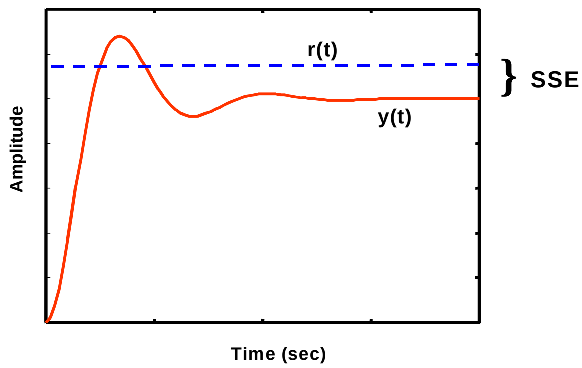
} pole locations



Steady State Error (SSE) Specification



$$e(t) = r(t) - y(t) \qquad SSE = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \quad \Big\} \text{ Final Value Theorem}$$



Example - Calculation of SSE

$$E(s) = R(s) - Y(s) \qquad Y(s) = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \cdot R(s) \quad \Big\} \text{ From Block Diagram}$$

$$E(s) = R(s) - \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \cdot R(s) \quad \Big\} \text{ Substitute Y(s)}$$

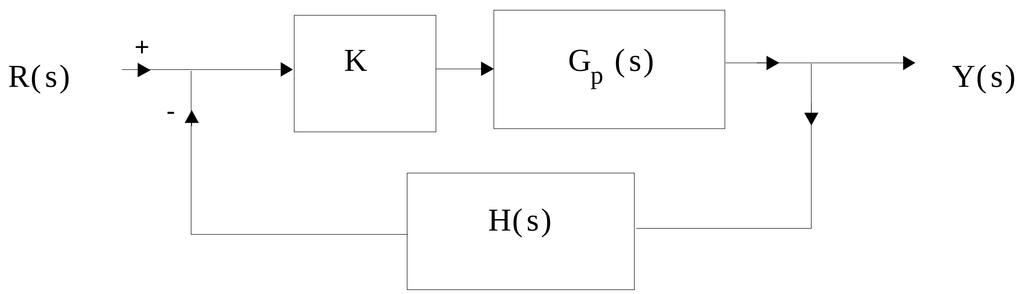
$$E(s) = \frac{R(s) \cdot \left[1 + K \cdot G_p(s)\right] - K \cdot G_p(s) \cdot R(s)}{1 + K \cdot G_p(s)} \quad \Big\} \text{ Common denominator}$$

If $K, G_p(s)$ and $R(s)$ were available, then SSE could be calculated numerically

$$E(s) = \left[\frac{1}{1 + K \cdot G_p(s)}\right] \cdot R(s) \quad \Big\} \text{ Simplify}$$

$$SSE = \lim_{s \to 0} s \cdot \left[\frac{1}{1 + K \cdot G_p(s)}\right] \cdot R(s) \quad \Big\} \text{ Final Value Theorem}$$

Example - Calculation of SSE



$$E(s) = R(s) - Y(s) \qquad Y(s) = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s) \cdot H(s)} \cdot R(s)$$ } From Block Diagram

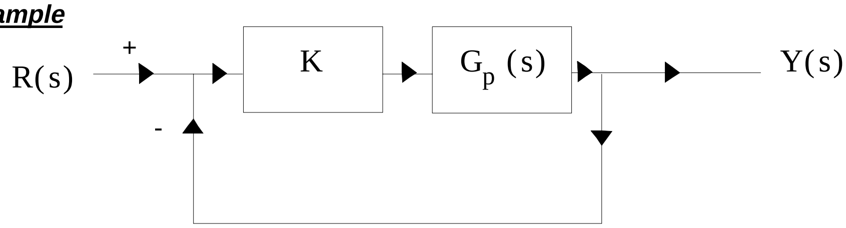
$$E(s) = R(s) \cdot \left[1 - \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s) \cdot H(s)} \right]$$ } substitute Y(s)

$$E(s) = \left[\frac{1 + K \cdot G_p(s) \cdot (H(s) - 1)}{1 + K \cdot G_p(s) \cdot H(s)} \right] \cdot R(s)$$ } common denominator

$$SSE = \lim_{s \to 0} s \cdot \left[\frac{1 + K \cdot G_p(s)\,(H(s) - 1)}{1 + K \cdot G_p(s)\,H(s)} \right] \cdot R(s)$$ } Final Value Theorem

If $K, G_p(s), H(s)$ and $R(s)$ were available, then the SSE could be calculated numerically

Design Example



$$G_p(s) = \frac{1}{(s+1)\cdot(s+5)}$$

1) Find K such that if r(t) is a unit step, then y(t) has the following characteristics

a) $t_s = \frac{4\cdot\sqrt{2}}{\sqrt{18}}$ (Settling Time: 2% criterion)

b) PO = 5% (Percent Overshoot)

2) Find y(t) for r(t) = step

3) Find the steady state error for r(t) = step

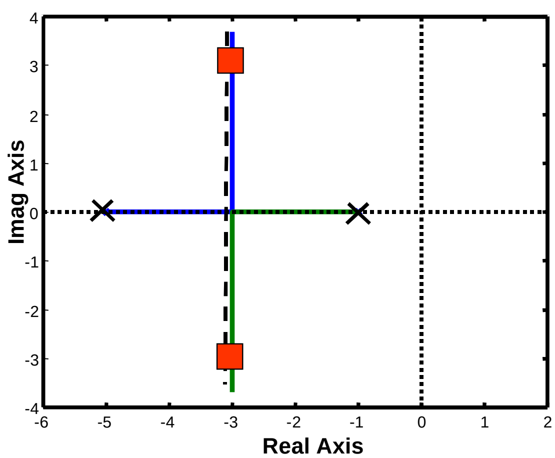
Calculate Pole location parameters.

$PO = 5\ \% \Rightarrow \delta = \frac{1}{\sqrt{2}}$ } plot $t_s = \frac{4}{\delta\cdot\omega_n} \Rightarrow \omega_n = \sqrt{18}$ } Direct Calculation

Formulate desired closed loop characteristic equation

$$\Delta_{dcl}(s) = s^2 + 2\cdot\delta\cdot\omega_n\cdot s + \left(\omega_n\right)^2 = s^2 + 6\cdot s + 18$$

Desired closed loop poles are are $s = -3 + j3$ and $s = -3 - j3$ } quadratic formula



} Root locus for

$$\frac{Y(s)}{R(s)} = \frac{K\cdot G_p(s)}{1 + KG_p(s)}$$

$$\Delta_{cl}(s) = 1 + K\cdot\frac{1}{(s+1)\cdot(s+5)}$$

Find K by Magnitude Condition

$$\frac{Y(s)}{R(s)} = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \qquad \Delta(s) = 1 + K \cdot G_p(s) \Big|_{s=-3+j3} = 0$$

$$K \cdot \left| \frac{1}{(s+1)\cdot(s+5)} \right| \Bigg|_{s=-3+j3} = 1 \quad \Rightarrow \quad K = 13 \quad \} \text{ Use Calculator}$$

2) Find y(t) when R(s) = $\frac{1}{s}$

$$Y(s) = \frac{1}{s} \cdot \left[\frac{\frac{13}{(s+1)\cdot(s+5)}}{1 + \frac{13}{(s+1)\cdot(s+5)}} \right] = \frac{13}{s \cdot \left(s^2 + 6 \cdot s + 18\right)}$$

} From Block Diagram K=13

$$Y(s) = 13 \cdot \left[\frac{1}{18 \cdot s} + \frac{A \cdot s + B}{(s+3)^2 + (3)^2} \right]$$

} PFE Formulation

$$A \cdot s^2 + B \cdot s + \frac{1}{18} \cdot s^2 + \frac{1}{3} \cdot s + 1 = 1 \qquad A = \frac{-1}{18} \qquad B = \frac{-1}{3}$$

} PFE Tool

$$Y(s) = 13 \cdot \left[\frac{1}{18\ s} - \frac{1}{18} \cdot \left[\frac{s+6}{(s+3)^2 + (3)^2} \right] \right]$$

$$Y(s) = 13 \cdot \left[\frac{1}{18 \cdot s} - \frac{1}{18} \cdot \left[\frac{s+3}{(s+3)^2 + (3)^2} + \frac{3}{(s+3)^2 + (3)^2} \right] \right]$$

} Make it look like the tables

$$y(t) = 13 \cdot \left(\frac{1}{18} - \frac{1}{18} \cdot e^{-3\ t} \cdot \cos(3 \cdot t) - \frac{1}{18} \cdot e^{-3 \cdot t} \sin(3 \cdot t) \right)$$

} Inverse Laplace Transform

Interesting check on PFE Calculation

$$y(\infty) = \lim_{s \to 0} s \cdot Y(s) = \frac{13}{18}$$

} Final Value Theorem

$$y(0) = \lim_{s \to \infty} s \cdot Y(s) = 0$$

} Initial Value Theorem

3) Find SSE when R(s) = 1/s

$$E(s) = R(s) - Y(s)$$ } Definition

$$Y(s) = \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \cdot R(s)$$ } From Block Diagram

$$E(s) = R(s) - \frac{K \cdot G_p(s)}{1 + K \cdot G_p(s)} \cdot R(s)$$ } substitute Y(s)

$$E(s) = \frac{1}{1 + K \cdot G_p(s)} \cdot R(s)$$ } Simplify

$$SSE = \lim_{s \to 0} s \cdot E(s) = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{1}{1 + \frac{13}{(s+1)\cdot(s+5)}}$$ } FVT $R(s) = \frac{1}{s}$

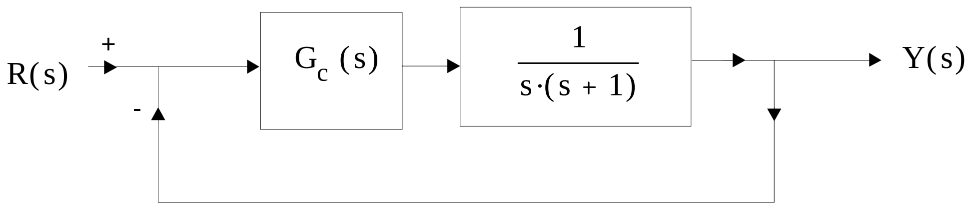
$$SSE = \frac{5}{18}$$ } answer

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} (r(t) - y(t)) = \lim_{t \to \infty} r(t) - \lim_{t \to \infty} y(t)$$

$$= 1 - \frac{13}{18} = \frac{5}{18}$$ } Another method

Note: $$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot Y(s) = \frac{13}{18}$$

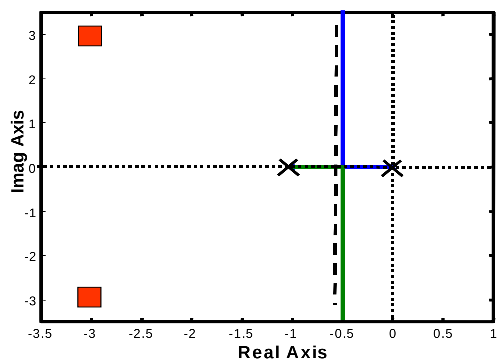
Lead Compensator (Using Magnitude and Angle Condition)



Find $G_c(s)$ such that the dominant closed loop poles are at $-3 \pm j3$

Let $G_c(s) = K$

$$\Delta_{cl}(s) = 1 + K \cdot \frac{1}{s \cdot (s+1)}$$



} Root Locus does not go through $-3 \pm j3$

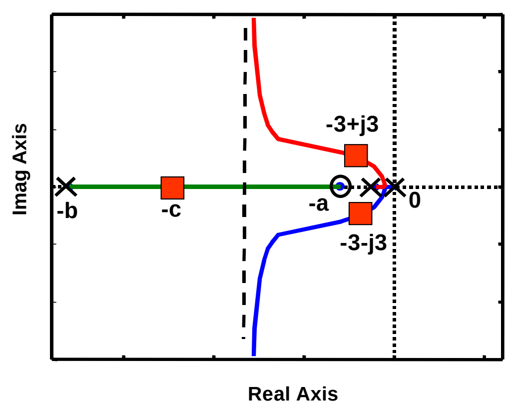
If $G_c(s) = K \cdot \dfrac{(s+a)}{(s+b)}$ } **Lead Compensation =>** $b > a > 0$

$$\Delta_{cl}(s) = 1 + K \cdot \left[\frac{(s+a)}{(s+b)} \cdot \frac{1}{s \cdot (s+1)} \right]$$

$$\text{Centroid} = \frac{-b - 0 - 1 + a}{2}$$

$RD = 2$

Root Locus goes through $-3 \pm j3$ {



Note: a closed loop pole at "-c" has been injected

By moving "a" and "b" around we can change the centroid and hence bend the root locus to go through $-3 \pm j3$

Note: $-3 \pm j3$ are more dominant that "-c"

How do we find the values for a, b and K which ensure that $-3 \pm j3$ **are the closed-loop poles ?**

Find Closed Loop Transfer Function

$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K(s+a)}{s\cdot(s+b)\cdot(s+1)}}{\left[1 + K\cdot\dfrac{s+a}{(s+b)\cdot s\cdot(s+1)}\right]}$$ } From the Block Diagram

$$\Delta(s) = 1 + K\cdot\frac{(s+a)}{s\cdot(s+b)\cdot(s+1)}\Bigg|_{s=-3+j3} = 0$$ } Achieves the desired result

Angle Condition: Find "a" and "b"

$$\text{Angle}\left[K\cdot\frac{(s+a)}{s\cdot(s+b)\cdot(s+1)}\right]\Bigg|_{s=-3+j\cdot 3} = -180$$ } One Equation Two Unknowns } Angle Conditions

$$\text{Angle}\left[\frac{(s+a)}{(s+b)}\cdot\frac{1}{s\cdot(s+1)}\right]\Bigg|_{s=-3+j\cdot 3} = -180$$ } One Equation Two Unknowns Let $a = 3$

$$-\text{Angle}(s+b)\Big|_{s=-3+j\cdot 3} = -180 + \text{Angle}\left[s\cdot\frac{(s+1)}{(s+3)}\right]\Bigg|_{s=-3+j\cdot 3}$$

$$-\tan^{-1}\left(\frac{3}{b-3}\right) = -11.3\ \text{deg} \quad \Rightarrow \quad b = 18$$

Note : $\text{Angle}\left(\frac{A}{B}\right) = \text{Angle}(A) - \text{Angle}(B) \qquad \text{Angle}(AB) = \text{Angle}(A) + \text{Angle}(B)$

$$\text{Angle}(\alpha + j\beta) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Selection of "a"

1) "a" is not unique
2) If "a" is picked too big, "b" will not exist (i.e., b may become negative)
3) If "a" is picked too small, the system might not exhibit the right dominant behavior (i.e., closed loop pole at -c might dominate the behavior)

Magnitude Condition - Find "K"

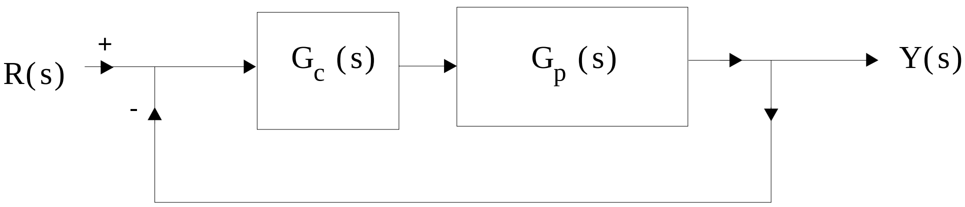
$$K\cdot\left|\frac{s+3}{(s+18)\cdot(s)\cdot(s+1)}\right|\Bigg|_{s=-3+j\cdot 3} = 1$$

$$K\cdot\left|\frac{|j\cdot 3|}{|15+j\cdot 3|\cdot|-3+j\cdot 3|\cdot|-2+j\cdot 3|}\right| = 1$$

$$K\cdot\frac{3}{(15.3)\cdot(4.24)\cdot(3.65)} = 1 \qquad K = 79 \ \text{(approximately)}$$

$$G_c(s) = 79\cdot\left(\frac{s+3}{s+18}\right) \qquad \Big\} \ \text{answer}$$

Lead Compensator (Using Coefficient Matching)



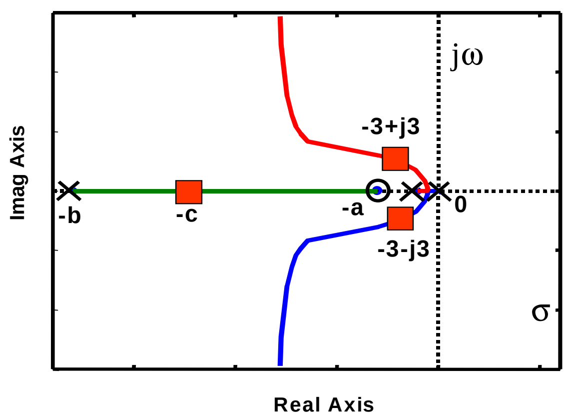
Find $G_c(s)$ so that the dominant closed loop poles are at $-3 \pm j3$

$$G_p(s) = \frac{1}{s\cdot(s+1)} \qquad G_c(s) = K\cdot\left(\frac{s+a}{s+b}\right)$$
} Select a Lead Compensator for the same reason as before

$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K\cdot(s+a)}{s\cdot(s+b)\cdot(s+1)}}{1 + K\cdot\dfrac{(s+a)}{(s+b)\cdot s\cdot(s+1)}} = \frac{K\cdot(s+a)}{s\cdot(s+1)\cdot(s+b) + K\cdot(s+a)}$$
} closed loop TF

Note: There are three closed loop poles given by the characteristic equation

$$\Delta_{cl}(s) = s\cdot(s+1)\cdot(s+b) + K\cdot(s+a)$$
} actual closed loop denominator



} Still need to draw the root locus to understand the problem

$$\Delta_{cl}(s) = s^3 + (b+1)\cdot s^2 + (b+K)\cdot s + a\cdot K$$
} actual closed loop denominator

$$\Delta_{dcl}(s) = (s+c)\cdot(s+3-j\cdot 3)\cdot(s+3+j\cdot 3)$$
} desired closed loop denominator

$$\Delta_{dcl}(s) = (s+c)\cdot\left(s^2 + 6\cdot s + 18\right)$$

$$\Delta_{dcl}(s) = s^3 + (6+c)\, s^2 + (18+6\, c)\, s + 18\, c$$
} simplification

Equate the desired denominator to the actual denominator

$\Delta_{dcl}(s) = \Delta_{cl}(s)$

$$\left.\begin{aligned} 6 + c &= b + 1 \\ 18 + 6c &= b + K \\ aK &= 18c \end{aligned}\right\}$$

3 equations, 4 unknowns (cannot solve)

Let $a = 3$ (same reason as before)

$$\left.\begin{aligned} K &= 6c \\ 5 + c = b \qquad b &= 6c + 18 - K \end{aligned}\right\} \quad \begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 1 \\ 1 & 1 & -6 \end{bmatrix} \cdot \begin{bmatrix} K \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 18 \end{bmatrix}$$

$b = 18 \qquad c = 13 \qquad K = 78$ } checks with previous answer

Note we have discovered the third closed loop pole in the process (i.e., s = - 13)

Interesting Side Note

Previous equation can be used to determine how big ''a'' can get

$$5 + c = b \qquad b = 6c + 18 - K \qquad K = \left(\frac{18}{a}\right)\cdot c \quad \}$$ eliminate K and b

$$5 + c = 6\cdot c + 18 - \left(\frac{18}{a}\right)\cdot c \quad \}$$ one equation and two unknowns

$$c\left(\frac{18}{a} - 5\right) = 13 \quad \}$$ simplified expression

From root locus, "c" must be a positive number for the problem to make sense

so if $c > 0$ then $\frac{18}{a} - 5 > 0$ therefore $a < \frac{18}{5}$

Any other selection of "a" forces "b" to be negative, and hence, the compensator would have an unstable pole.

Find the Structure of the Time Response For the Previous Problem Quickly

$$\frac{Y(s)}{R(s)} = \frac{78\cdot(s+3)}{(s+13)\cdot\left[(s+3)^2+(3)^2\right]} \qquad R(s) = \frac{1}{s} \quad \text{where} \quad r(t) \text{ is a step input}$$

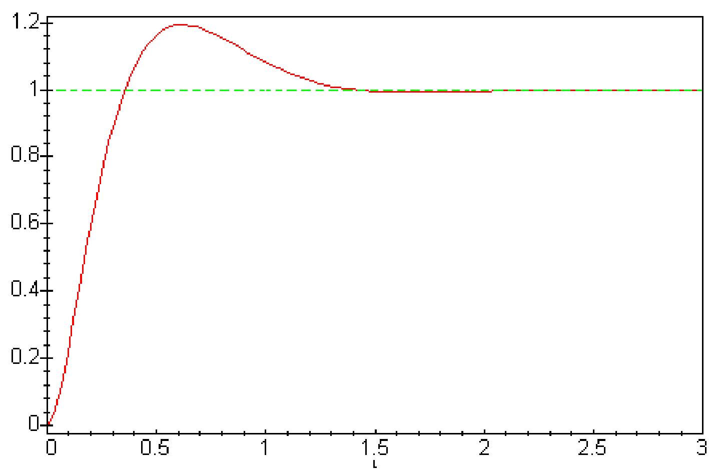
$$y(t) = A + B\cdot e^{-13t} + C\cdot e^{-3t}\cdot\cos(3t) + D\cdot e^{-3t}\cdot\sin(3t) \quad \Big\} \text{ write this immediately}$$

A, B,C, D can be found with the PFE tool

How can we find A ?

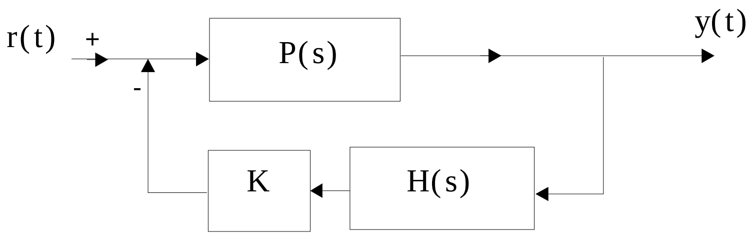
$$\lim_{s \to 0} sY(s) = \frac{78\cdot(3)}{(13)\cdot(18)} = 1 = \lim_{t \to \infty} y(t) \quad \Big\} \text{ final value theorem}$$

$$\lim_{t \to \infty} y(t) = A = 1 \quad \text{so} \quad A = 1$$



} Closed Loop Step Response

Example Find the range of K which ensures closed loop stability

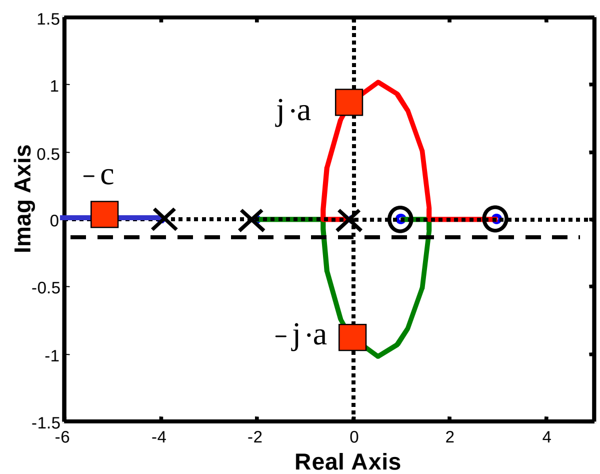


$$H(s) = \frac{s-1}{(s+4)\cdot(s+2)} \qquad P(s) = \frac{s-3}{s+0.1}$$

$$\frac{Y(s)}{R(s)} = \frac{P(s)}{1 + P(s)\cdot H(s)\,K}$$ } Closed-loop Transfer Function

$$\Delta_{cl}(s) = 1 + K\cdot P(s)\cdot H(s) = 1 + K\cdot\left(\frac{s-3}{s+0.1}\right)\cdot\left[\frac{s-1}{(s+2)\cdot(s+4)}\right]$$ } Denominator in Root Locus Form

Draw the Root Locus



Use coefficient matching to find the value of K which places closed loop poles at $\pm j.a$

$$\Delta_{cl}(s) = (s+0.1)\cdot(s+2)\cdot(s+4) + K\cdot(s-3)\cdot(s-1)$$ } Denominator in Polynomial Form

$$\Delta_{cl}(s) = (s+0.1)\cdot\left(s^2 + 6s + 8\right) + K\cdot\left(s^2 - 4s + 3\right)$$

$$\Delta_{cl}(s) = s^3 + 6.1s^2 + 8.6s + 0.8 + Ks^2 - 4\cdot K\cdot s + 3\cdot K$$

$$\Delta_{cl}(s) = s^3 + (6.1 + K)s^2 + (8.6 - 4\cdot K)\cdot s + (0.8 + 3\cdot K)$$

} Simplification

$\Delta_{cl}(s) = s^3 + (6.1 + K)\, s^2 + (8.6 - 4\cdot K)\cdot s + (0.8 + 3\cdot K)$ } actual denominator

$\Delta_{dcl}(s) = (s + c)\cdot\left(s^2 + a^2\right)$ } desired denominator (see root locus)

$\Delta_{dcl}(s) = s^3 + cs^2 + a^2\, s + a^2\, c$ } simplified

After equating the coefficients, we obtain

1) $6.1 + K = c$

2) $8.6 - 4\cdot K = a^2$

3) $0.8 + 3\cdot K = a^2\, c$

} 3 equations, 3 unknowns

After multiplying equation 1) by equation 2), and equating the result to equation 3), we obtain one equation for K

$$(6.1 + K)\cdot(8.6 - 4\, K) = 0.8 + 3\, K = a^2\, c$$

$$-4\, K^2 + (8.6 - 24.4)\cdot K + 52.46 = 0.8 + 3\, K$$

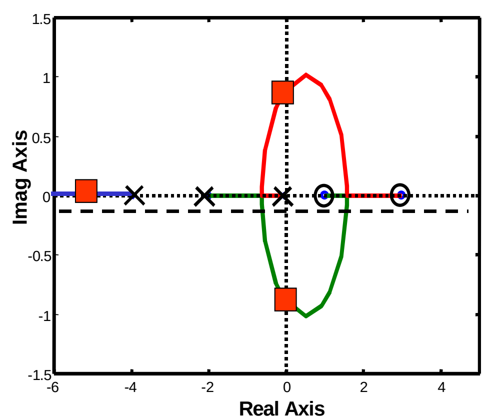
$-4\, K^2 - 18.8\, K + 51.66 = 0$ } simplified

$(K - 1.94)\cdot(K + 6.657) = 0$ } factored

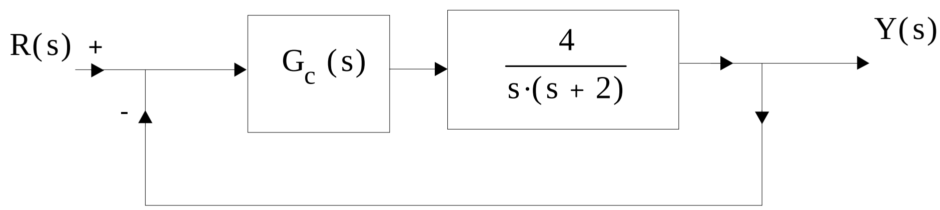
$K = 1.94$ or $K = -6.657$ } roots

Since K cannot be negative $K = 1.94$ is the answer

For $0 < K < 1.94$, the system is stable



Lead Compensator: Design Example



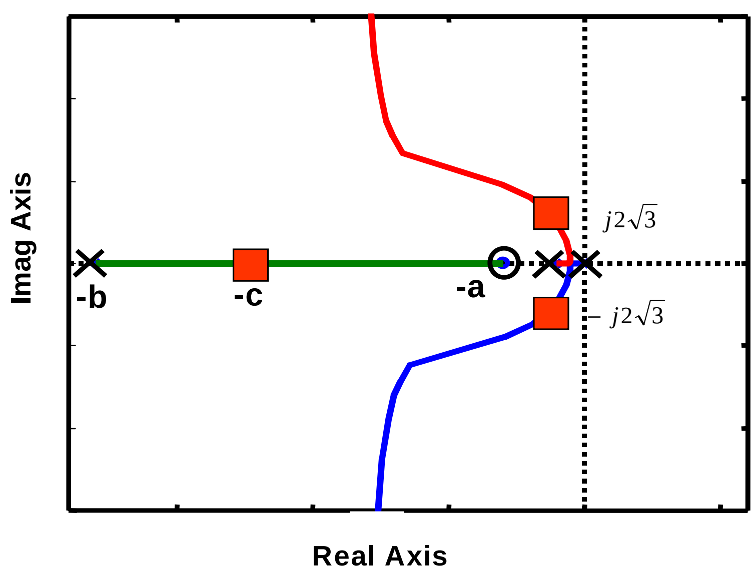
Design $G_c(s)$ so that the dominant closed loop poles are at $s = -2 + j\cdot 2\sqrt{3}$

and $s = -2 - j\cdot 2\sqrt{3}$

$$G_c(s) = K\cdot\left(\frac{s+a}{s+b}\right)$$ } Lead Compensator

$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{(s+a)}{(s+b)}\cdot\frac{4}{s\cdot(s+2)}}{1 + K\cdot\left[\frac{(s+a)}{(s+b)}\cdot\frac{4}{s\cdot(s+2)}\right]}$$ } Closed Loop Transfer Function

$$\Delta_{cl}(s) = 1 + K\cdot\left(\frac{s+a}{s+b}\right)\cdot\frac{4}{s\cdot(s+2)}$$



} Draw the Root Locus

Method I Angle/Magnitude Condition

Angle Condition

Let $a = 2.9$ } Do you know why ?

$$\text{Angle}\left[\left(\frac{s+2.9}{s+b}\right)\cdot\left[\frac{4}{(s+2)\cdot(s)}\right]\right]\Bigg|_{s=-2+j\cdot 2\cdot\sqrt{3}} = -180$$

$$-\text{Angle}(s+b)\Bigg|_{s=-2+j\cdot 2\sqrt{3}} = -180 - \text{Angle}\left[\frac{4\cdot(s+2.9)}{s\cdot(s+2)}\right]\Bigg|_{s=-2+j\cdot 2\sqrt{3}}$$

$$-\tan^{-1}\frac{2\sqrt{3}}{b-2} = -45 \quad \Rightarrow \quad b = 5.46$$

Magnitude Condition

$$K\cdot\left|\left(\frac{s+2.9}{s+5.46}\right)\cdot\left[\frac{4}{s\cdot(s+2)}\right]\right|\Bigg|_{s=-2+j2\sqrt{3}} = 1 \quad \Rightarrow \quad K = 4.68$$

Method II: Coefficient Matching

$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{(s+a)}{(s+b)}\,\frac{4}{s\cdot(s+2)}}{1+K\cdot\left(\frac{s+a}{s+b}\right)\cdot\frac{4}{s(s+2)}} = \frac{4\,K\,(s+a)}{s\cdot(s+b)\cdot(s+2)+(Ks+Ka)\,4}$$ } Closed-Loop Transfer Function

$$\Delta_{cl}(s) = s^3 + (2+b)\cdot s^2 + (2\,b + 4\,K)\cdot s + 4\,Ka$$ } simplified actual denominator

$$\Delta_{dcl}(s) = \left(s + 2 + j\cdot 2\sqrt{3}\right)\cdot\left(s + 2 - j\cdot 2\sqrt{3}\right)\cdot(s+c)$$ } desired denominator (see root locus)

$$\Delta_{dcl}(s) = s^3 + (4+c)\,s^2 + (16 + 4\cdot c)\cdot s + 16\cdot c$$ } simplified

$$4 + c = 2 + b$$

$$2 \cdot b + 4 \cdot K = 16 + 4 \cdot c$$

$$16 \cdot c = 4 \cdot K \cdot a$$

} 3 equations

4 unknowns

Let $a = 2.9$ } Do you know why ?

$$\begin{bmatrix} 1 & -1 & 0 \\ -4 & 2 & 4 \\ 16 & 0 & -11.6 \end{bmatrix} \cdot \begin{bmatrix} c \\ b \\ K \end{bmatrix} = \begin{bmatrix} -2 \\ 16 \\ 0 \end{bmatrix}$$

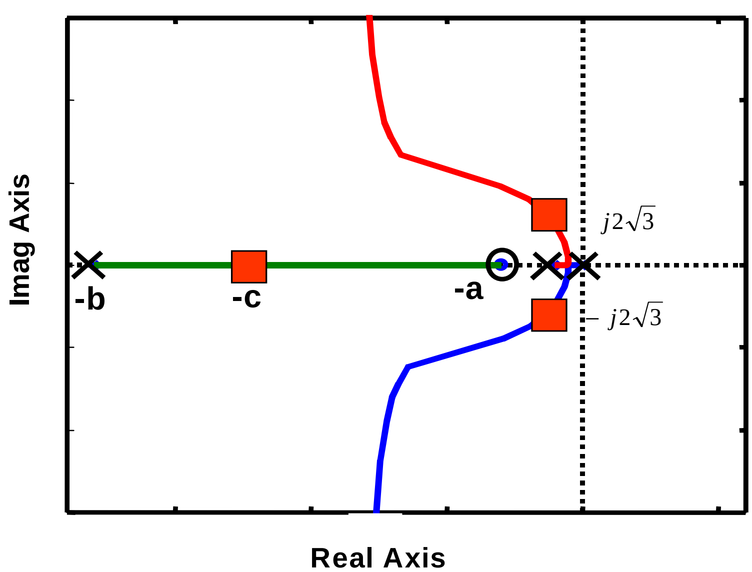
} 3 equations

3 unknowns

$$\begin{bmatrix} c \\ b \\ K \end{bmatrix} = \begin{bmatrix} 3.4 \\ 5.4 \\ 4.68 \end{bmatrix}$$

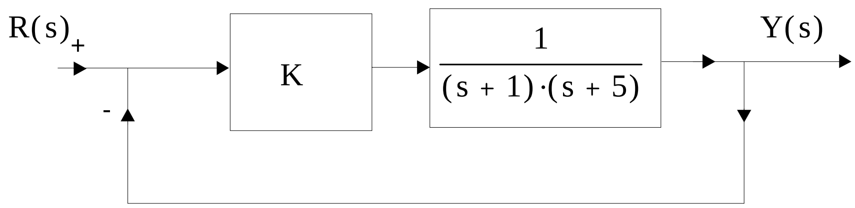
} Use calculator to solve (same answer as before)

Note: "c" is between "a" and "b"



Steady State Error Specification

Decrease the steady state error so that the output goes close to the input
(Remember that e(t) = r(t) - y(t))



Let $R(s) = \frac{1}{s}$

$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{1}{(s+1)\cdot(s+5)}}{1 + K\cdot\frac{1}{(s+1)\cdot(s+5)}}$$ } Closed Loop Transfer Function

$$E(s) = R(s) - Y(s)$$ } By Defintion

$$E(s) = R(s) - \left[\frac{K\cdot\frac{1}{(s+1)\cdot(s+5)}}{1 + K\cdot\frac{1}{(s+1)\cdot(s+5)}}\right] R(s)$$

$$E(s) = \frac{1}{1 + K\cdot\left[\frac{1}{(s+1)\cdot(s+5)}\right]}\cdot R(s)$$ } Simplify

$$SSE = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\cdot E(s) = \frac{1}{1 + \frac{K}{5}}$$ } Apply the final value theorem $R(s) = \frac{1}{s}$

<u>so as K goes up SSE goes down</u>

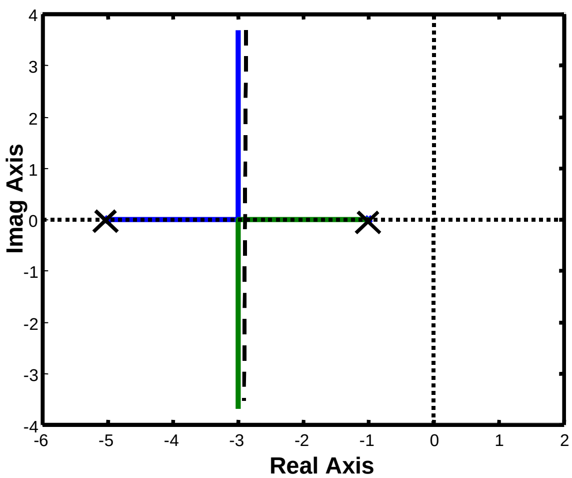
Same Problem

$$\frac{Y(s)}{R(s)} = \frac{K \cdot \frac{1}{(s+1)\cdot(s+5)}}{1 + K \cdot \frac{1}{(s+1)\cdot(s+5)}}$$

Find K so that the closed loop poles are at $-3 \pm j3$

$$\Delta_{cl}(s) = 1 + \frac{K}{(s+1)\cdot(s+5)}$$

} Root Locus Form



} Draw the Root Locus

Magnitude Condition (Find K)

$$K\left|\frac{1}{(s+1)(s+5)}\right|_{s=-3+3j} = 1$$

=> $K = 13$

$$SSE = \frac{1}{1 + \frac{K}{5}} = \frac{5}{18}$$

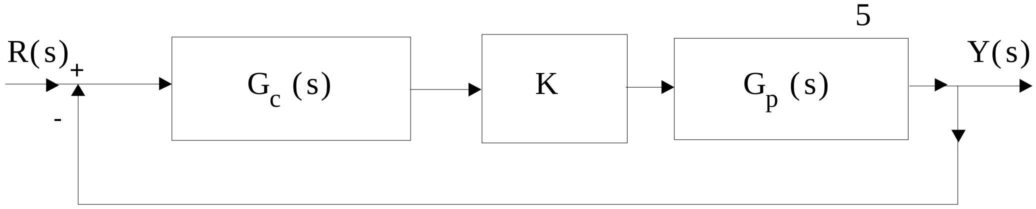
} From the previous slide with K = 13

From the above problem, we can see that the SSE is fixed since we have selected a value of K to place the poles at $-3 \pm j3$ It would be desireable to place the pole and reduce the SSE at the same time

Lag Compensation

(The preferred method for reducing SSE)

$$SSE = \frac{1}{1+\frac{13}{5}} = \frac{5}{18}$$ } For K = 13



Let $R(s) = \frac{1}{s}$

$$G_p(s) = \frac{1}{(s+1)\cdot(s+5)}$$

Let $K = 13$ } To meet a transient specification

$$G_c(s) = \left(\frac{s+c}{s+d}\right)$$ } $c > d > 0$

Typical Lag Compensator $c = 0.1$ $d = 0.01$

$$E(s) = \frac{1}{1 + K\cdot G_p(s)\cdot G_c(s)}$$ } From block diagram $E(s) = R(s) - Y(s)$

$$SSE(s) = \lim_{s \to 0} s\cdot E(s) = \lim_{s \to 0} s\cdot\frac{1}{s}\cdot\frac{1}{1 + K\cdot G_p(s)\cdot G_c(s)}$$ } Final Value Theorem

$$SSE(s) = \lim_{s \to 0} \frac{1}{1 + 13\cdot\left(\frac{s+0.1}{s+0.01}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+5)}\right]}$$ } Plug in the transfer functions

$$SSE(s) = \frac{1}{1+\frac{130}{5}} = \frac{1}{27} \quad \text{(approximately)}$$ } From previous slide

If $G_c(s) = 1$ and $K = 13 \Rightarrow SSE = \frac{1}{3}$ (approximately)

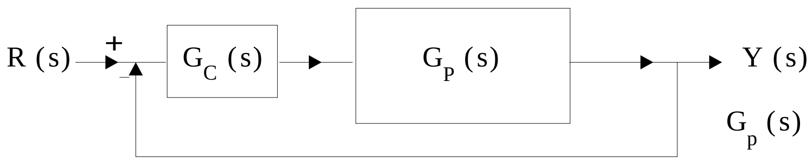
with $G_c(s) = \frac{s+0.1}{s+0.01}$ SSE is a factor of 7.5 better than with $G_c(s) = 1$

Question : Have we created a problem by adding a lag compensator ?

That is, will the output still behave according to the dominant specifications ?

Lead/Lag Compensators

Example



$$G_p(s) = \frac{1}{(s+1)\cdot(s+5)}$$

Design a lead/lag compensator that has

1) Dominent closed loop poles as s = - 4 +/- j5

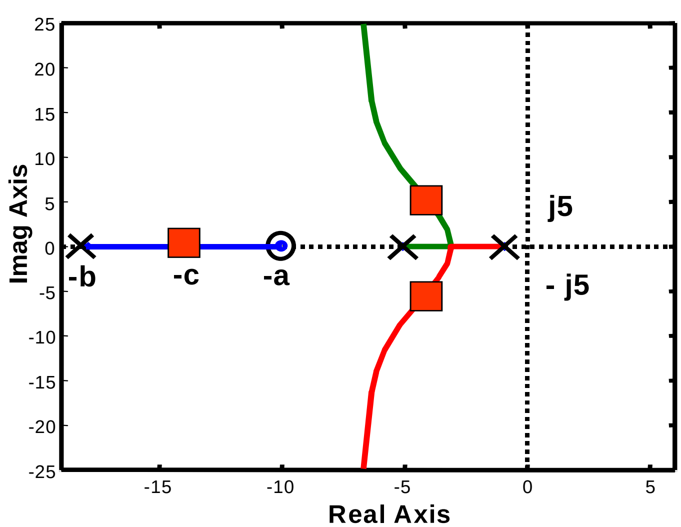
2) SSE = 0.01   when R(s) = 1/s

Design Lead Compensator to place "dominant" closed-loop poles at $-4 \pm j5$ while neglecting the effects of the lag compensator. Then design lag to achieve SSE specification.
Note - if we don't neglect the lag during the lead design, the problem becomes fairly difficult to solve.

Let $G_c(s) = \frac{K(s+a)}{(s+b)}$ } Lead Compensator

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)\cdot G_p(s)}{1 + G_c(s)\cdot G_p(s)}$$ } closed loop TF

$$\Delta(s) = 1 + K\cdot\left(\frac{s+a}{s+b}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+5)}\right]$$ } Denominator form for Root Locus



} Draw Root Locus

Angle Condition

$$\text{angle}\left[\frac{s+10}{(s+b)\,(s+1)\,(s+5)}\right]_{(s=-4+5j)} = -180$$ } Let a = 10 **Why ?**

$$-\tan^{-1}\left(\frac{5}{-4+b}\right)=-19.3 \quad \Rightarrow \quad b = 18.3$$

Magnitude Condition

$$K\left|\frac{s+10}{(s+18)\,(s+1)\,(s+5)}\right|_{s=-4+5j} = 1$$

$$K = \left|\frac{(s+18)\,(s+1)\,(s+5)}{(s+10)}\right|_{s=-4+5j} = 56.6$$

Lag Compensator Design:

$$G_c(s)=56.6\cdot\underbrace{\frac{s+10}{s+18}}_{\text{Lead}}\cdot\underbrace{\frac{s+c}{s+d}}_{\text{Lag}}$$ } General Form for the Lead/Lag

$$SSE=\lim_{s\to 0} s\cdot\frac{1}{1+G_c(s)\,G_p(s)}\cdot R(s)=0.01$$ } From Block Diagram

$$SSE=\frac{1}{1+\frac{56.6\,(10)}{1+(18)\,(1)\,(5)}\frac{c}{d}}$$ } $R(s)=\frac{1}{s}$

$$SSE=\frac{1}{1+6.3\left(\frac{c}{d}\right)}=0.01$$ } simplify

$$\frac{c}{d} = 15.7$$ } obtained a ratio for "c" and "d"

"c" is typically selected to be 0.1; hence, d = 0.00625

$$G_c(s)=56.6\cdot\frac{s+10}{s+18.3}\cdot\frac{s+0.1}{s+0.00625}$$ } Final Compensator Design

Why can we neglect the Lag dynamics when we design the Lead compensator ?
That is, how close was the above calculation ?
Let us re-examine the Lead calculation with the Lag Dynamics inserted

$$\Delta_{cl}(s) = \left[1 + 56.6\cdot\left(\frac{s+10}{s+18.3}\right)\cdot\left(\frac{s+0.1}{s+.00625}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+5)}\right]\right]\Bigg|_{s=-4+j5} = 0$$ } Exact Characteristic Equation

Angle Condition

$$\left(\frac{s+10}{s+18.3}\right)\cdot\left(\frac{s+0.1}{.00625}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+5)}\right]\Bigg|_{s=-4+j5}$$ should be equal to -180;

however, it is equal to -179.798

Magnitude Condition

$$56.6\left[\left(\frac{s+10}{s+18.3}\right)\cdot\left(\frac{s+0.1}{s+.00625}\right)\cdot\frac{1}{(s+1)\cdot(s+5)}\right]_{s-4+5j}$$ should be equal to 1;

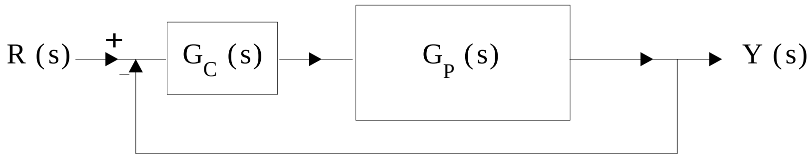
however, it is equal to 0.973

The reason that the above calculations are still close is because } $\left(\frac{s+0.1}{s+.00625}\right)_{s=-4+5j} = 0.9819\ \angle -0.66^\circ$

which is close to $1\ \angle 0$

Hence, by selecting the lag compensator zero close to the $j\omega$ axis and the lag compensator pole relatively close to the lag compensator zero, we change the angle and magnitude calculation very little. The above approximation method is fairly accurate.

Example



Design a lead/lag compensator that has

$$G_p(s) = \frac{1}{(s+1)\,(s+3)\,(s+10)}$$

1) Dominant closed loop poles at s = -2 +/- 2j

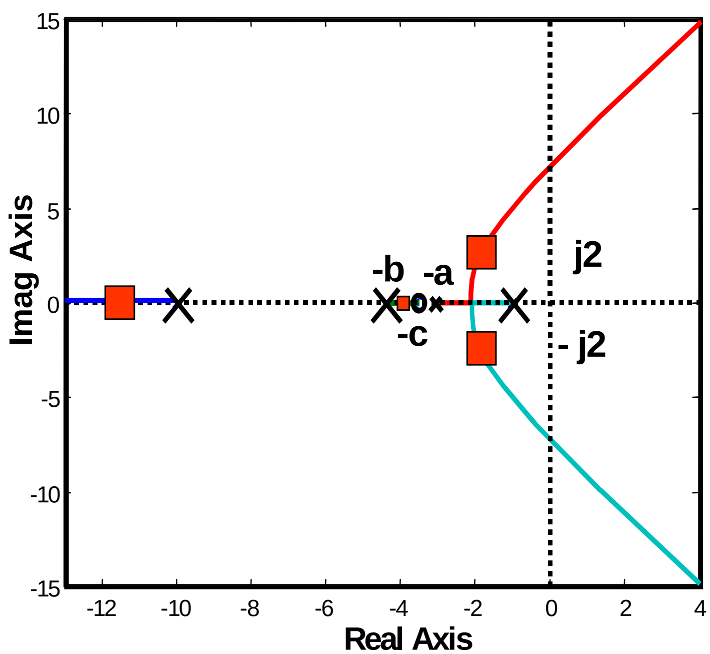
2) SSE = 0.01 when $R(s) = \frac{1}{s}$

Design Lead to place the "dominant" closed-loop poles at -2 +/- 2j while neglecting the effects of the Lag compensator.

$$G_c(s) = K\cdot\left(\frac{s+a}{s+b}\right)$$ } Lead Compensator

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)\cdot G_p(s)}{1+G_c(s)\cdot G_p(s)}$$ } closed loop TF

$$\Delta(s) = 1 + K\cdot\left(\frac{s+a}{s+b}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+3)\cdot(s+10)}\right]$$ } Denominator form for Root Locus



} Draw Root Locus

Angle Condition

$$\text{angle}\left[\frac{s+3.5}{(s+b)\,(s+1)\,(s+3)\cdot(s+10)}\right]_{(s=-2+2j)} = -180$$ } Let a = 3.5 Why ?

$$\Rightarrow \quad b = 4.38$$

Magnitude Condition

$$K\left|\frac{s+3.5}{(s+4.38)\,(s+1)\,(s+3)\cdot(s+10)}\right|_{s=-2+2j} = 1 \quad \Rightarrow \quad K = 50$$

Lag Compensator Design:

$$G_c(s) = 50.0\cdot\underbrace{\frac{s+3.5}{s+4.38}}_{\text{Lead}}\cdot\underbrace{\frac{s+c}{s+d}}_{\text{Lag}}$$ } General Form for the Lead/Lag

$$SSE = \lim_{s\to 0} s\cdot E(s) = \lim_{s\to 0}\frac{1}{1+G_c(s)\,G_p(s)} \quad R(s) = 0.01$$ } From Block Diagram

$$SSE = \frac{1}{1+\frac{1\,(50)\,(3.5)\,(c)}{30\,(4.38)\,d}} = \frac{1}{1+\frac{1.332\cdot c}{d}} = 0.01$$

$$\frac{c}{d} = 74.32$$ } obtained a ratio for "c" and "d"

"c" is typically chosen to be 0.1; hence, d = .001345

Why can we neglect the Lag dynamics when we design the Lead compensator ?
That is, how close was the above calculation ?
Let us re-examine the Lead calculation with the Lag Dynamics inserted

$$\Delta_{cl}(s) = \left[1+50.0\cdot\left(\frac{s+3.5}{s+4.38}\right)\cdot\left(\frac{s+0.1}{s+.001345}\right)\cdot\left[\frac{1}{(s+1)\cdot(s+3)\cdot(s+10)}\right]\right]\Bigg|_{s=-2+j2} = 0$$

..... Exact Characteristic Equation

Angle Condition

angle $\left[\left(\frac{s + 3.5}{s + 4.38}\right)\cdot\left(\frac{s + 0.1}{s + .001345}\right)\cdot\frac{1}{(s + 1)\cdot(s + 3)\cdot(s + 10)}\right]_{s = -2 + j2}$ should be equal to -180;

however, it is equal to -182.398

Magnitude Condition

$50\cdot\left|\left[\left(\frac{s + 3.5}{s + 4.38}\right)\cdot\left(\frac{s + 0.1}{s + .001345}\right)\cdot\frac{1}{(s + 1)\cdot(s + 3)\cdot(s + 10)}\right]\right|_{s = -2 + j2}$ should be equal to 1;

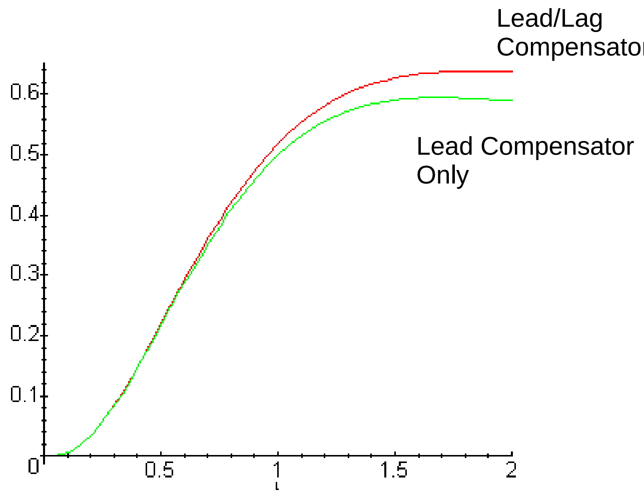
however, it is equal to 0.9034

The reason the above calculations are still close is because $\left.\left(\frac{s + 0.1}{s + 0.001345}\right)\right|_{s = -2+2j} = 0.9512\angle -1.43707^{\circ}$

which is close to $1 \ \angle 0$

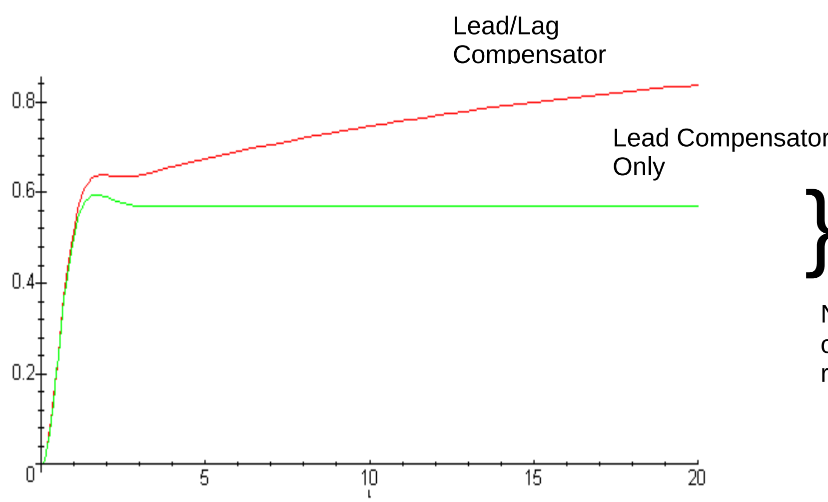
Hence, by selecting the lag compensator zero close to the $j\omega$ axis and the lag compensator pole relatively close to the lag compensator zero, we change the angle and magnitude calculation very little. The above approximation method is fairly accurate.

Note: The transient response of the Lead / Lag compensator is similar to the Lead / No Lag compensator; however, it is clear that the steady state responses are very different



} output response to a step input

Note the **similarities** of the **transient** responses



} output response to a step input

Note the **differences** of the **steady state** responses

Time for a Movie

Click on image to start movie

Bode Plot Construction

$$H(s) = K\cdot\frac{(s+a)\,(s+b)}{s^n\cdot(s+c)\cdot(s+d)\cdot(s+e)}$$

} Typical Transfer Function a,b,c,d,e are real positive numbers

Bode Form : Let $s = j\omega$ and normalize

$$H(s) = K\cdot\frac{ab}{ecd}\cdot\frac{\left(\frac{j\omega}{a}+1\right)\cdot\left(\frac{j\omega}{b}+1\right)}{(j\omega)^n\cdot\left(\frac{j\omega}{c}+1\right)\cdot\left(\frac{j\omega}{d}+1\right)\cdot\left(\frac{j\omega}{e}+1\right)}$$

} Bode Form

<u>Magnitude Plot</u>

1) Curve breaks { up / down } by 20 dB/dec at { a,b / c,d,e}

<u>Angle Plot</u>

2) Curve breaks { up / down } by 45 degrees one decade before { a,b / c,d,e } and breaks { down / up } by 45 degrees one decade after { a,b / c,d,e }

<u>Initial Magnitude Plot</u>

<u>Case 1: n=0</u>

Calculate $H(j\omega)\Big|_{\omega=0}$ and then convert to dB i.e., $20\cdot\log(H(j\omega))\Big|_{\omega=0}$

<u>Case 2: n > 0</u>

Calculate the ω - intercept $\omega - \text{intercept} = \left(\frac{Kab}{cde}\right)^{\frac{1}{n}}$

Calculate the initial slope $-20\cdot n$ dB / dec

Final Magnitude Slope = -20 . RD dB / dec

Initial and Final Phase Plot

1) Plot a pole / zero plot for the original transfer function

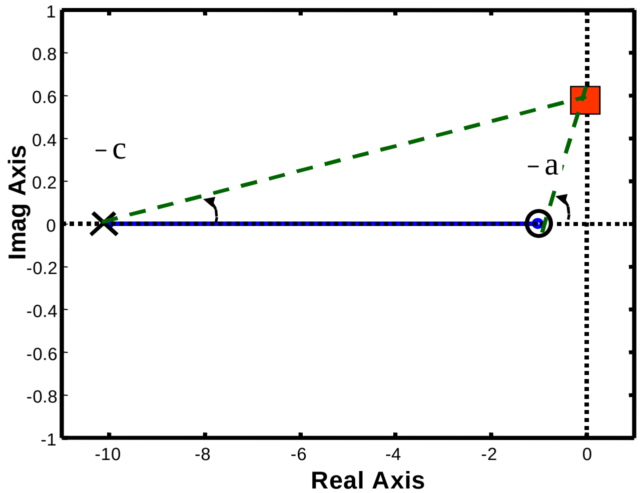
2) Calculate the angle from the picture at $\omega = 0$

Illustrative Example

$$\mathrm{Angle}\left(\frac{s+a}{s+c}\right)$$

Initial Angle = $\left(\mathrm{Angle}(j\omega + a) - \mathrm{Angle}(j\omega + c)\right)\Big|_{\omega = 0}$ = 0 degrees

Final Angle = $\left(\mathrm{Angle}(j\omega + a) - \mathrm{Angle}(j\omega + c)\right)\Big|_{\omega = \infty}$ = 90 - 90 = 0 degrees



} Pole / Zero Plot

Example - Bode Plot Construction

$$H(s) = \frac{s + 1}{s + 10} \qquad H(j\omega) = \frac{\frac{1}{10}\cdot\left(\frac{j\omega}{1} + 1\right)}{\left(\frac{j\omega}{10} + 1\right)}$$ } Bode Form

<u>Magnitude Plot</u>

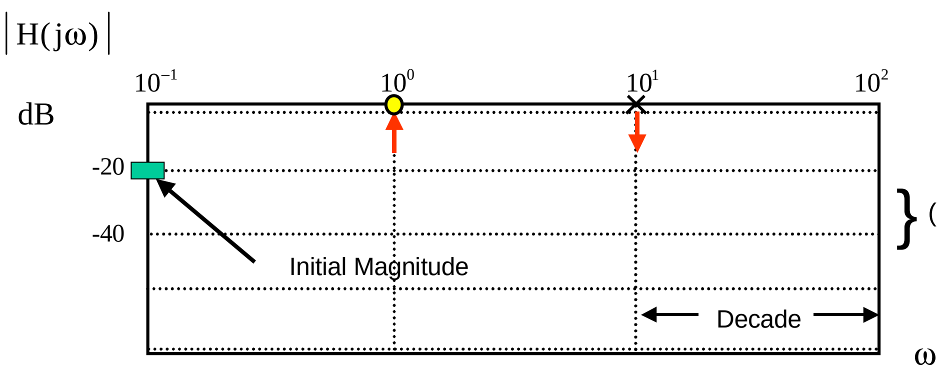
1) Initial Magnitude => n = 0 => $\left|H(j\omega)\right| \Big|_{\omega = 0} = \frac{1}{10}$

Initial Magnitude = $20\cdot\log\left(\frac{1}{10}\right) = -20\ \text{dB}$ } dB Conversion

2) Plot the negative of the poles and the zeros on the $j\omega$ axis

○ zero at -1 X pole at -10

3) Use an "up" arrow to denote a slope of +20 dB/dec and a "down" arrow to denote a slope of -20 dB/dec

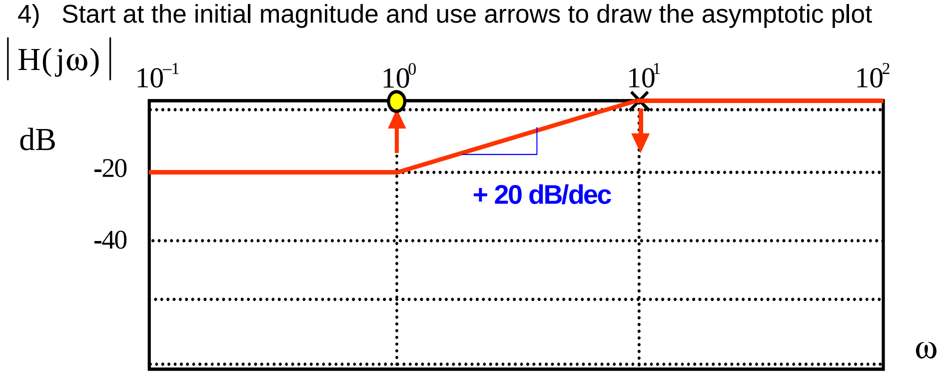


$RD = 0$

Final Slope = 0 dB / dec

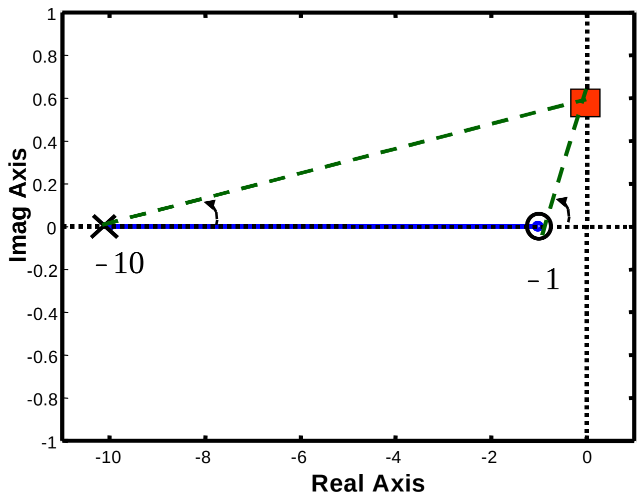
} (use semilog paper)

4) Start at the initial magnitude and use arrows to draw the asymptotic plot



Phase Plot for $H(s) = \frac{s+1}{s+10}$

$$H(j\omega) = \frac{\frac{1}{10}\cdot\left(\frac{j\omega}{1}+1\right)}{\left(\frac{j\omega}{10}+1\right)}$$ } Bode Form



} Pole / Zero Plot

1) Initial Phase

$$\text{Angle}(H(j\omega))\Big|_{\omega=0} = \left(\text{Angle}\left(\frac{j\omega}{1}+1\right) - \text{Angle}\left(\frac{j\omega}{10}+1\right)\right)\Big|_{\omega=0}$$

Note : $= 0 - 0 = 0$ } see picture

Final Angle

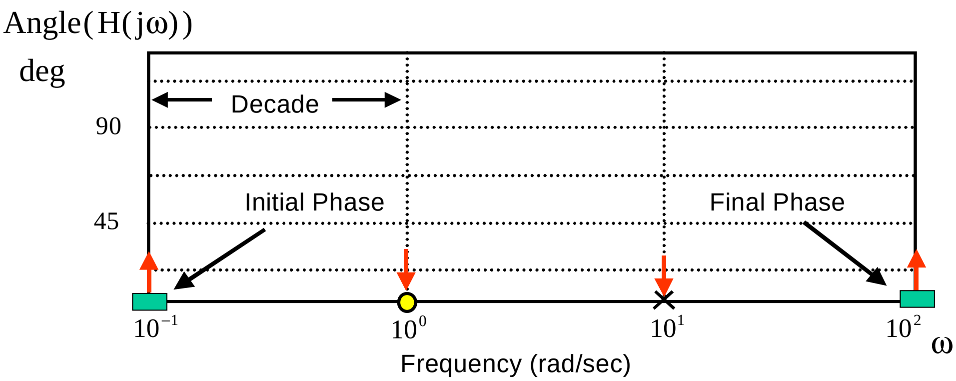
$$\text{Angle}(H(j\omega))\Big|_{\omega=\infty} = \left(\text{Angle}\left(\frac{j\omega}{1}+1\right) - \text{Angle}\left(\frac{j\omega}{10}+1\right)\right)\Big|_{\omega=\infty}$$

Note : $= 90 - 90 = 0$ } see picture

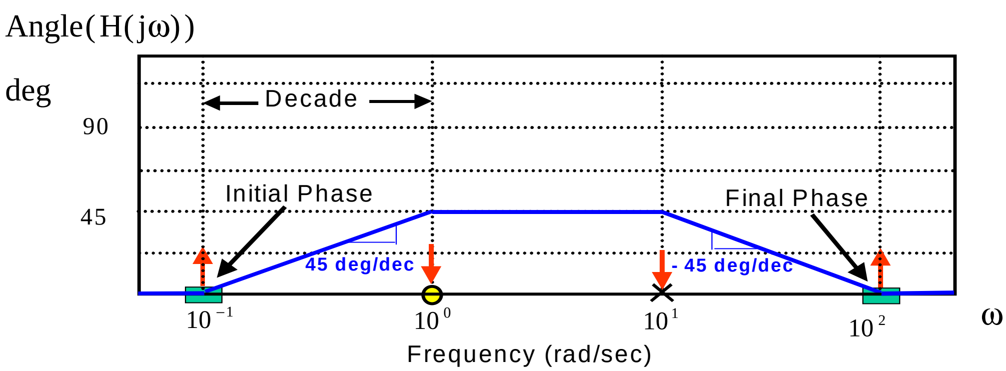
2) Plot the negative of the poles and the zeros on the $j\omega$ axis

zero at -1 pole at -10

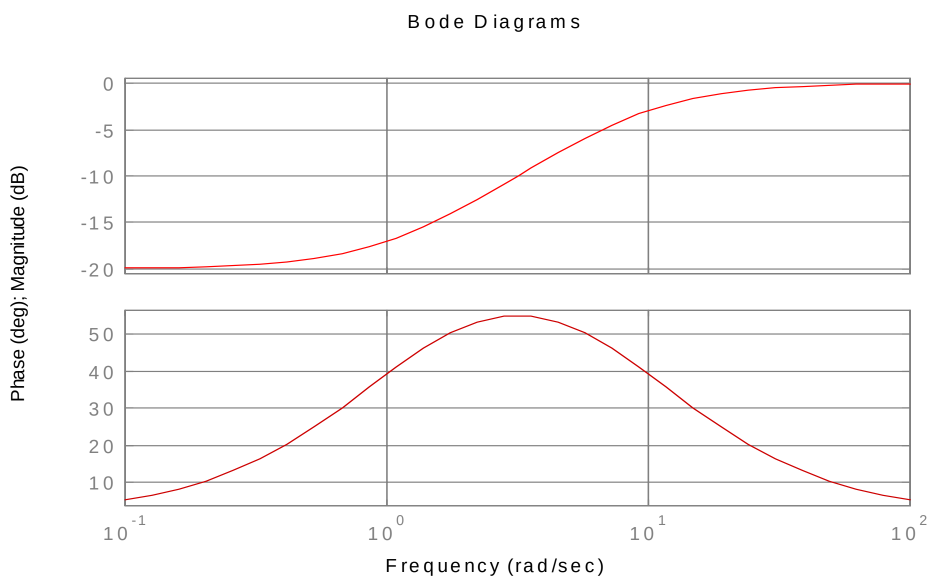
3) Use an "up" arrow to denote a slope of +45 deg/dec and a "down" arrow to denote a slope of - 45 deg/dec



4) Start at the initial phase and use arrows to draw the asymptotic plot



Use the Asymptotic Plots to draw the actual Bode Plot free hand (rough out the edges)



Example Bode Plot Construction

$$H(s) = \frac{10(s + 10)}{s(s + 1)(s + 100)}$$

$$H(j\omega) = \frac{10(10)}{100} \cdot \frac{\left(\frac{j\omega}{10} + 1\right)}{j\omega\left(\frac{j\omega}{1} + 1\right)\left(\frac{j\omega}{100} + 1\right)} \quad \Big\} \text{ Bode Form}$$

Magnitude Plot

1) Initial Magnitude => n = 1 => Calculate the ω intercept (ω_I)

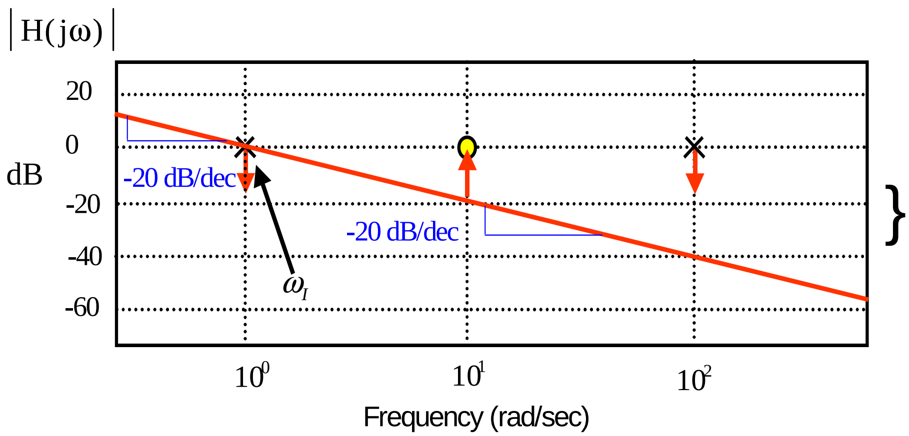
$$\omega_I = \left(\frac{10(10)}{100}\right)^{\frac{1}{n}} = 1 = 10^0 \qquad \Big\} \quad n = 1$$

Initial slope = -20.n dB / dec = -20 dB / dec $\Big\}$ $n = 1$

2) Plot the negative of the poles and the zeros on the jω axis

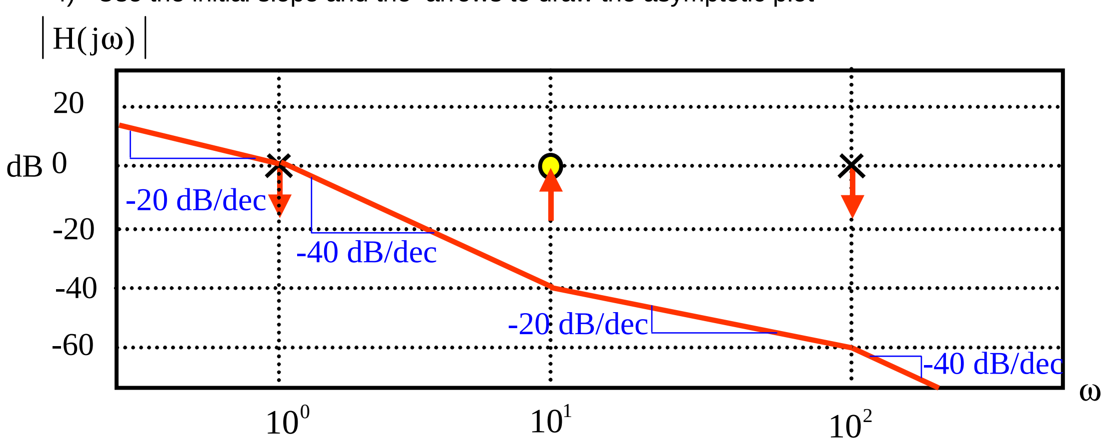
zero at -10   X poles at 0 , -1 and -100

3) Use an "up" arrow to denote a slope of +20 dB/dec and a "down" arrow to denote a slope of -20 dB/dec



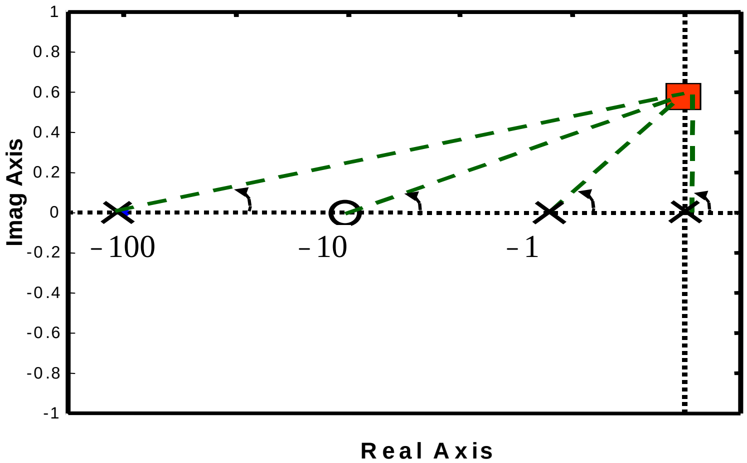
(use semilog paper)

4) Use the initial slope and the arrows to draw the asymptotic plot



Phase Plot for $H(s) = \dfrac{10(s+10)}{s(s+1)(s+100)}$

$$H(s) = \frac{10(10)}{100}\,\frac{\left(\frac{j\omega}{10}+1\right)}{j\omega\left(\frac{j\omega}{1}+1\right)\left(\frac{j\omega}{100}+1\right)}$$ } Bode Form



} Pole / Zero Plot

1) Initial Phase

$$\left.\text{Angle}(H(j\omega))\right|_{\omega=0}$$

$$= \left.\text{Angle}\left(\frac{j\omega}{10}+1\right) - \text{Angle}(j\omega) - \text{Angle}\left(\frac{j\omega}{1}+1\right) - \text{Angle}\left(\frac{j\omega}{100}+1\right)\right|_{\omega=0}$$

$$= 0° - 90° - 0° - 0°$$

$$= -90°$$

Final Angle

$$\left.\text{Angle}(H(j\omega))\right|_{\omega=\infty}$$

$$= \left.\text{Angle}\left(\frac{j\omega}{10}+1\right) - \text{Angle}(j\omega) - \text{Angle}\left(\frac{j\omega}{1}+1\right) - \text{Angle}\left(\frac{j\omega}{100}+1\right)\right|_{\omega=\infty}$$

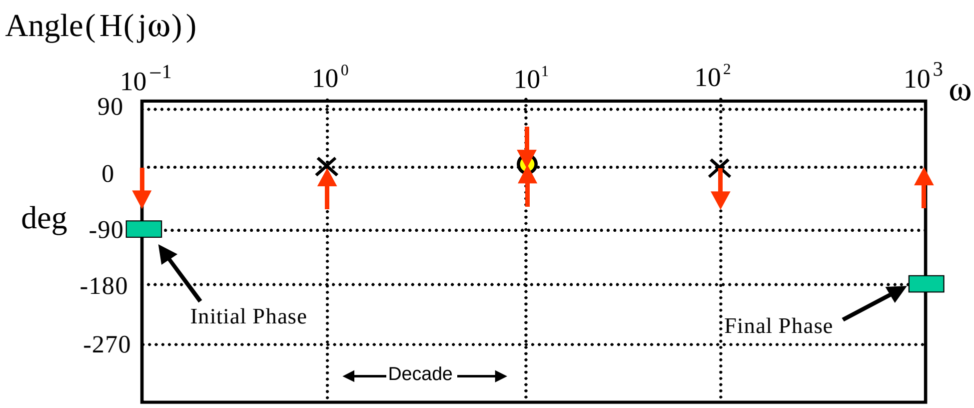
$$= 90° - 90° - 90° - 90°$$

$$= -180°$$

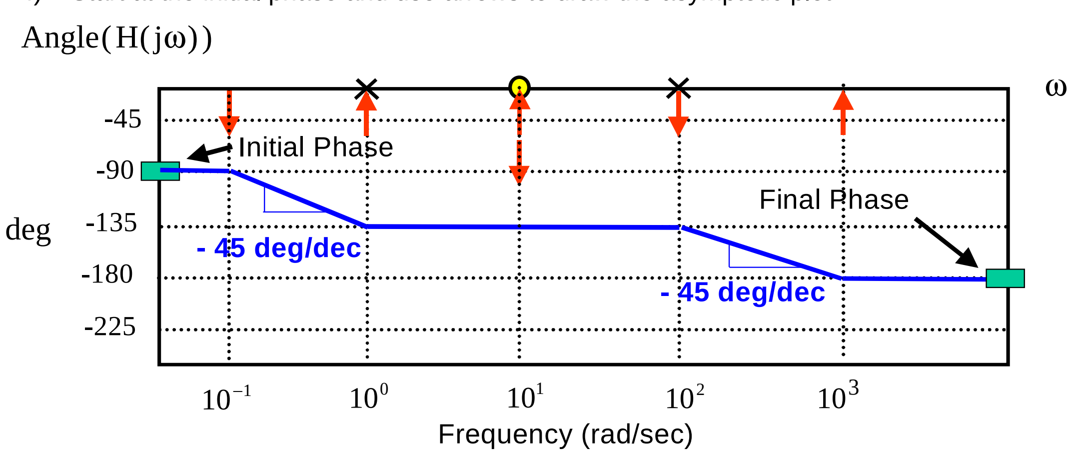
2) Plot the negative of the poles and the zeros on the jω axis

zero at -10 X poles at 0 , -1 and -100

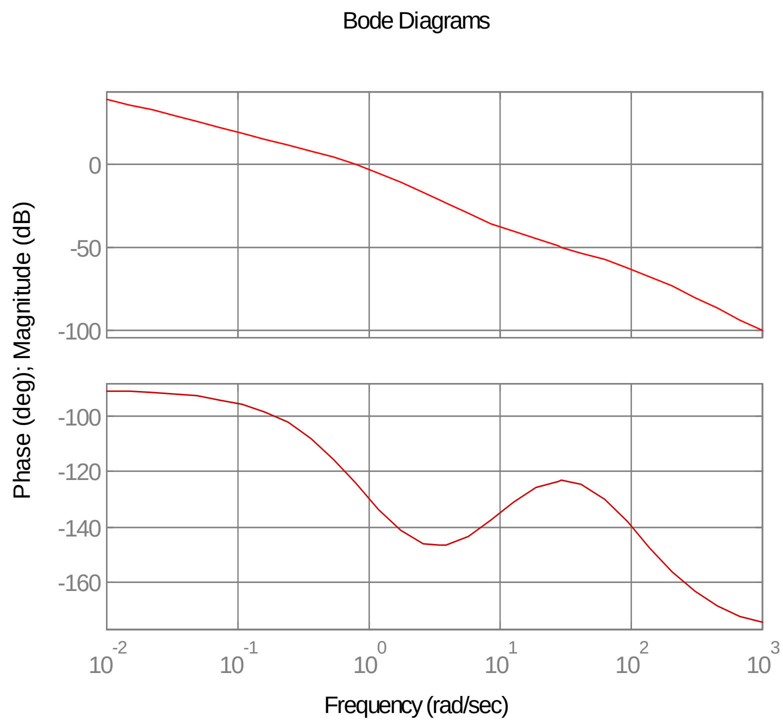
3) Use an "up" arrow to denote a slope of +45 deg/dec and a "down" arrow to denote a slope of - 45 deg/dec



4) Start at the initial phase and use arrows to draw the asymptotic plot



Use the Asymptotic Plots to draw the actual Bode Plot free hand (rough out the edges)



Bode Plots for Repeated Roots

$$H(s) = \frac{1}{\left(s + \omega_n\right)^2}$$

$$H(s) = \frac{1}{\omega_n^{\,2}} \cdot \frac{1}{\left(\frac{j\omega}{\omega_n} + 1\right)\left(\frac{j\omega}{\omega_n} + 1\right)}$$ } Bode Form

Magnitude Plot

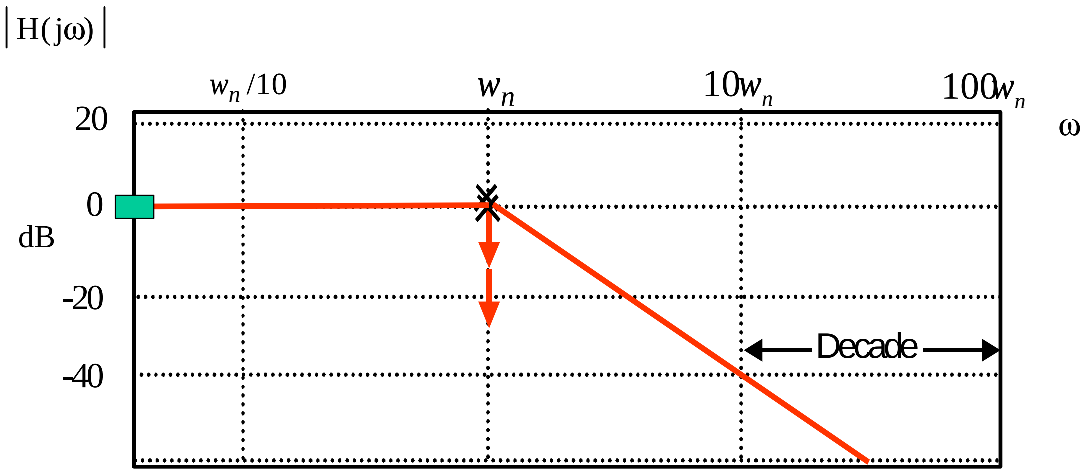
1) Initial Magnitude => n = 0 => Calculate initial magnitude $\left|H(j\omega)\right| \Big|_{\omega = 0} = \frac{1}{\left(\omega_n\right)^2}$

Initial Magnitude = $20 \cdot \log\left[\frac{1}{\left(\omega_n\right)^2}\right]$ dB } dB Conversion

2) Plot the negative of the poles on the $j\omega$ axis

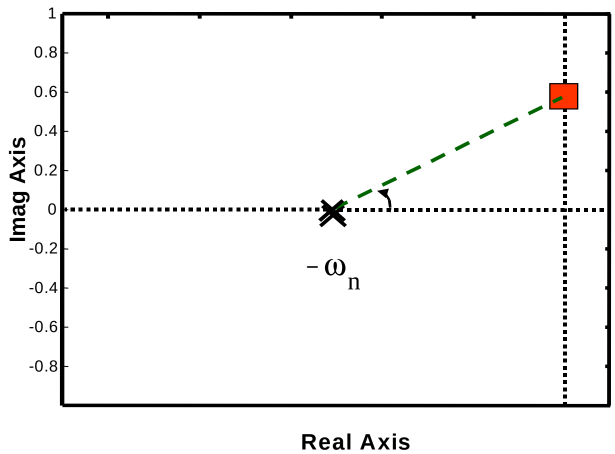
X 2 poles at $-\omega_n$

3) Use an "up" arrow to deonte a slope of +20 dB/dec and a "down" arrow to denote a slope of -20 dB/dec



<u>Phase Plot</u> for $H(s) = \dfrac{1}{\left(s + \omega_n\right)^2}$

$$H(s) = \frac{1}{\left(\omega_n\right)^2} \frac{1}{\left(\frac{j\omega}{\omega_n} + 1\right)\left(\frac{j\omega}{\omega_n} + 1\right)}$$ } Bode Form



} Pole / Zero Plot

1) Initial Phase

$$\text{Angle}(H(j\omega))\Big|_{\omega=0} = -\text{Angle}\left(\frac{j\omega}{\omega_n} + 1\right) - \text{Angle}\left(\frac{j\omega}{\omega_n} + 1\right)\Big|_{\omega=0}$$
$$= -0^\circ - 0^\circ$$
$$= 0^\circ$$

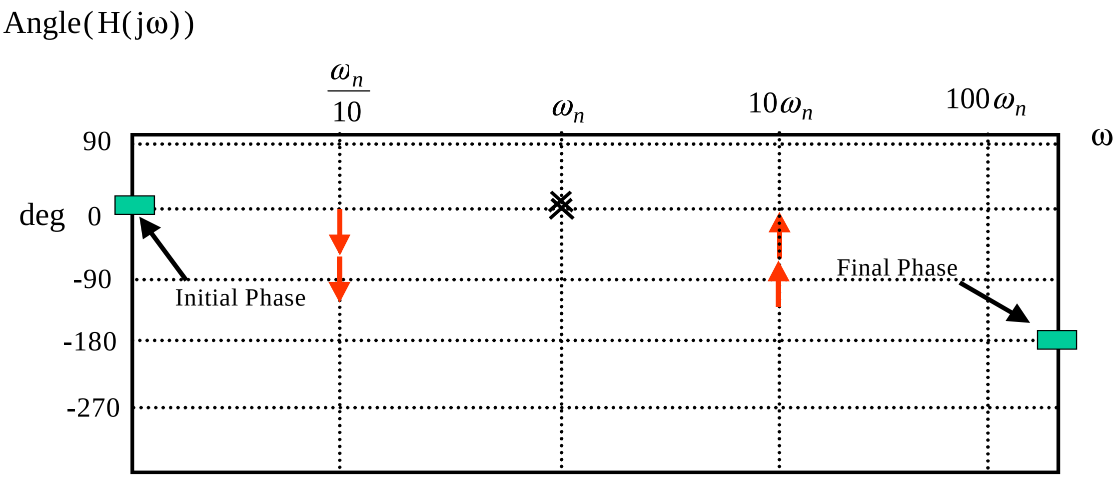
Final Angle

$$\text{Angle}(H(j\omega))\Big|_{\omega=\infty} = -\text{Angle}\left(\frac{j\omega}{\omega_n} + 1\right) - \text{Angle}\left(\frac{j\omega}{\omega_n} + 1\right)\Big|_{\omega=\infty}$$
$$= -90^\circ - 90^\circ$$
$$= -180^\circ$$

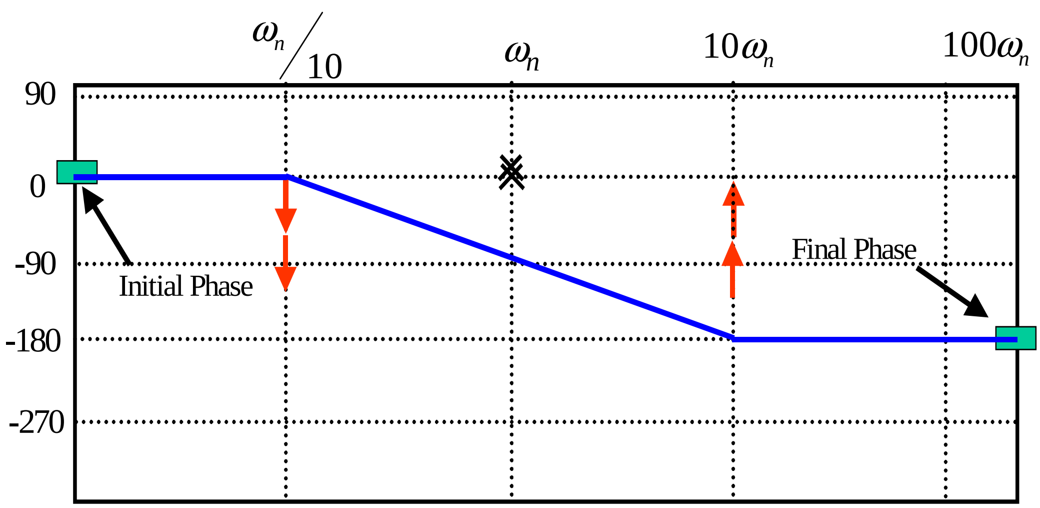
2) Plot the negative of the poles and the zeros on the jω axis

X 2 poles at $-\omega_n$

3) Use an "up" arrow to denote a slope of +45 deg/dec and a "down" arrow to denote a slope of - 45 deg/dec

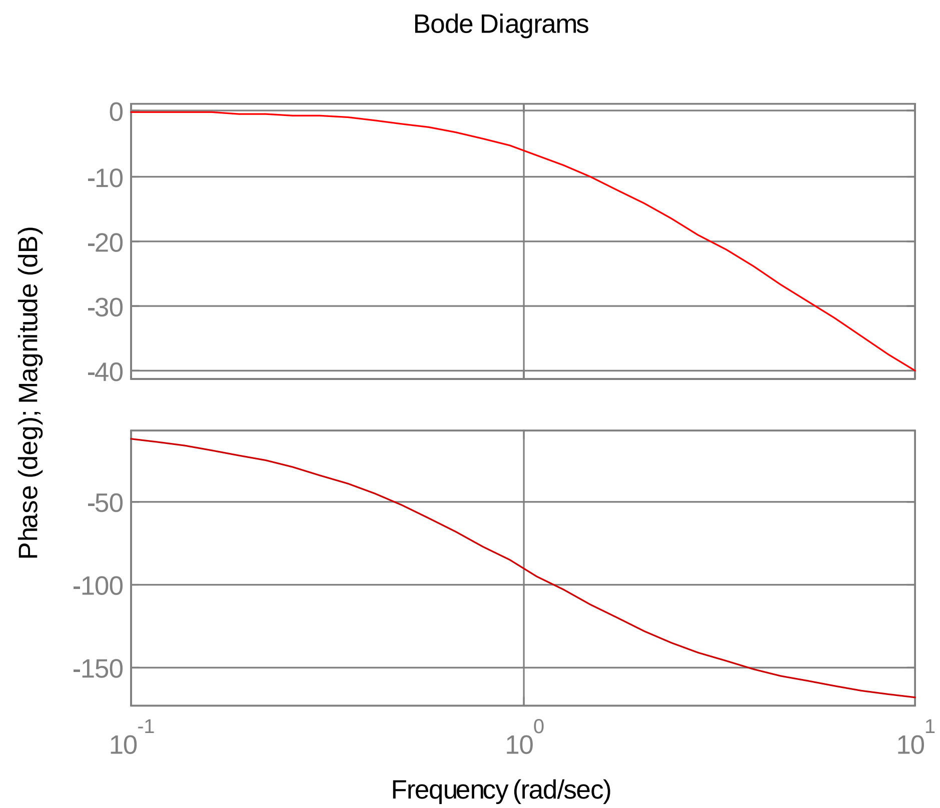


4) Start at the initial phase and use arrows to draw the asymptotic plot



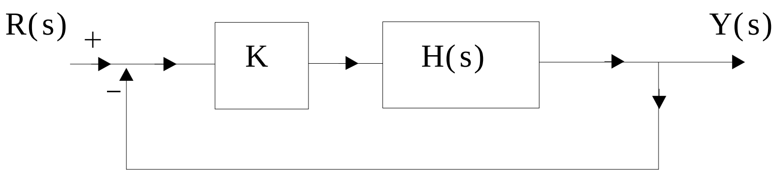
Use the Asymptotic Plots to draw the actual Bode Plot free hand (rough out the edges)

In the following plot $\omega_n = 1$



Example

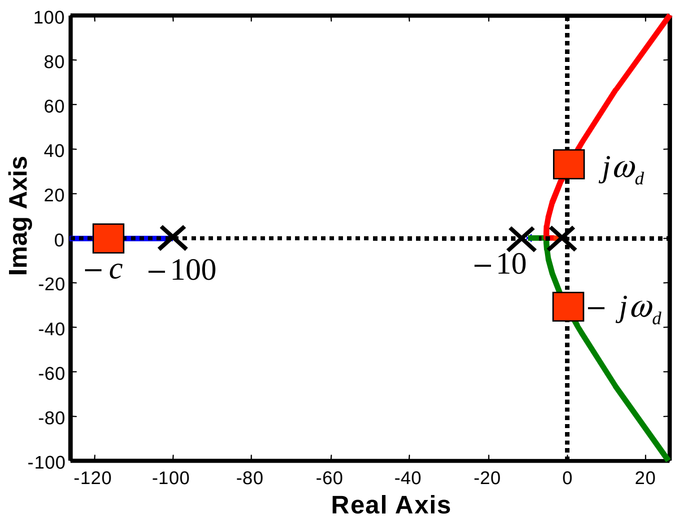
Using the Bode Plot for determining Stability of a Closed-Loop System



$$\frac{Y(s)}{R(s)} = \frac{K \cdot H(s)}{1 + K \cdot H(s)}$$ } Closed Loop Transfer Function

$$H(s) = \frac{1}{\left(\frac{s}{1}+1\right)\cdot\left(\frac{s}{10}+1\right)\cdot\left(\frac{s}{100}+1\right)}$$

$$\Delta(s) = 1 + K \cdot H(s)$$ } Root Locus form for denominator



} Draw the Root Locus

For $0 < K < K^u$, the closed-loop system is marginally stable if at $K = K^u$

the closed-loop poles are at $-c, -j\omega_d, j\omega_d$ (see Root Locus)

Angle Condition

$$\text{Angle}(H(s))\Big|_{s = j\omega_d} = -180$$

Magnitude Condition

$$K \cdot |H(s)|\Big|_{s = j\omega_d} = 1$$

} These two equations can be used to find ω_d and K^u ; however, the angle condition equation would be difficult to solve **Why ?**

Bode Solution for Finding K^u **and** ω_d

$$\text{Angle}(H(s))\Big|_{s=j\omega_d} = -180 \quad \Rightarrow \quad \text{Angle}(H(j\omega_d)) = -180$$

$$K^u \cdot |H(s)|\Big|_{s=j\omega_d} = 1 \quad \Rightarrow \quad K^u \cdot |H(j\omega_d)| = 1$$

A Bode Plot is a graph of $|H(j\omega)|$ and $\text{Angle}(H(j\omega))$ for all ω ; hence

ω_d can be directly read off the phase plot

With ω_d , we can read off $|H(j\omega_d)|$ directly off the magnitude plot ; K^u is calculated as follows

$$K^u = \frac{1}{|H(j\omega_d)|} \quad \Big\} \quad K^u \text{ is in dB}$$

$$K^u = \frac{1}{10^{\left(\frac{|H(j\omega_d)|}{20}\right)}} \quad \Big\} \quad K^u \text{ is "regular" units}$$

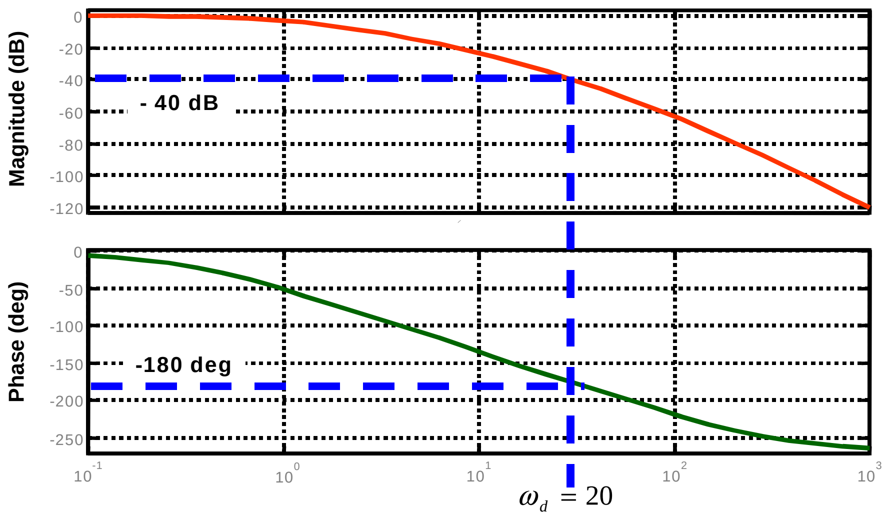
<u>Bottom Line</u>: The Bode Plot gives us a graphical solution for finding K^u and ω_d

Let us continue with the problem of finding K^u for the closed loop system

characterized by

$$\Delta(s) = 1 + K \cdot \frac{1}{\left(\frac{s}{1} + 1\right) \cdot \left(\frac{s}{10} + 1\right) \cdot \left(\frac{s}{100} + 1\right)}$$

Bode Solution for finding K^u and ω_d



} Bode Plot from previous problem

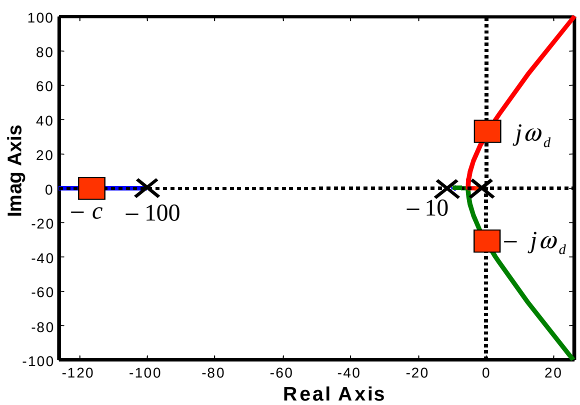
$$\left|H\left(j\omega_d\right)\right| = -40 \text{ dB}$$ } Read from the Bode Plot

$$K^u = 10^{\frac{-\left|H\left(j\omega_d\right)\right|}{20}}$$ } in dB

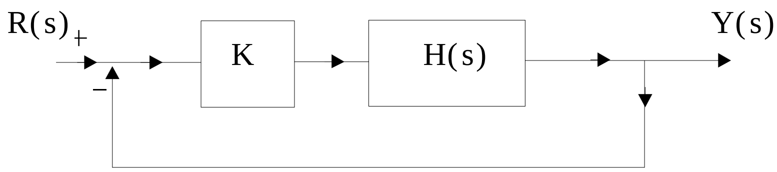
For

$$K^u = 100$$ } answer

$0 < K < 100$, the closed-loop is stable



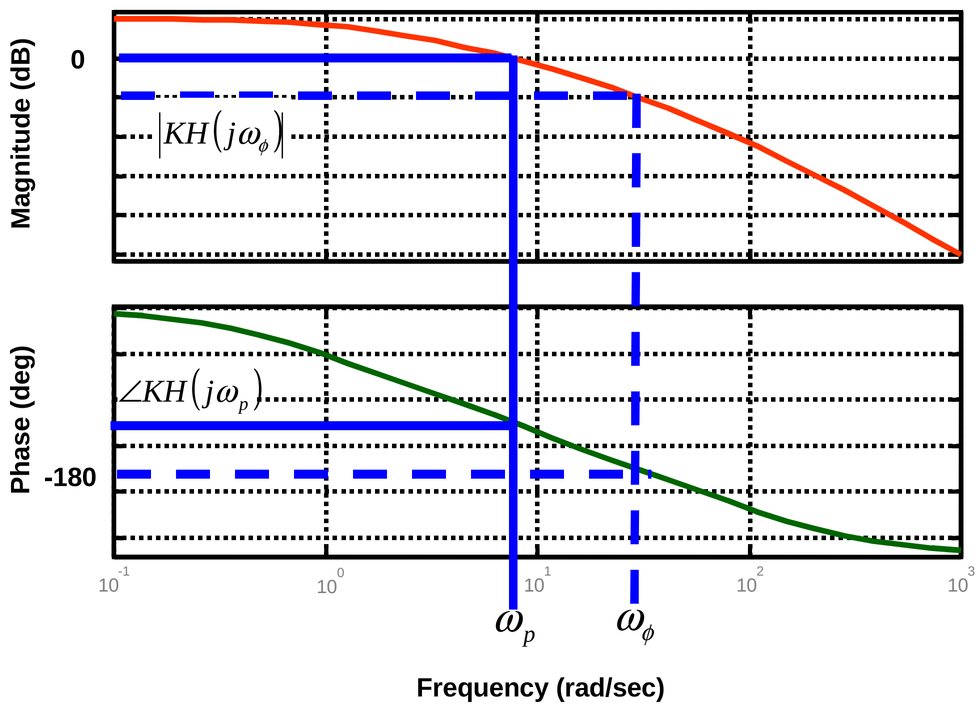
Bode Compensator Design



Similar to the root-locus, we use the characteristic equation in the form

$$\Delta_{cl}(s) = 1 + K \cdot H(s)$$

to examine the stability / performance of Bode Performance Parameters



ω_p (gain crossover frequency) is equal to the frequency where $|K \cdot H(j\omega)| = 0$ dB

ω_ϕ (phase crossover frequency) is equal to the frequency where $\text{Angle}(H(j\omega)) = -180$

Phase Margin - $\phi_{pm} = 180 + \text{Angle}\left(H\left(j \cdot \omega_p\right)\right)$

(If $\phi_{pm} < 0$ then the closed-loop system is unstable)

Gain Margin - $GM = \left| K \cdot H\left(j \cdot \omega_\phi\right) \right|$

The Bode design parameters are related to the root locus design parameters according to

$$\omega_p = \omega_n \cdot \left(\sqrt{4\cdot\delta^4 + 1} - 2\cdot\delta^2\right)^{\frac{1}{2}}$$

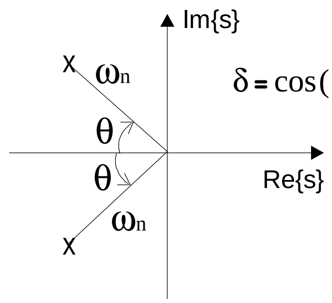
$$\phi_{pm} = \tan^{-1}\left[2\cdot\delta\cdot\left(\sqrt{4\cdot\delta^4 + 1} - 2\cdot\delta^2\right)^{\frac{-1}{2}}\right]$$

} Root Locus design parameters
δ - damping ratio
ω_n - natural frequency

The phase margin is also approximately related to the damping ratio as follows

$\delta = 0.01\cdot\phi_{pm}$ degrees for $\phi_{pm} < 60$ degrees } Rule of thumb

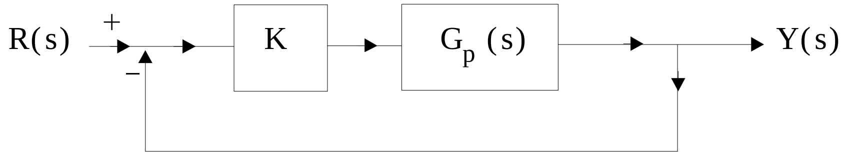
Remember δ is a measure of relative stability



$\delta = \cos(\theta)$

Hence, ϕ_{pm} is a measure of relative stability

Bode Design Problem



$$G_p(s) = \frac{1}{s\cdot\left(\frac{s}{2}+1\right)}$$

Find K so that $\phi_{pm} = 45$ degrees for the closed loop system

Outline of the Solution

1) Find the closed-loop denominator in the form $\Delta(s) = 1 + K\cdot G_p(s)$

2) Plot $\left|K\cdot G_p(j\omega)\right|$ and $\text{Angle}\left[G_p(j\omega)\right]$ } Assume K = 1

3) From ϕ_{pm} , calculate the desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right]$ } From given ϕ_{pm}

i.e. $\text{Desired}\ \ \text{Angle}\left[G_p\left(j\omega_p\right)\right] = \phi_{pm} - 180$

4) From the desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right]$ and the $\text{Angle}\left[G_p(j\omega)\right]$ plot , find the desired ω_p

5) Raise or lower magnitude plot so that it crosses the 0 dB level at the desired value of ω_p

6) Find the ⟶ between the desired magnitude plot and the original magnitude plot at the desired value of ω_p

desired + original = shift dB

7) Calculate K

$$K = 10^{\left(\frac{-\text{shift}}{20}\right)}$$

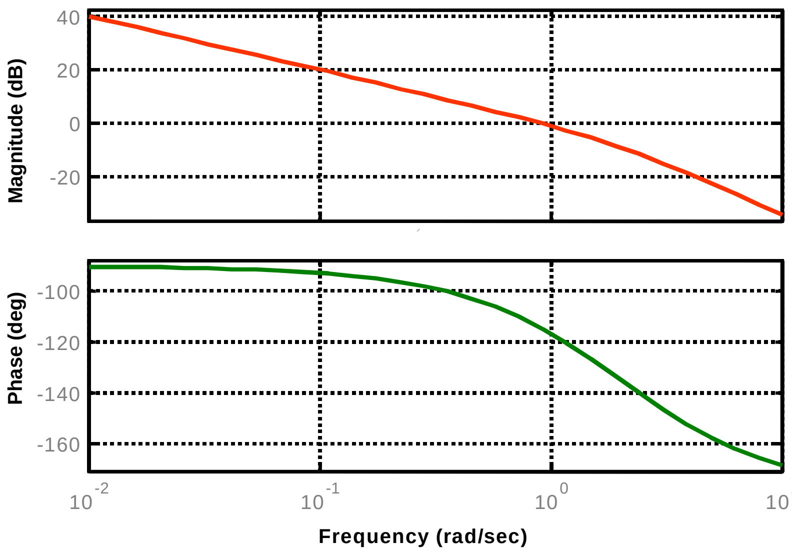
} This formula is derived from the magnitude condition as done before

Step 1

$$\Delta(s) = 1 + K \cdot \frac{1}{s \cdot \left(\frac{s}{2} + 1\right)}$$

} closed loop denominator Bode Plot form

Step 2



} $K = 1$

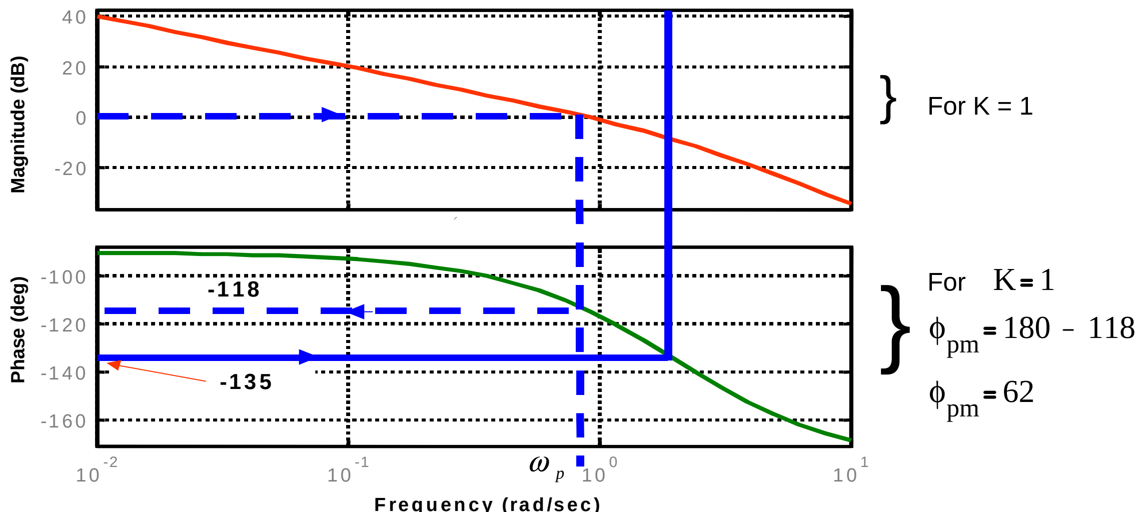
} Draw Bode Plot for $KG_p(s)$

Step 3

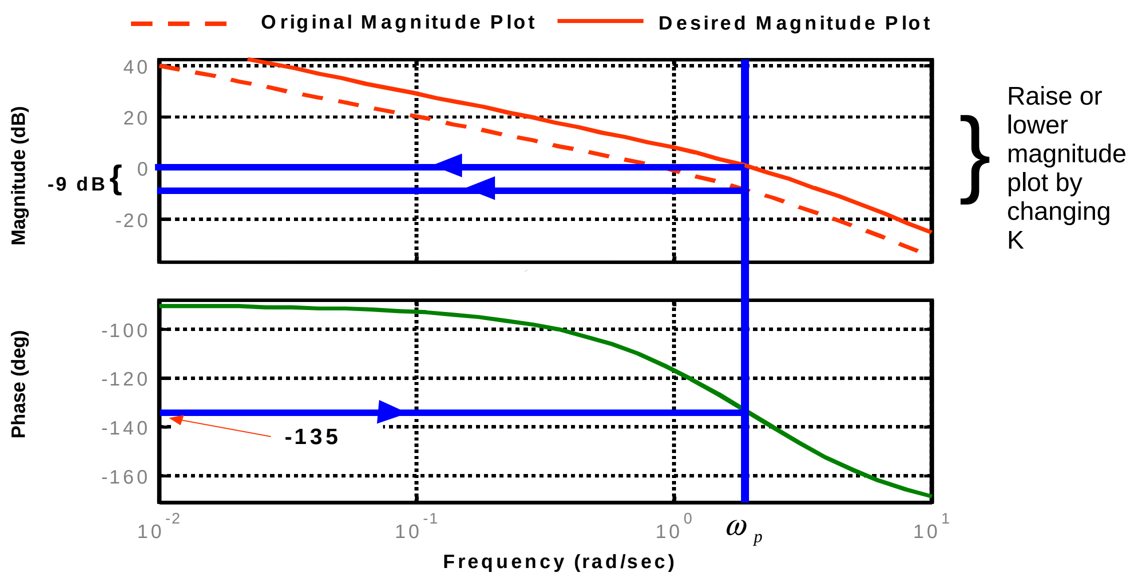
$$\text{Desired} \quad \text{Angle}\left[G_p\left(j\omega_p\right)\right] = \phi_{pm} - 180$$

$$\text{Desired} \quad \text{Angle}\left[G_p\left(j\omega_p\right)\right] = 45 - 180 = -135$$

Step 4 Given Desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right] = -135$ Find ω_p



Step 5

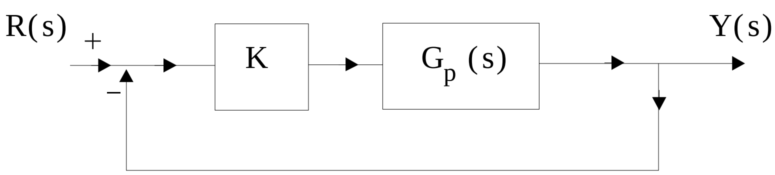


Step 6 The difference between the desired magnitude plot and the original magnitude plot is $\text{desired} + \text{original} = 0 - 9 = -9\ \text{dB}$; therefore

$$\text{shift} = -9\ \text{dB}$$

Step 7 $K = 10^{\frac{-\text{shift}}{20}}$ => $K = 10^{-\left(\frac{-9}{20}\right)}$ => $K = 2.8$

Root Locus Design for the same Problem



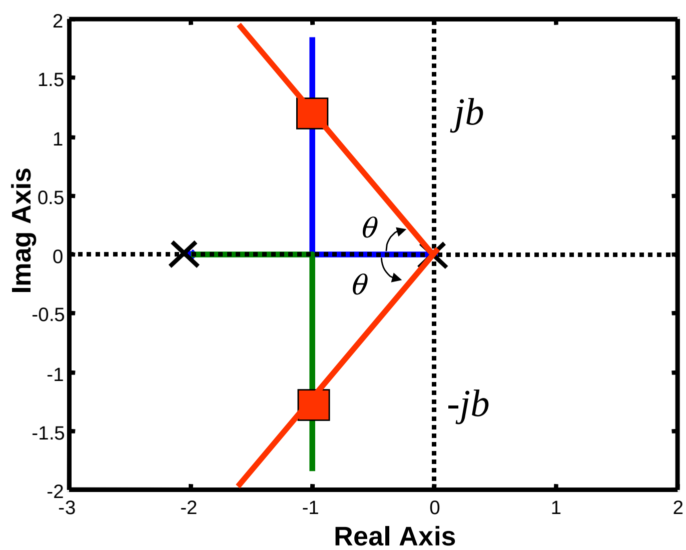
$$G_p(s) = \frac{1}{s\cdot\left(\frac{s}{2}+1\right)}$$

Find K so that the closed loop system has a $\phi_{pm} = 45\ \text{deg}$

$$\frac{Y(s)}{R(s)} = \frac{K\cdot G_p(s)}{1 + K\cdot G_p(s)}$$ } From Block Diagram

$$\frac{Y(s)}{R(s)} = \frac{K\cdot\frac{2}{s\cdot(s+2)}}{1 + K\cdot\frac{2}{s\cdot(s+2)}}$$ } Closed Loop Transfer Function

$$\Delta(s) = 1 + K\cdot\frac{2}{s\cdot(s+2)}$$ } Root Locus From for the denominator



} Draw Root Locus

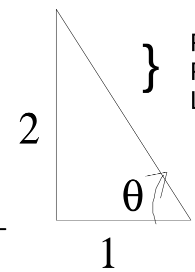
If $\phi_{pm} = 45$ => $\delta = 0.45$ } From conversion formulas

Note: $\delta = \cos(\theta)$

Desired closed loop poles must have a damping ratio of $\delta = 0.45$

$\delta = \cos(\theta)$ => $\theta = 63\ \text{deg}$ } Formulas

$\frac{b}{\sin(63)} = \frac{1}{\sin(27)}$ } Law of sines => $b = 2$



} From Root Locus

Desired closed loop poles $-1 \pm j2$

Magnitude Condition

$$K \cdot \left| \frac{2}{s(s+2)} \right| \Bigg|_{s = -1 + j2} = 1 \quad \Rightarrow \quad K = 2.5$$

} Why is this number different from K = 2.8 obtained from the Bode Design ?

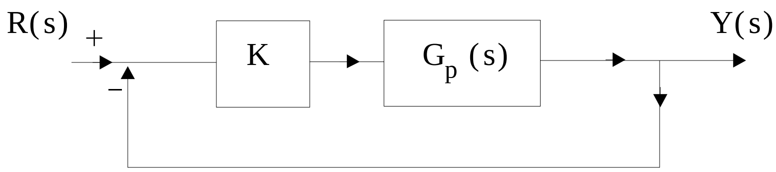
For this problem, these three specifications are equivalent

1) $\phi_{pm} = 45\ \text{deg}$

2) $\delta \cong 0.45$

3) desired closed loop poles = $-1 \pm j2$

Example

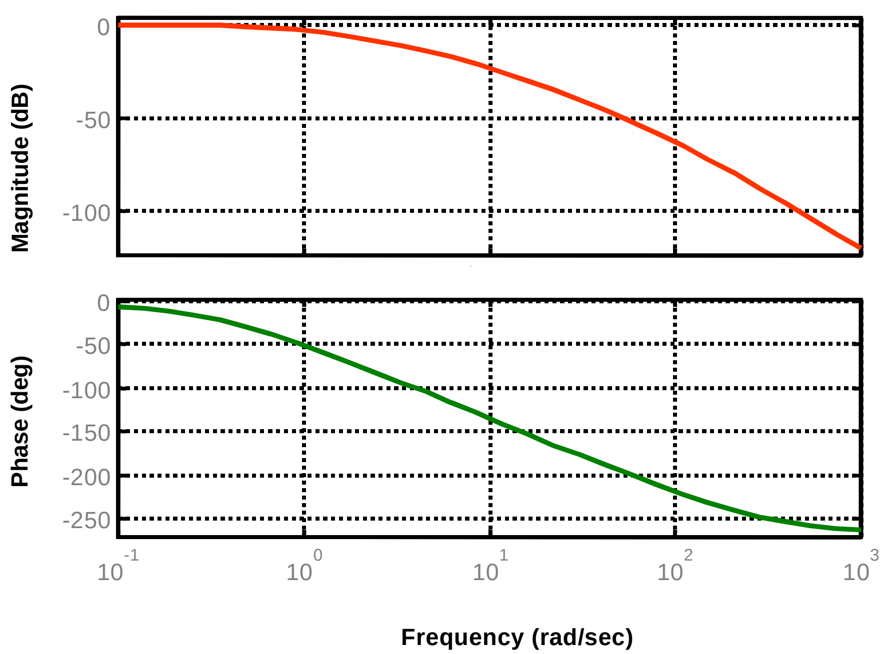


Find K so that $\phi_{pm} = 45$ deg

$$G_p(s) = \frac{1}{\left(\frac{s}{1}+1\right)\cdot\left(\frac{s}{10}+1\right)\cdot\left(\frac{s}{100}+1\right)}$$

Step 1 $$\Delta(s) = 1 + K\cdot\frac{1}{\left(\frac{s}{1}+1\right)\cdot\left(\frac{s}{10}+1\right)\cdot\left(\frac{s}{100}+1\right)}$$ } closed loop denominator Bode Plot form

Step 2



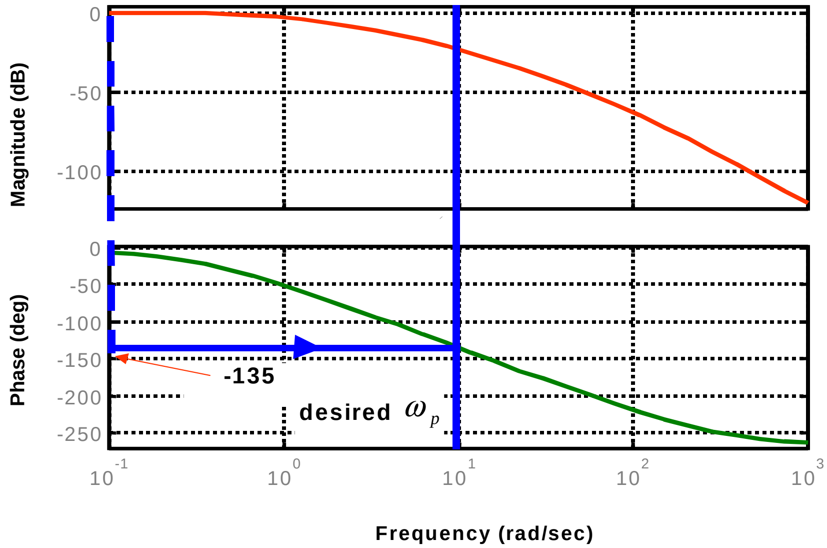
} For $K = 1$

} Draw Bode Plot

Step 3 Desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right] = \phi_{pm} - 180$

Desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right] = 45 - 180 = -135$

Step 4 Given Desired $\text{Angle}\left[G_p\left(j\omega_p\right)\right] = -135$ Find ω_p



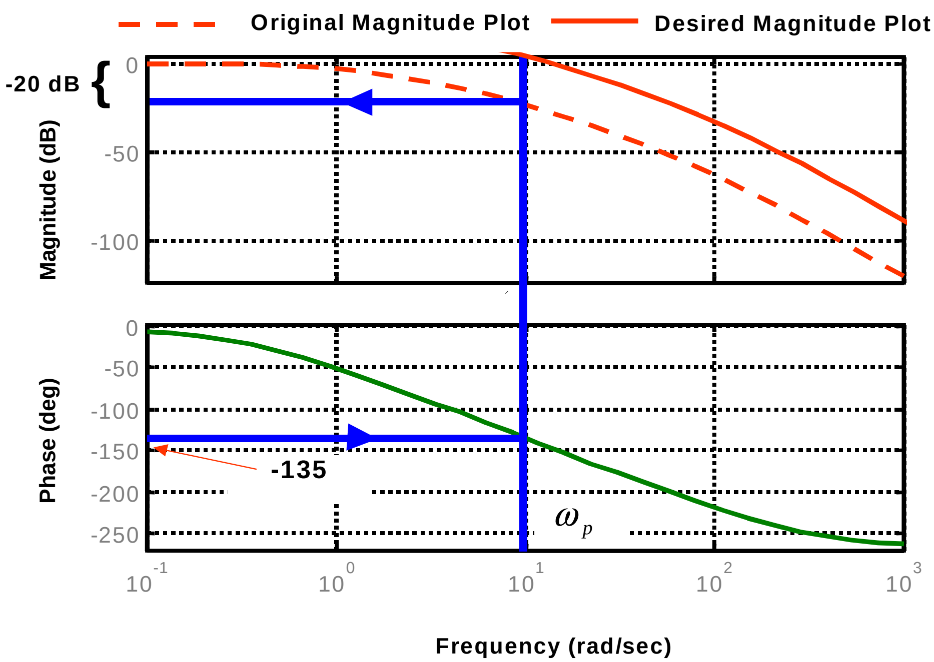
} For $K = 1$

} For $K = 1$

$\phi_{pm} = 180 - 0$

$\phi_{pm} = 180$

Step 5



} Raise or lower magnitude plot by changing K

Step 6 The difference between the desired magnitude plot and the original magnitude plot is

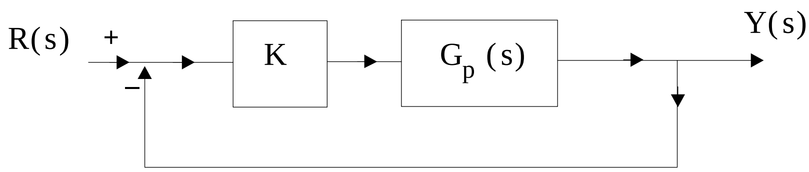
$\text{desired} + \text{original} = 0 - 20 = -20\ \text{dB}$; therefore $\text{shift} = -20\ \text{dB}$

Step 7 $K = 10^{\frac{-\text{shift}}{20}}$ => $K = 10^{-\left(\frac{-20}{20}\right)}$ => $K = 10$

Time for a Movie

Click on image to start movie

Nyquist Plots/Nyquist Stability Criterion



$$\frac{Y(s)}{R(s)} = \frac{KG_p(s)}{1 + KG_p(s)}$$ } Closed loop TF

$$\Delta(s) = 1 + KG_p(s)$$ } Denominator Form for Nyquist Analysis

1) Plot Nyquist Plot (i.e., a Polar Plot) for $\mathrm{Im}\left[K \cdot G_p(j\omega)\right]$ versus $\mathrm{Re}\left[K \cdot G_p(j\omega)\right]$ using the Nyquist contour. (Assume K = 1)

2) Apply the Nyquist Criterion to the Nyquist Plot and then determine the stability of the closed-loop system.

Nyquist Criterion

P_0 denotes the number of poles of $G_p(s)$ inside s-plane contour.

N denotes the net number CW encirclements of the point $1\angle -180$

If the contour in the s-plane is CW, then CW is the positive direction.

i.e., 1 CW $\Rightarrow N = 1$ 1 CCW $\Rightarrow N = -1$

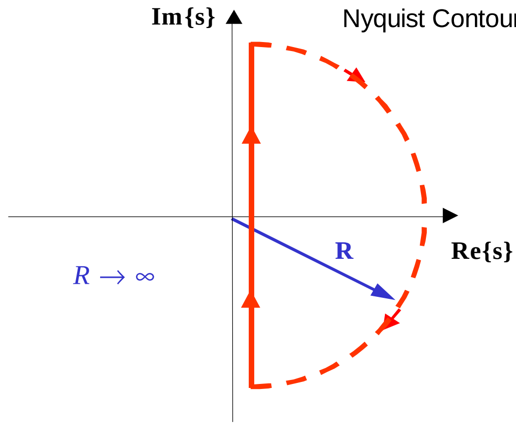
If the contour in the s-plane is CCW, then CCW is the positive direction.

i.e., 1 CCW $\Rightarrow N = 1$ 1 CW $\Rightarrow N = -1$

Z_c denotes the number of closed-loop poles inside the s-plane contour.

$$Z_c = N + P_0$$ } Formula for calculation purposes

Example



Draw the Nyquist Plot for $H(s) = \dfrac{1}{s + 1}$

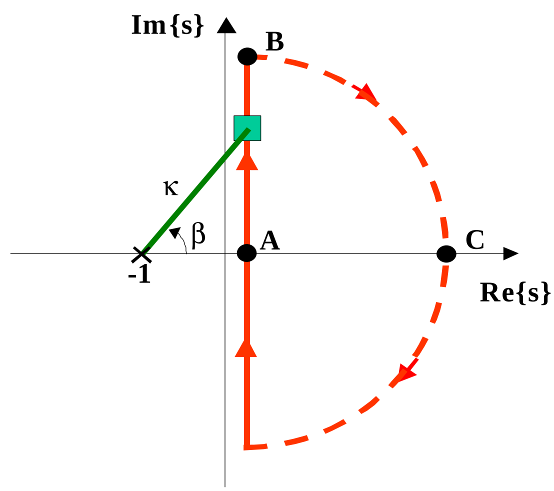
for the following s-plane contour.

Step 1) Let $s = j\omega$ $\quad H(j \cdot \omega) = \dfrac{1}{j \cdot \omega + 1}$

Step 2) Plot a pole/zero plot for H(s) on the s-plane contour picture and include measurement scheme to the $j\,\omega$ - axis along with the necessary critical points.

Step 3) Enter a magnitude/phase table entry for each critical point.

Point	r mag	θ Phase
A	1	0
B	0	-90
C	0	0

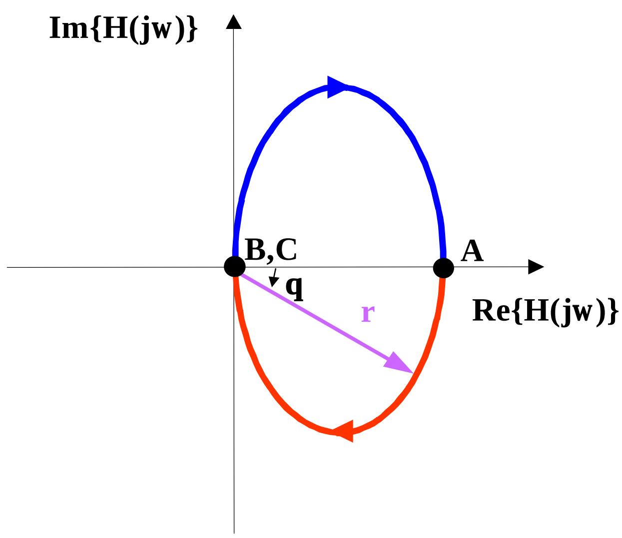


$$|H(j\omega)| = \frac{1}{|j\omega + 1|} = r = \frac{1}{\kappa}$$

$$\text{Angle}(H(j\omega)) = -\text{Angle}(j\omega + 1) = \theta = -\beta$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis. Draw other half.

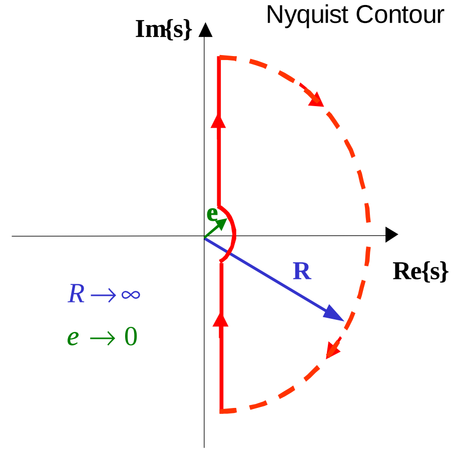


} Draw a Polar plot on a rectangular coordinate system

Example

Draw the Nyquist Plot for $H(s) = \dfrac{1}{s(s+1)}$

for the following s-plane contour.



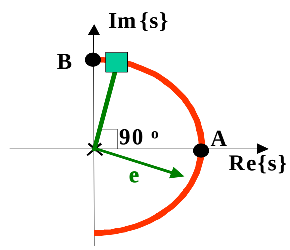
Step 1) Let $s = j\omega$ $\quad H(j\cdot\omega) = \dfrac{1}{j\omega(j\cdot\omega + 1)}$

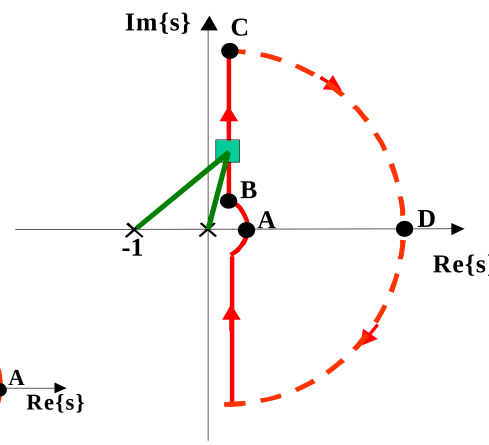
Step 2) Plot a pole/zero plot for H(s) on the s-plane contour picture and include measurement scheme to the $j\omega$ axis along with the necessary critical points.

Step 3) Enter a magnitude/phase table entry for each critical point.

Point	r mag	θ Phase
A	∞	0
B	∞	-90
C	0	-180
D	0	0

"Blow-up"



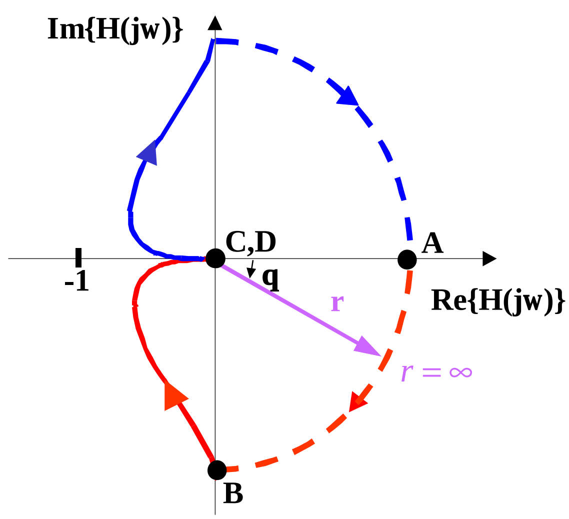


$$|H(j\omega)| = \frac{1}{|j\omega| \cdot |j\omega + 1|}$$

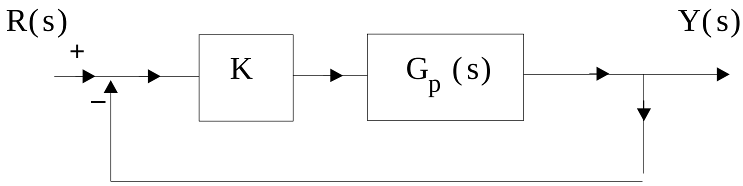
$$\text{Angle}(H(j\omega)) = -\text{Angle}(j\omega) - \text{Angle}(j\omega + 1)$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis.
Draw other half.

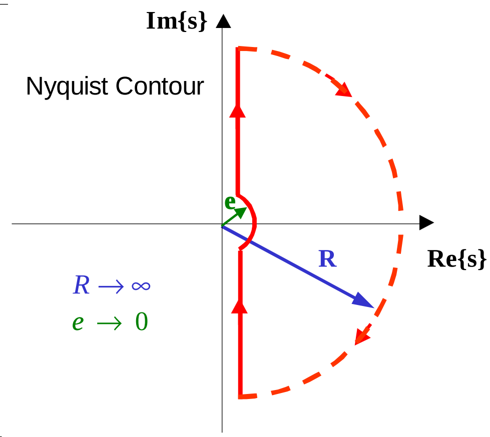


Example - Determine stability with Nyquist Plot



$$G_p(s) = \frac{1}{s\cdot(s+1)\cdot(s+10)}$$

Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.



$$\frac{Y(s)}{R(s)} = \frac{K\cdot G_p(s)}{1 + K\cdot G_p(s)}$$ } Closed Loop Transfer Function

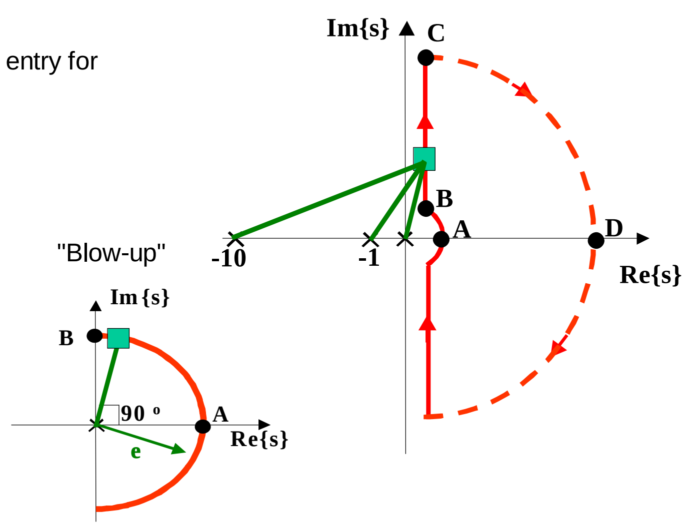
$$\Delta(s) = 1 + K\cdot\frac{1}{s\cdot(s+1)\cdot(s+10)}$$ } Nyquist Form for the denominator

Step 1) Let $s = j\omega$ $$K\cdot G_p(j\omega) = \frac{K}{(j\omega)\cdot(j\omega+1)\cdot(j\omega+10)}$$ } Assume K = 1

Step 2) Plot a pole/zero plot for $G_p(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis along with the necessary critical points.

Step 3) Enter a magnitude/phase table entry for each critical point.

point	r mag	θ phase
A	∞	0
B	∞	-90
C	0	-270
D	0	0



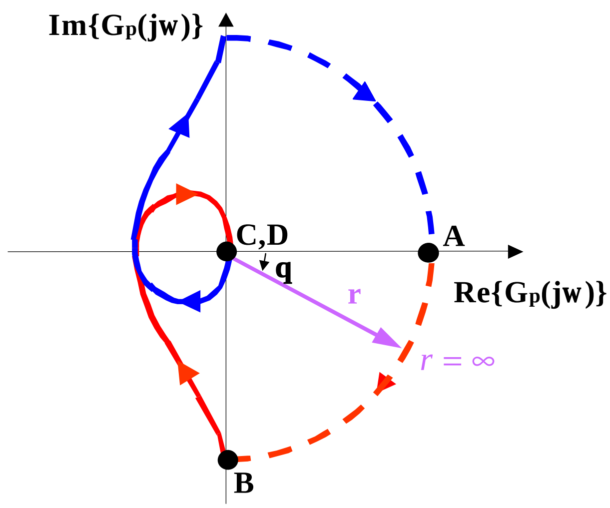
$$\left|G_p(j\omega)\right| = \frac{1}{|j\omega|\cdot|j\omega+1|\cdot|j\omega+10|}$$

$$\text{Angle}\left[G_p(j\omega)\right] = -\text{Angle}(j\omega) - \text{Angle}(j\omega+1) - \text{Angle}(j\omega+10)$$

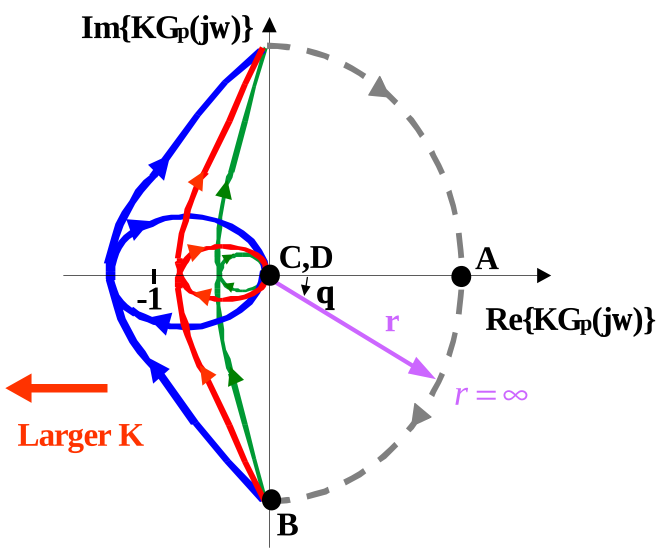
Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis. Draw other half.

point	r mag	θ phase
A	∞	0
B	∞	-90
C	0	-270
D	0	0

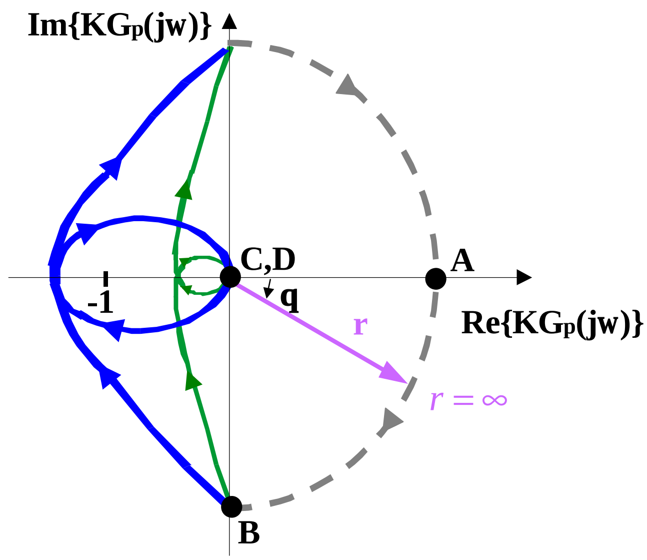


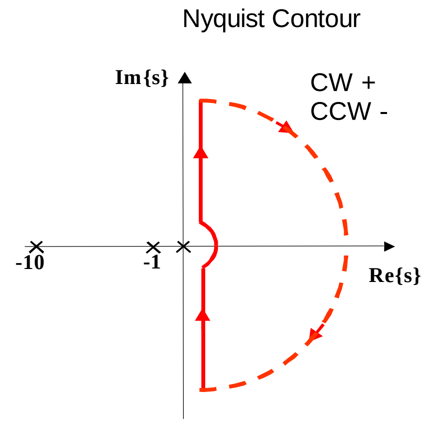
} Nyquist Plot with K = 1



} Nyquist Plot for different values of K

As K is increased, the shape of the Nyquist plot does not change because K is a real number (i.e., K only affects the magnitude (size); K does NOT affect the phase (shape))





Nyquist Criterion

1) Find P_0 => $P_0 = 0$ (Number of poles of $G_p(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

   Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

   Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	0	Yes
big K	2	2	No

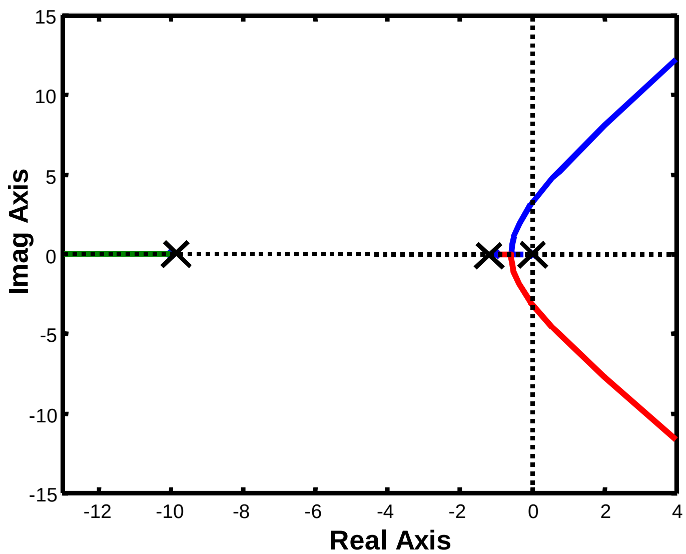
} Nyquist Criterion table

4) Understanding the above table:

   For **small K**, no closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

   For **big K**, two closed loop poles are inside the s-plane contour; hence, for big K, the closed loop system is unstable

Root Locus agrees with the Nyquist Criterion



$$G_p(s) = \frac{1}{s\cdot(s+1)\cdot(s+10)}$$

} Root Locus for $\Delta(s) = 1 + K\cdot G_p(s)$

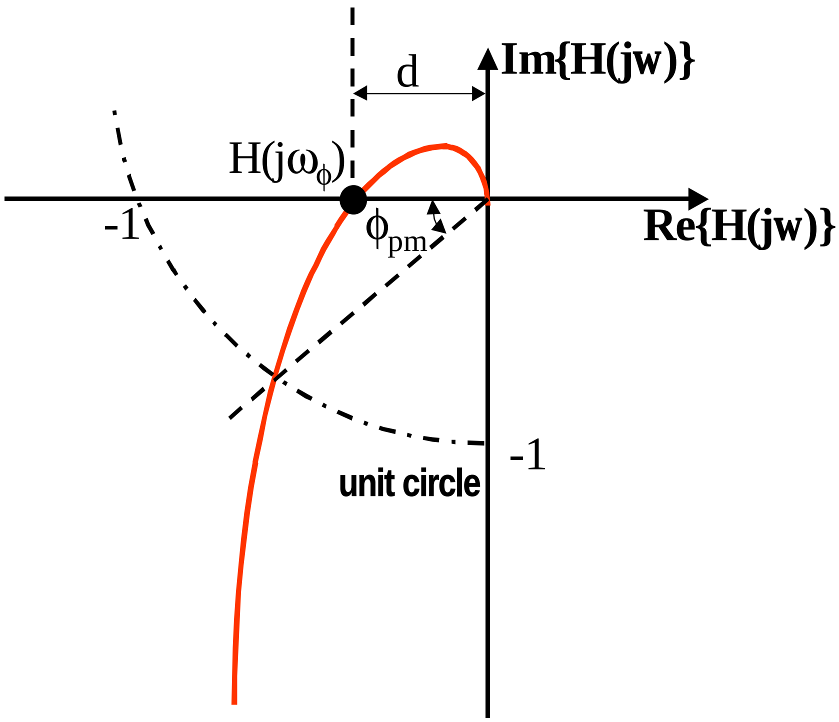
small K: closed loop system is stable

big K: closed loop system is unstable

	N	Z_c	Stability
small K	0	0	Yes
big K	2	2	No

} Nyquist Criterion table

Nyquist Performance Specification Parameters



Nyquist Plot

ϕ_{pm} = phase margin ω_p = gain crossover frequency

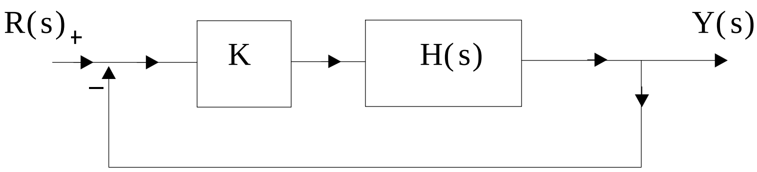
$$\left|H\left(j\cdot\omega_p\right)\right| = 1$$

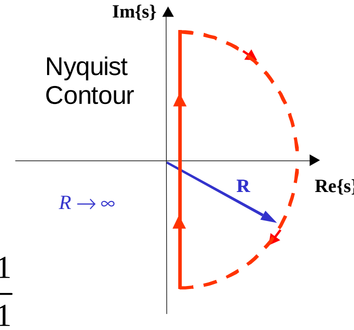
ω_ϕ = Phase Crossover frequency

$$\text{Gain Margin} = \frac{1}{\left|H\left(j\cdot\omega_\phi\right)\right|} = \frac{1}{d}$$

} Nyquist Plot contains the same type of performance specifications as the Bode Plot and the Root Locus

Example - Clockwise Contour



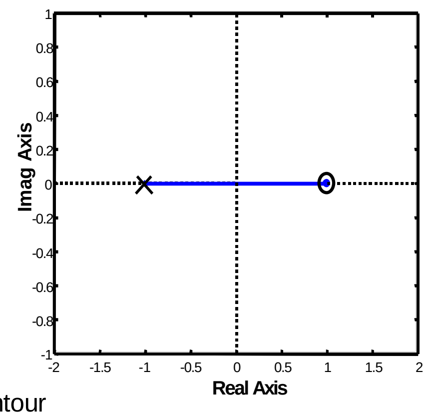


Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

$$H(s) = \frac{s - 1}{s + 1}$$

$$\frac{Y(s)}{R(s)} = \frac{K \cdot H(s)}{1 + K \cdot H(s)}$$ } Closed Loop Transfer Function

We can see from the root locus that the system goes unstable



$$\Delta(s) = 1 + K \cdot \frac{s - 1}{s + 1}$$ } Nyquist Form for the denominator

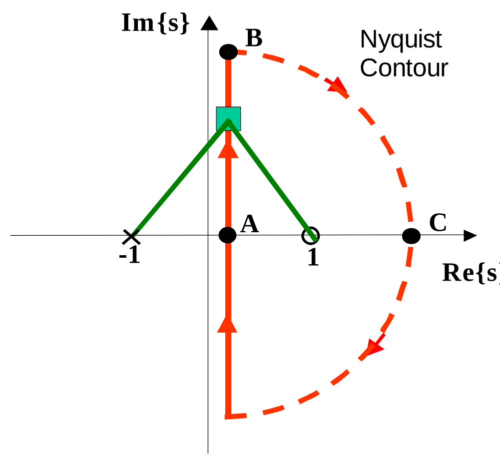
Step 1) Let $s = j\omega$

$$K \cdot H(s) = \frac{K \cdot (j\omega - 1)}{(j\omega + 1)}$$ } Assume K = 1

Step 2) Plot a pole/zero plot for $H(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	1	180
B	1	0
C	1	0

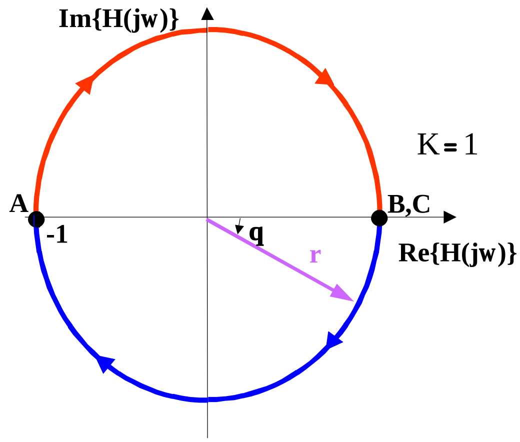


$$|H(j\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|}$$

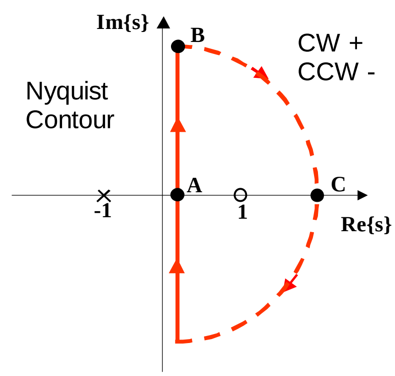
$$\text{Angle}(H(j\omega)) = \text{Angle}(j\omega - 1) - \text{Angle}(j\omega + 1)$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis. Draw other half.



point	r mag	θ phase
A	1	180
B	1	0
C	1	0



As K is decreased below 1, the Nyquist Plot intersects the Re{j ω} axis before -1

As K is increased beyond 1, the Nyquist Plot intersects the Re{j ω} axis beyond -1

Nyquist Criterion

1) Find P_0 => $P_0 = 0$ (Number of poles of $H(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	0	Yes
big K	1	1	No

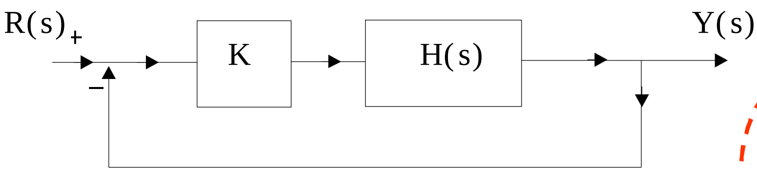
} Nyquist Criterion table

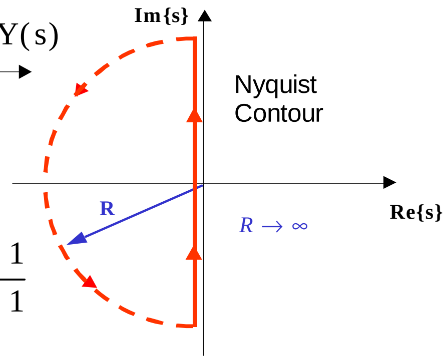
4) Understanding the above table:

For **small K**, no closed loop pole is inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

For **big K**, one closed loop pole is inside the s-plane contour; hence, for big K, the closed loop system is unstable

Same Example - Counterclockwise Contour





Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

$$H(s) = \frac{s - 1}{s + 1}$$

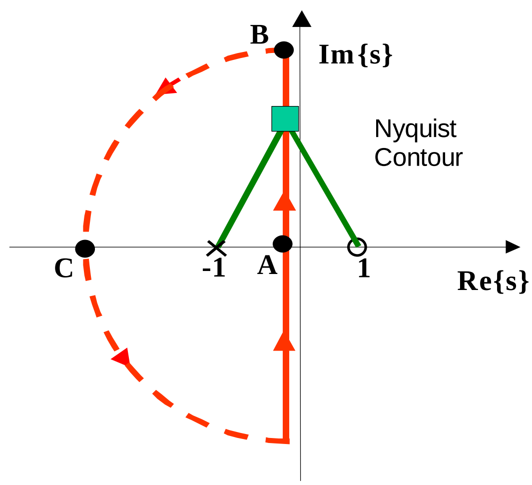
Step 1) Let $s = j\omega$

$$K \cdot H(s) = \frac{K \cdot (j\omega - 1)}{(j\omega + 1)}$$ } Assume K = 1

Step 2) Plot a pole/zero plot for $H(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	1	180
B	1	0
C	1	0



$$|H(j\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|}$$

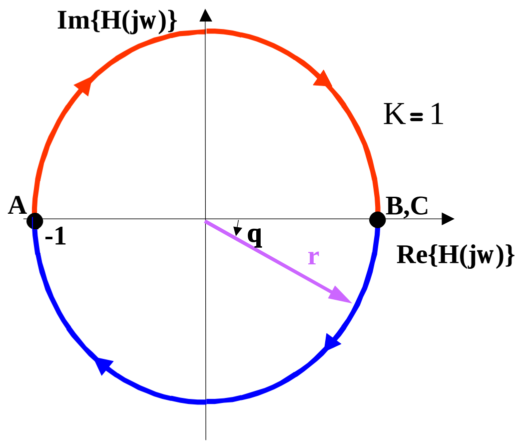
$$\text{Angle}(H(j\omega)) = \text{Angle}(j\omega - 1) - \text{Angle}(j\omega + 1)$$

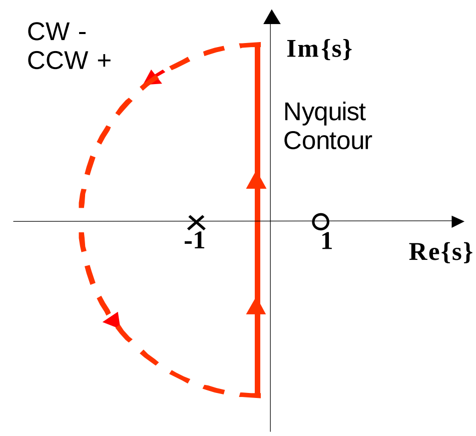
Note: Numbers in the table are exactly the same as before

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis.
Draw other half.

	r	θ
point	mag	phase
A	1	180
B	1	0
C	1	0





As K is decreased below 1, the Nyquist Plot intersects the Re{j ω} axis before -1

As K is increased beyond 1, the Nyquist Plot intersects the Re{j ω} axis beyond -1

Nyquist Criterion

1) Find P_0 => $P_0 = 1$ (Number of poles of $H(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CCW so, CW encirclements are negative and CCW encirclements are positive

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	1	Yes
big K	-1	0	No

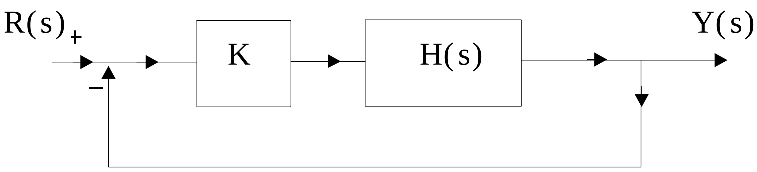
} Nyquist Criterion table

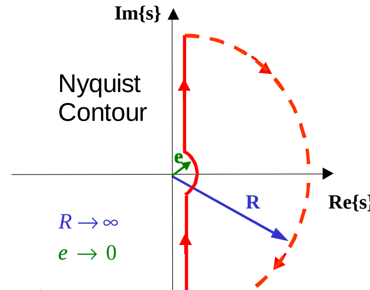
4) Understanding the above table:

For **small K**, one closed loop pole is inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

For **big K**, no closed loop pole is inside the s-plane contour; hence, for big K, the closed loop system is unstable

Example - Clockwise Contour





Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

$$H(s) = \frac{1}{s\cdot(s+1)\cdot(s+2)}$$

$$\frac{Y(s)}{R(s)} = \frac{K\cdot H(s)}{1 + K\cdot H(s)}$$ } Closed Loop Transfer Function

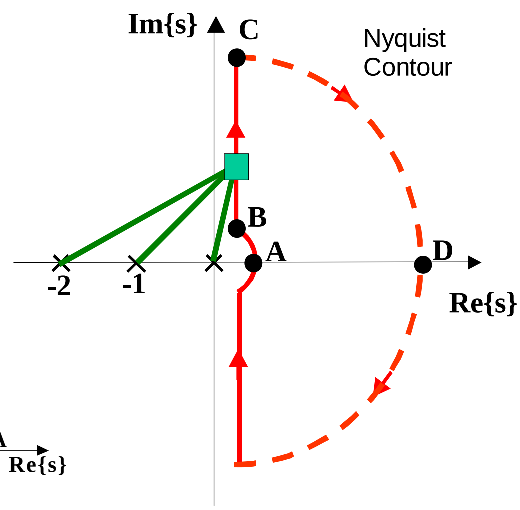
$$\Delta(s) = 1 + K\cdot\frac{1}{s\cdot(s+1)\cdot(s+2)}$$ } Nyquist Form for the denominator

Step 1) Let $s = j\omega$ $\quad K\cdot H(s) = \frac{K}{(j\omega)\cdot(j\omega+1)\cdot(j\omega+2)}$ } Assume K = 1

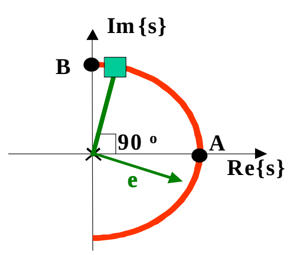
Step 2) Plot a pole/zero plot for $H(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	∞	0
B	∞	-90
C	0	-270
D	0	0



"Blow-up"



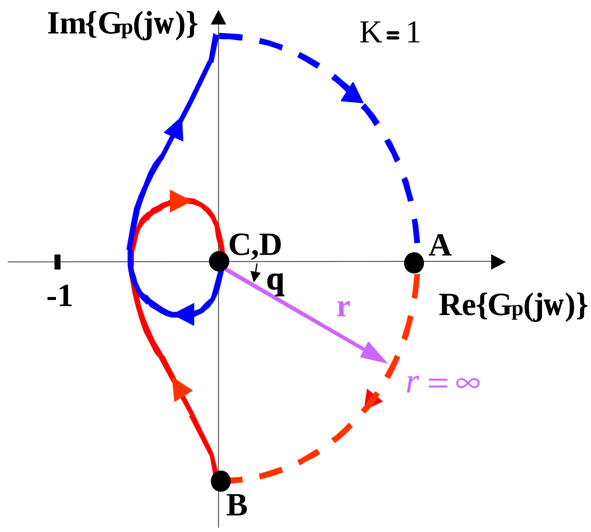
$$|H(j\omega)| = \frac{1}{|j\omega|\cdot|j\omega+1|\cdot|j\omega+2|}$$

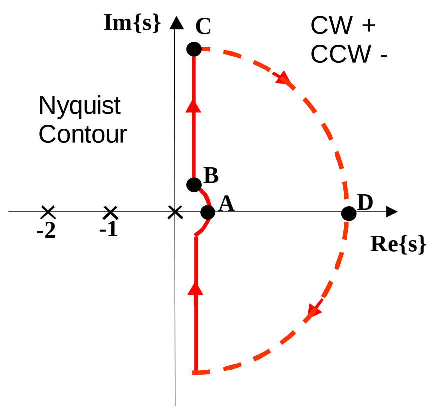
$$\text{Angle}(H(j\omega)) = -\text{Angle}(j\omega) - \text{Angle}(j\omega+1) - \text{Angle}(j\omega+2)$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis. Draw other half.

	r	θ
point	mag	phase
A	∞	0
B	∞	-90
C	0	-270
D	0	0





Nyquist Criterion

1) Find P_0 => $P_0 = 0$ (Number of poles of $H(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	0	Yes
big K	2	2	No

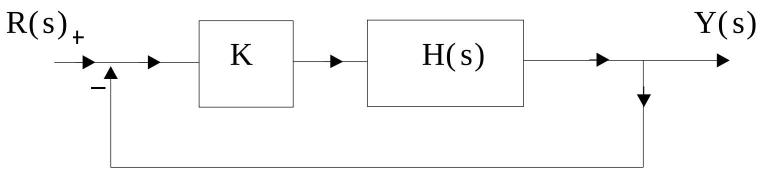
} Nyquist Criterion table

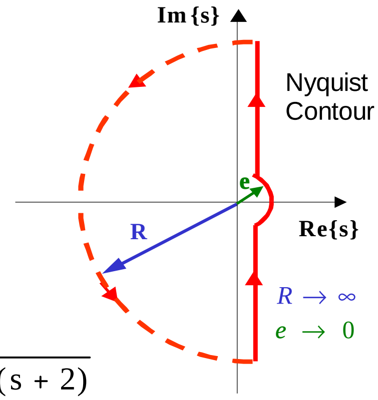
4) Understanding the above table:

For **small K**, no closed loop pole is inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

For **big K**, two closed loop poles are inside the s-plane contour; hence, for big K, the closed loop system is unstable

Same Example - Counterclockwise Contour





Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

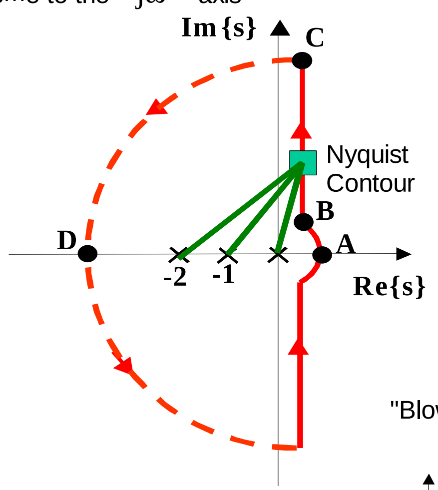
$$H(s) = \frac{1}{s \cdot (s+1) \cdot (s+2)}$$

Step 1) Let $s = j\omega$

$$K \cdot H(s) = \frac{K}{(j\omega) \cdot (j\omega + 1) \cdot (j\omega + 2)}$$

} Assume K = 1

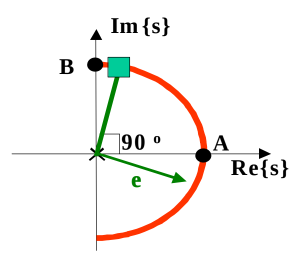
Step 2) Plot a pole/zero plot for $H(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis



Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	∞	0
B	∞	-90
C	0	-270
D	0	-540

"Blow-up"



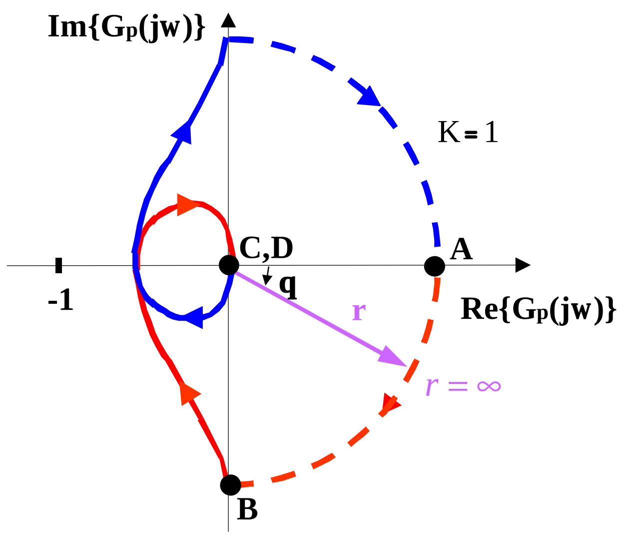
$$|H(j\omega)| = \frac{1}{|j\omega| \cdot |j\omega + 1| \cdot |j\omega + 2|}$$

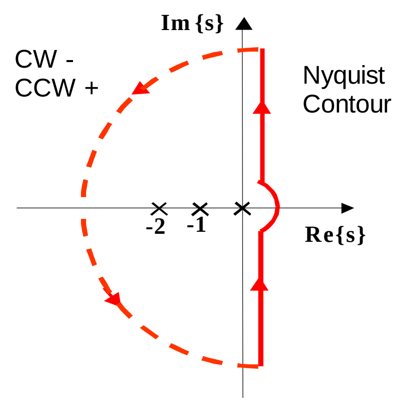
$$\text{Angle}(H(j\omega)) = -\text{Angle}(j\omega) - \text{Angle}(j\omega + 1) - \text{Angle}(j\omega + 2)$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis. Draw other half.

point	mag (r)	phase (θ)
A	∞	0
B	∞	-90
C	0	-270
D	0	-540





Nyquist Criterion

1) Find P_0 => $P_0 = 3$ (Number of poles of $H(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CCW so, CW encirclements are negative and CCW encirclements are positive

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	3	Yes
big K	-2	1	No

} Nyquist Criterion table

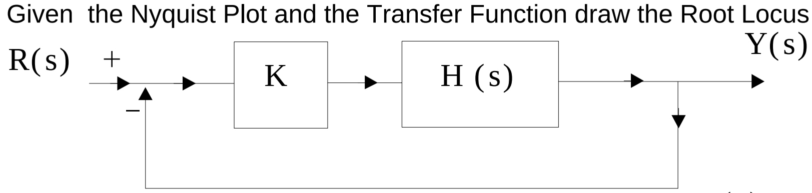
4) Understanding the above table:

For **small K**, all three closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

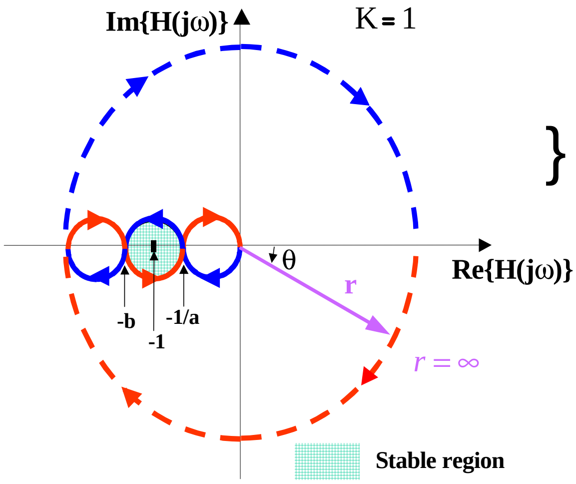
For **big K**, only one closed loop pole is inside the s-plane contour; hence, for big K, the closed loop system is unstable

Example: Using Nyquist Plots to Draw Root Loci

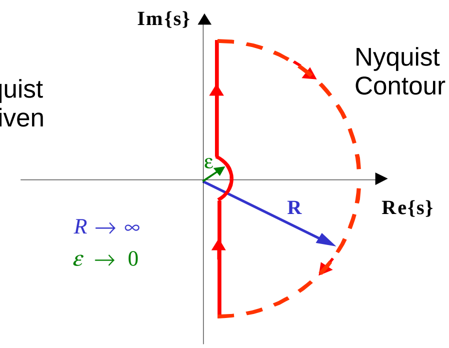
Given the Nyquist Plot and the Transfer Function draw the Root Locus



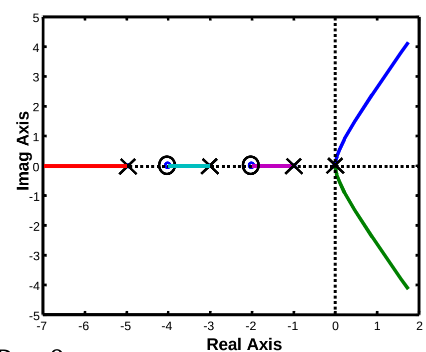
$$H(s) = \frac{(s+2)\cdot(s+4)}{s^2\cdot(s+1)\cdot(s+3)\cdot(s+5)}$$



} The Nyquist Plot is given



This is how you might be tempted to draw the root locus without the Nyquist plot {



Does the Nyquist Plot concur ?

1) Find P_0 ; the number of poles inside the contour $P_0 = 0$

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$

	N	Z_c	Stability
small K	2	2	No
medium K	0	0	Yes
big K	2	2	No

} Nyquist Criterion table => **The above root locus is wrong**

	N	Z_c	Stability
small K	2	2	No
medium K	0	0	Yes
big K	2	2	No

} Nyquist Criterion table

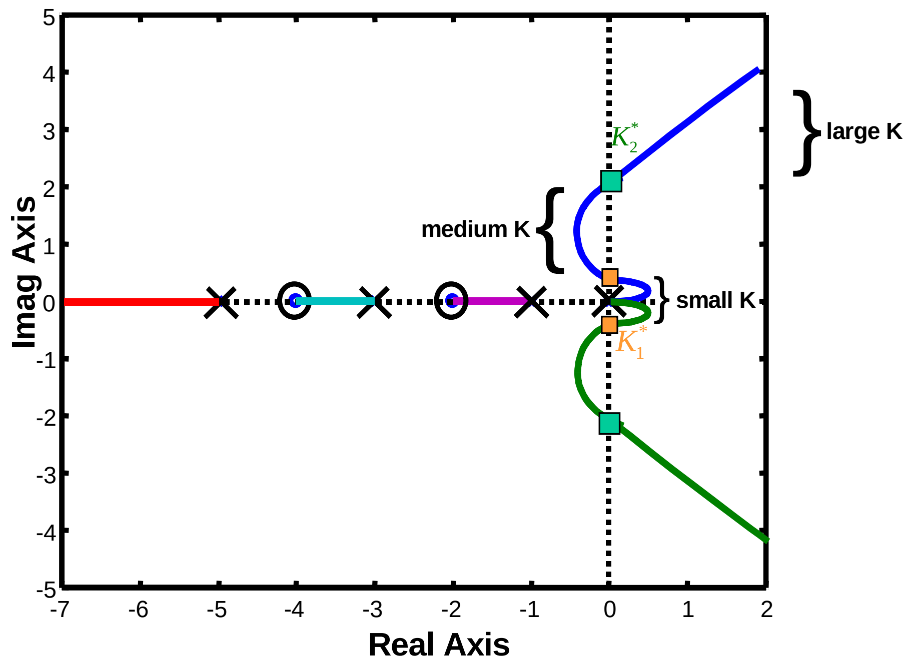
4) Understanding the above table:

For **small K**, two closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is unstable

For **medium K**, no closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for medium K the closed loop system is stable

For **big K**, two closed loop poles are inside the s-plane contour; hence, for big K, the closed loop system is unstable

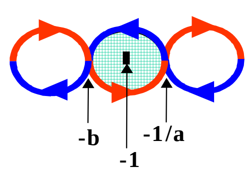
The above table assists in the Root Locus sketch



From the above root locus, we can conclude that the system is stable for

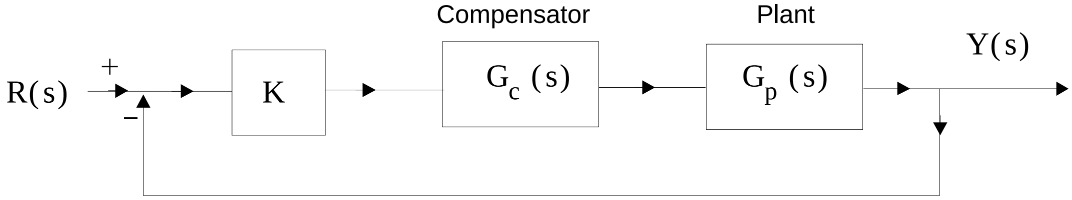
$$K_1^* < K < K_2^*$$

The above result matches that of the Nyquist criterion

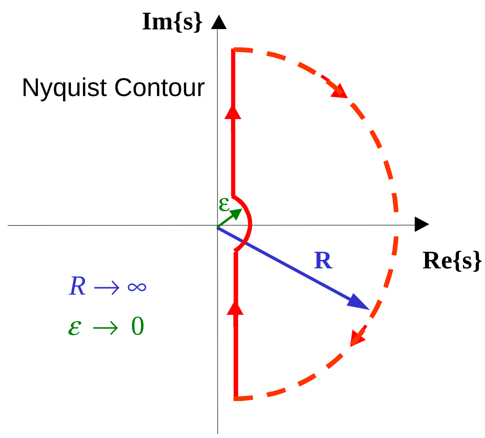


$$K_1^* = \frac{1}{b} \quad \text{and} \quad K_2^* = a$$

Example: Using Compensators



Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.



$$G_p(s) = \frac{1}{s \cdot (s - 1)}$$ } Plant Transfer Function

Assume $G_c(s) = 1$ and $K = 1$

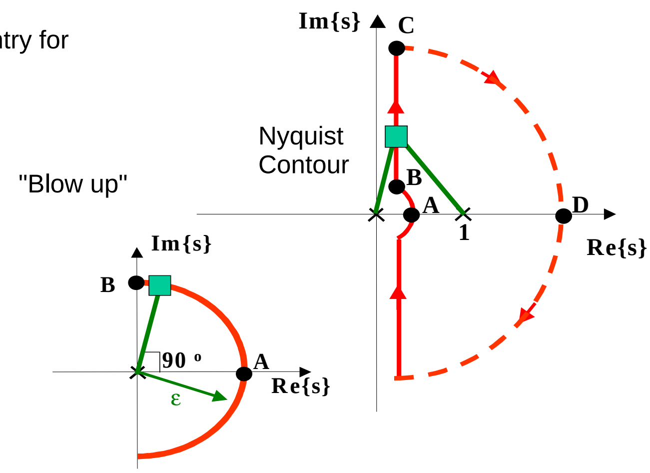
$$\Delta(s) = 1 + K \cdot \frac{1}{s \cdot (s - 1)}$$ } Nyquist Form for the denominator

Step 1) Let $s = j\omega$ $\quad K \cdot G_p(j\omega) = \dfrac{K}{j\omega \cdot (j\omega - 1)}$ } Assume K = 1

Step 2) Plot a pole/zero plot for $G_p(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	∞	−180
B	∞	−270
C	0	−180
D	0	0

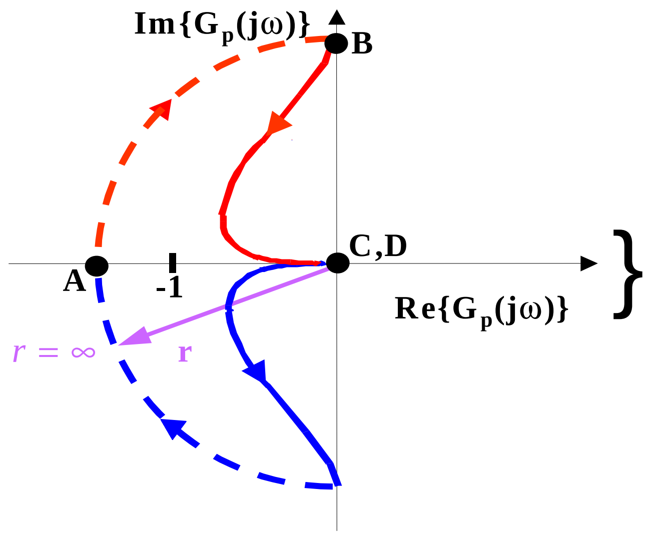


$$\left| G_p(j\omega) \right| = \frac{1}{\left| j\omega \right| \cdot \left| j\omega - 1 \right|} \qquad \text{Angle}\left[G_p(j\omega) \right] = -\text{Angle}(j\omega) - \text{Angle}(j\omega - 1)$$

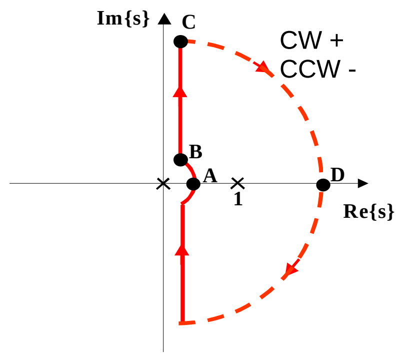
Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis.
Draw other half.

	r	θ
point	mag	phase
A	∞	- 180
B	∞	- 270
C	0	- 180
D	0	0



} Nyquist Plot with K = 1



Nyquist Criterion

1) Find P_0 => $P_0 = 1$ (Number of poles of $G_p\ (s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

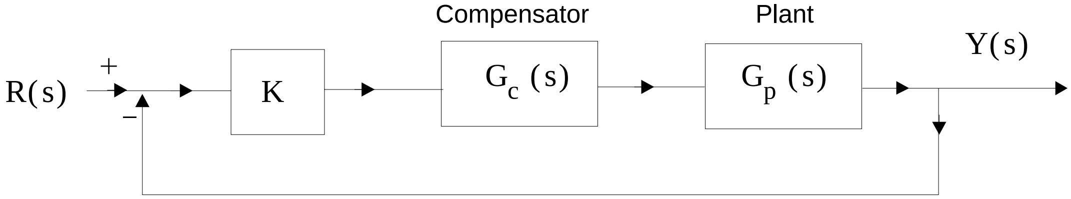
	N	Z_c	Stability
small K	1	2	No
big K	1	2	No

} Nyquist Criterion table

4) Understanding the above table:

For **small and big K**, two closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is always unstable

Same Example: Different Compensator



Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

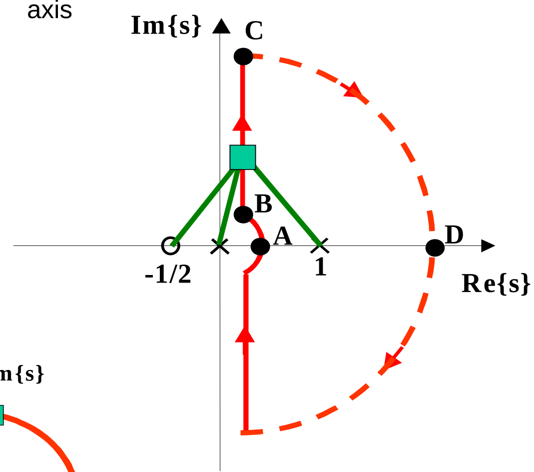
Let $G_c(s) = 2s + 1$

$$\Delta(s) = 1 + K \cdot \frac{2s+1}{s \cdot (s-1)}$$ } Nyquist Form for the denominator

Step 1) Let $s = j\omega$ $\quad K \cdot G_p(j\omega) \cdot G_c(j\omega) = K \cdot \frac{(2j\omega + 1)}{j\omega \cdot (j\omega - 1)}$ } Assume K = 1

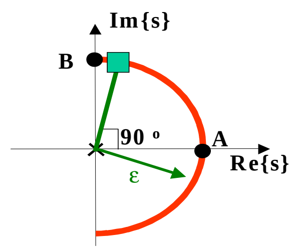
Step 2) Plot a pole/zero plot for $G_p(s) \cdot G_c(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.



	r	θ
point	mag	phase
A	∞	−180
B	∞	−270
C	0	−90
D	0	0

"Blow up"



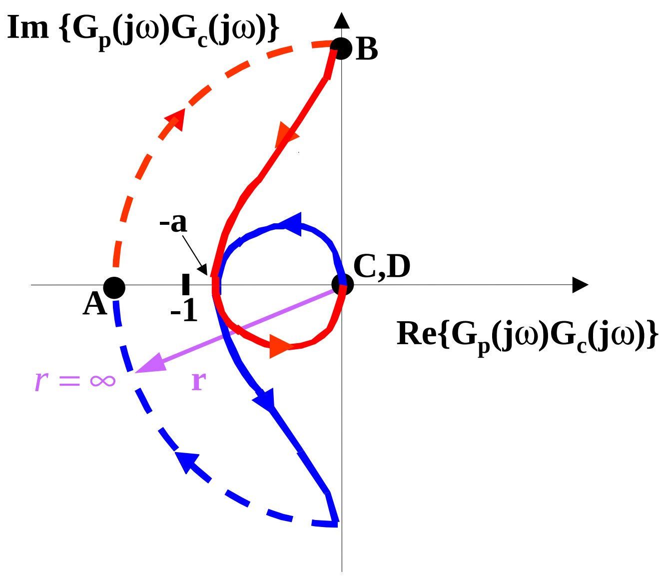
$$\left| G_p(j\omega) \cdot G_c(j\omega) \right| = \frac{\left| j2\omega + 1 \right|}{\left| j\omega \right| \cdot \left| j\omega - 1 \right|}$$

$$\text{Angle}\left[G_p(j\omega) \cdot G_c(j\omega) \right] = \text{Angle}(j2\omega + 1) - \text{Angle}(j\omega) - \text{Angle}(j\omega - 1)$$

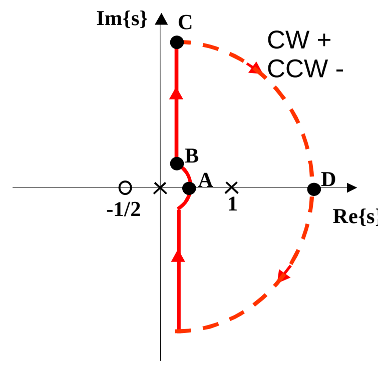
Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis.
Draw other half.

point	r mag	θ phase
A	∞	- 180
B	∞	- 270
C	0	- 90
D	0	0



} Nyquist Plot with K = 1



Nyquist Criterion

1) Find P_0 => $P_0 = 1$ (Number of poles of $G_p(s) \cdot G_c(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $\left(Z_c = N + P_0\right)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	1	2	No
big K	- 1	0	Yes

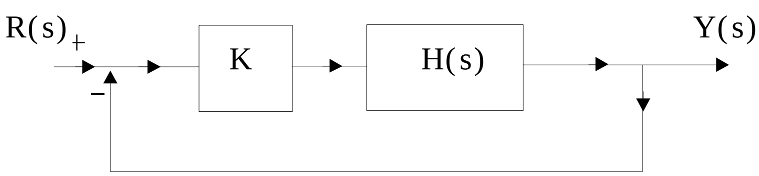
} Nyquist Criterion table

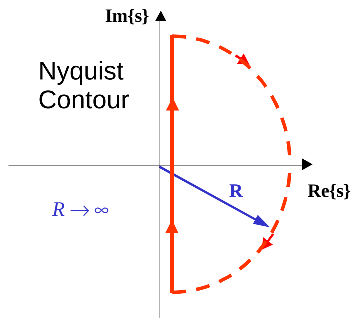
4) Understanding the above table:
For **small K**, both closed loop poles are inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is unstable

For **big K**,no closed loop poles are inside the s-plane contour; hence, for big K, the closed loop system is stable

Stable for $K > \frac{1}{a}$ } "a" can be obtained off of the Bode plot

Example





Use Nyquist Plot, Contour and Criterion to discuss quantitatively how the control gain K affects closed-loop stability.

$$H(s) = \frac{s - 1}{(s + 1)\cdot(s + 10)}$$

$$\frac{Y(s)}{R(s)} = \frac{K\cdot H(s)}{1 + K\cdot H(s)}$$ } Closed Loop Transfer Function

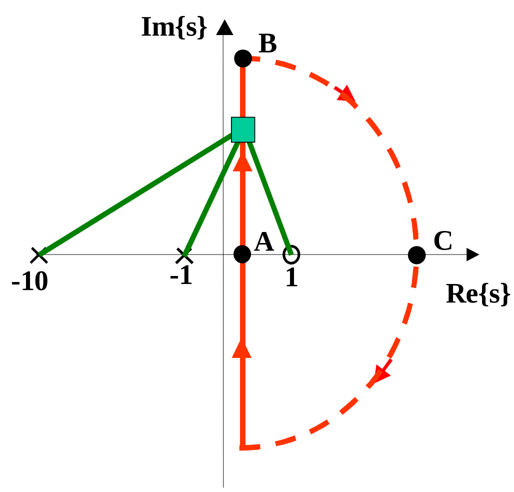
$$\Delta(s) = 1 + K\cdot\frac{s - 1}{(s + 1)\cdot(s + 10)}$$ } Nyquist Form for the denominator

Step 1) Let $s = j\omega$ $K\cdot H(s) = \frac{K\cdot(j\omega - 1)}{(j\omega + 1)\cdot(j\omega + 10)}$ } Assume K = 1

Step 2) Plot a pole/zero plot for $H(s)$ on the s-plane contour picture and include measurement scheme to the $j\omega$ axis

Step 3) Enter a magnitude/phase table entry for each critical point.

	r	θ
point	mag	phase
A	0.1	180
B	0	-90
C	0	0



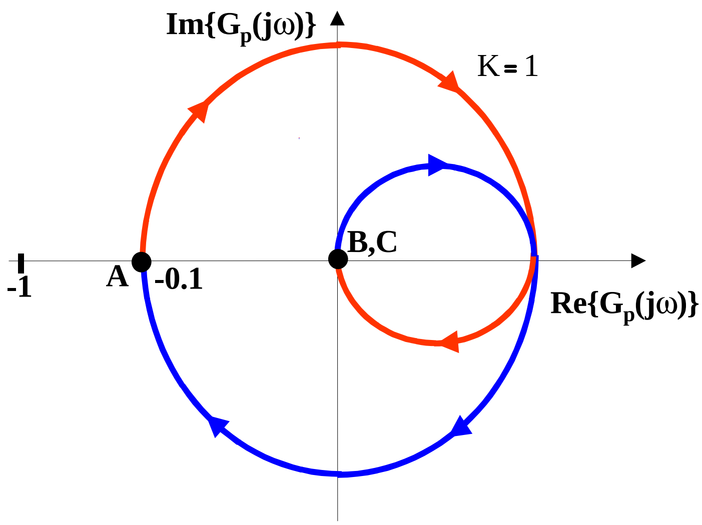
$$|H(j\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|\cdot|j\omega + 10|}$$

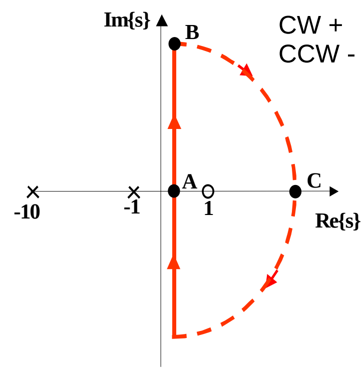
$$\text{Angle}(H(j\omega)) = \text{Angle}(j\omega - 1) - \text{Angle}(j\omega + 1) - \text{Angle}(j\omega + 10)$$

Step 4) Use critical points to draw Nyquist Plot.

Step 5) Nyquist plot is symmetric about real axis.
Draw other half.

	r	θ
point	mag	phase
A	0.1	180
B	0	-90
C	0	0





Nyquist Criterion

1) Find P_0 => $P_0 = 0$ (Number of poles of $H(s)$ inside the contour)

2) Determine the sign notation for the encirclements.

Contour is CW so, CW encirclements are positive and CCW encirclements are negative

3) Find N (i.e., the number of encirclements of (-1,0) for different values of K

Also compute Z_c $(Z_c = N + P_0)$ } Number of closed loop poles inside the contour

	N	Z_c	Stability
small K	0	0	Yes
big K	1	1	No

} Nyquist Criterion table

4) Understanding the above table:

For **small K**, no closed loop pole is inside the s-plane contour. From the contour, we can see that this means that for small K the closed loop system is stable

For **big K**, one closed loop pole is inside the s-plane contour; hence, for big K, the closed loop system is unstable

For $0 < K < 10$ the closed loop system is stable (see Nyquist Plot)

Time for a Movie

Click on image to start movie

State Space Analysis

All of the control problems we have studied in this class have been solved in the frequency domain. We will now concentrate on developing control solutions in the time domain for the same type of systems.

$$a_n\, y^{(n)} + a_{n-1}\cdot y^{(n-1)} + \ldots\ldots\ldots\ldots\ldots + .\ a_1\, y^{(1)} + a_0\, y = b_{n-1}\cdot u^{(n-1)} + b_{n-2}\cdot u^{(n-2)} + \ldots\ldots\ldots\ b_0\, u$$

} Typical DE

This type of system (i.e., a linear, causal, time invariant system, can also be written as follows

$$\frac{d}{dt}x = A\cdot x + B\cdot u$$

$$y = C\cdot x$$

} A is an n x n matrix B is n x 1 vector

C is 1 x n vector

} contains the DE coefficients

where y(t) denotes the system output
u(t) denotes the system input
x(t) - an n x 1 state vector is used to account for "derivatives" in the system.

The "n" dimensions of the matrix equation takes care of the "n" derivatives found in the original "nth" order scalar differential equation.

Example

Show that the differential equation

$$y^{(3)} + a_1 \cdot y^{(2)} + a_2 \cdot y^{(1)} + a_3 \cdot y = b_1 \cdot u^{(2)} + b_2 \cdot u^{(1)} + b_3 \cdot u \quad \Big\} \quad n = 3 \text{ third-order DE}$$

can be written as follows.

$$\begin{bmatrix} \frac{d}{dt} x_1 \\ \frac{d}{dt} x_2 \\ \frac{d}{dt} x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot u \qquad y = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\}$$

A is a 3 x 3 matrix
B is a 3 x 1 vector
C is a 1 x 3 vector
x is a 3 x 1 vector

From the original differential equation, we can obtain the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \quad \Big\}$$

Initial Conditions are equal to zero.

We now calculate the transfer function from the state equation by decomposing the state equation into its scalar components

$$\frac{d}{dt} x_1 = x_2 \qquad \frac{d}{dt} x_2 = x_3 \qquad \frac{d}{dt} x_3 = -a_3 \cdot x_1 - a_2 \cdot x_2 - a_1 \cdot x_3 + u$$

$$y = b_3 \cdot x_1 + b_2 \cdot x_2 + b_1 \cdot x_3$$

} Multiply matrices out

From the above equations we can see that

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \end{aligned}\right\} \quad \Rightarrow \ddot{x}_1 = x_3 \Rightarrow \dddot{x}_1 = \dot{x}_3$$

Therefore

$$y = b_3 x_1 + b_2 \dot{x}_1 + b_1 \ddot{x}_1 \quad \Big\}$$ Direct Substitution

$$\frac{Y(s)}{X_1(s)} = \frac{\left(b_1 s^2 + b_2 \cdot s + b_3\right)}{1}$$ } Take Laplace Transform

$$\left(x_1\right)^{(3)} = -a_3 \cdot x_1 - a_2 \cdot \left(x_1\right)^{(1)} - a_1 \cdot \left(x_1\right)^{(2)} + u$$ } Direct Substitution

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^3 + a_1 \cdot s^2 + a_2 \cdot s + a_3}$$ } Take Laplace Transform

$$\frac{Y(s)}{X_1(s)} \cdot \frac{X_1(s)}{U(s)} = \frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 \cdot s + b_3}{s^3 + a_1 \cdot s^2 + a_2 \cdot s + a_3}$$ } Cross Multiply

Hence, the state space form has the same transfer function as the original differential equation.

There are an infinite number of state space forms for a given differential equation. The state space form for this example is called Reachable Canonical Form (RCF).

Example

Put $y^{(4)} + 6\,y^{(2)} = 3\,u^{(3)} + 7\,u^{(1)}$ into RCF.

$y^{(4)} + (0)\,y^{(3)} + 6\,y^{(2)} + (0)\,y^{(1)} + (0)\,y =$ } Output side of DE

$3 \cdot u^{(3)} + (0)\,u^{(2)} + (7)\,u^{(1)} + (0)\,u$ } Input side of DE

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \quad \left.\begin{matrix} x - 4 \times 1 & B - 4 \times 1 \\ A - 4 \times 4 & C - 1 \times 4 \end{matrix}\right\} \quad n = 4$$

$$\left.\begin{matrix} A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 0 \end{bmatrix} & B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ C = \begin{bmatrix} 0 & 7 & 0 & 3 \end{bmatrix} & \end{matrix}\right\} \text{ State Space Form (RCF)}$$

<u>Step 1</u>

First place zeros everywhere except for the circled areas. The circled areas are the (i) super diagonal and the last row of "A", (ii) the last element of "B", and (iii) the entire "C" vector

<u>Step 2</u>

Put "ones" in the Super Diagonal.

<u>Step 3</u>

Put a one in the last enty of B.

<u>Step 4</u>

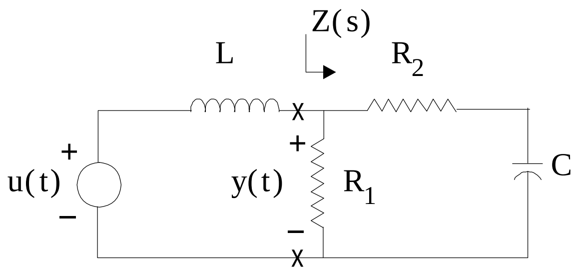
Take the last "n" coefficients of the output side of the DE, reverse their order, make them negative, and then put them in the last row of A.

<u>Step 5</u>

Take the "n" coefficients of the input side of the DE, reverse their order, and then put them in C.

Example

Find a state-space representation for the circuit below



} Z(s) is the impedance to the right of the two x's

Note: Capacitor Impedance = $\frac{1}{Cs}$ Inductor Impedance = Ls

First get a transfer function for the circuit

$$\frac{Y(s)}{U(s)} = \frac{Z(s)}{Z(s) + Ls}$$ } Voltage Division

$$Z(s) = R_1 \parallel \left(R_2 + \frac{1}{Cs}\right)$$ } Calculate Z(s) from the circuit

$$Z(s) = \frac{R_1 R_2 + \frac{R_1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}}$$ } Apply Parallel Law

$$Z(s) = \frac{CR_1 R_2 s + R_1}{C\left(R_1 + R_2\right) s + 1}$$ } Simplify

$$\frac{Y(s)}{U(s)} = \frac{sR_1 R_2 C + R_1}{s^2 \left(R_1 + R_2\right) LC + s\left(L + R_1 R_2 C\right) + R_1}$$ } Final Form for TF

$R_1 = 3\Omega$, $L = 3.8\ H$, $R_2 = 1\Omega$, $C = .066\ F$ } Typical Values

$$\frac{Y(s)}{U(s)} = H(s) = \frac{.2 \cdot s + 3}{s^2 + 4 \cdot s + 3}$$ } Substitute numerical values

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x$$

} answer via RCF pattern

Matrix Review

Transpose of a matrix — Exchange rows and columns

$$(A + B)^T = A^T + B^T \qquad (AB)^T = B^T A^T$$ } Standard Operations

Symmetric Matrix $A^T = A$ } By Definition

Determinants

2 x 2 $$|A| = \left|\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\right| = a_1 \cdot b_2 - b_1 \cdot a_2$$

3 x 3 $$|A| = \left|\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right| = \begin{matrix} a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 \\ -c_1 b_2 a_3 - b_1 a_2 c_3 - a_1 b_3 c_2 \end{matrix}$$

Inverse of a Matrix

$A \cdot A^{-1} = I$ } Note A^{-1} exists if $|A| \neq 0$

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

$$(AB)^{-1} = B^{-1} A^{-1} \qquad \left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$ } Standard Operations

2 x 2 $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \frac{1}{|A|}$$

3 x 3 $$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} ei - hf & -(bi - hc) & bf - ec \\ -(di - gf) & ai - gc & -(af - dc) \\ dh - ge & -(ah - gb) & ae - db \end{bmatrix} \cdot \frac{1}{|A|}$$

Rank of a Matrix

A matrix "A" is called rank "m" if the number of linearly independent columns is equal to "m"

x, y and z are linearly independent vectors if

$\alpha x \neq y$ and $\alpha x + \beta y \neq z$ for some scalar constants α, β

A square matrix n x n has rank n if and only if $|A| \neq 0$

Eigenvalues of a Matrix

$$|\lambda I - A| = 0$$ } Equation for determining eigenvalues

$$\lambda^n + a_1 \cdot \lambda^{n-1} + \ldots\ldots + a_{n-1} \cdot \lambda + a_n = 0$$ } above expression always reduces to finding the roots.

Note: Eigenvalues and poles are equivalent

Procedure for determining Eigenvectors

Note: The number of eigenvectors = number of eigenvalues = "n"

Case 1) All of the eigenvalues are distinct $(A - \lambda I) \cdot v = 0$

Case 2) Some eigenvalues are repeated - Use generalized procedure

Case 1: Distinct Eigenvalues Example

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$ } Find the eigenvalues

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| - \left| \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda & -1 \\ 6 & \lambda + 5 \end{bmatrix} \right| =$$ } Find roots

$$\lambda^2 + 5\lambda + 6 = (\lambda + 3) \cdot (\lambda + 2) = 0$$

$$\lambda_1 = -3 \qquad \lambda_2 = -2$$ } Two distinct eigenvalues

$$(A - \lambda_1 I) \cdot v_1 = \left[\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \right] \cdot v_1 = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$ } Eigenvector for $\lambda_1 = -3$

$$(A - \lambda_2 I) \cdot v_2 = \left[\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right] \cdot v_2 = \begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$ } Eigenvector for $\lambda_1 = -2$

Note any multiple of v_1 or v_2 would be an eigenvector

$$\begin{bmatrix} 2 \\ -6 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$ } Examples

Case 2: Procedure for finding Eigenvectors for Repeated Eigenvalues

Step 1: For an eigenvalue repeated m times in an n dimensional matrix A,

Find the smallest p such that Note $m \le n$

Rank $(A - \lambda \cdot I)^p = n - m$ } This step gives us "p"

Step 2: For $1 \le k \le p$

Find N_k = Rank $(A - \lambda I)^{k-1}$ - Rank $(A - \lambda I)^k$ } This step gives us the N_k's

Step 3: Find a x_p such that

$(A - \lambda I)^p \cdot x_p = 0$ but $(A - \lambda I)^{p-1} \cdot x_p \ne 0$ then find

$x_j = (A - \lambda I) \cdot x_{j+1}$ where $(j = p - 1, p - 2, ..., 1)$ } This step gives us eigenvectors.

All the x vectors you find are generalized eigenvectors

Step 4: Reduce each N_k by 1

If all $N_k = 0$ then the procedure is complete } This step checks for special cases.

If not continue to step 5

Step 5: Find the largest k for which N_k is not zero. Call it κ

Find a vector x_κ that is linearly independent of previous vector x. Then find

$x_j = (A - \lambda I) \cdot x_{j+1}$ where $j = \kappa - 1, \kappa - 2, ..., 1$

Example Find the eigenvectors of

$$A=\begin{bmatrix}1&1&2\\0&1&3\\0&0&1\end{bmatrix} \qquad |\lambda I - A| = \left|\begin{bmatrix}\lambda-1&-1&-2\\0&\lambda-1&-3\\0&0&\lambda-1\end{bmatrix}\right| = (\lambda-1)^3=0$$

} Gives Eigenvalues

Repeated Roots $n=3$ $m=3$ $\lambda=1$ } Eigenvalue

Step 1: Find smallest "p" so the Rank $\left[(A-\lambda I)^p\right] = n - m = 3 - 3 = 0$ $\lambda = 1$

$$p=1\ (A-\lambda I)^1=\begin{bmatrix}0&1&2\\0&0&3\\0&0&0\end{bmatrix} \quad \text{Rank}=2 \qquad p=2\ (A-\lambda I)^2=\begin{bmatrix}0&0&3\\0&0&0\\0&0&0\end{bmatrix} \quad \text{Rank}=1$$

$$p=3\ (A-\lambda I)^3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \quad \text{Rank}=0 \qquad \boxed{p=3}$$

} answer

Step 2: $N_k = \text{Rank}\cdot(A-\lambda I)^{k-1} - \text{Rank}\cdot(A-\lambda I)^k$ for $1 \le k \le p = 3$

$$N_1 = \text{Rank}\cdot(A-\lambda I)^0 - \text{Rank}\cdot(A-\lambda I)^1 = 3 - 2 = 1$$

$$N_2 = \text{Rank}\cdot(A-\lambda I)^1 - \text{Rank}\cdot(A-\lambda I)^2 = 2 - 1 = 1$$

$$N_3 = \text{Rank}\cdot(A-\lambda I)^2 - \text{Rank}\cdot(A-\lambda I)^3 = 1 - 0 = 1$$

Step 3: Find a x_p such that $(A-\lambda I)^p\cdot x_p = 0$ but $(A-\lambda I)^{p-1}\cdot x_p \ne 0$

$$(A-\lambda I)^3\cdot x_3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{but} \quad \underbrace{(A-\lambda I)^2}_{j=1}\cdot x_3=\begin{bmatrix}0&0&3\\0&0&0\\0&0&0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}\ne\begin{bmatrix}0\\0\\0\end{bmatrix}$$

so a possible solution is $x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$ } First Eigenvector

Now find $x_j = (A - \lambda I) \cdot x_{j+1}$ for $j = p - 1, p - 2, \ldots, 1$ $p = 3$

$$\underbrace{x_2 = (A - \lambda I)}_{j=2}\ x_3 \qquad \underbrace{x_1 = (A - \lambda I)}_{j=3}\ x_2$$ for j = 2, 1

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ } From above calculation

$$x_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$ } Last two eigenvectors

Step 4: After reducing all N_k by one, all the results become zero. Hence we stop at this stage

x_3, x_2, x_1 are the eigenvectors

Example Find the eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad |\lambda I - A| = \left| \begin{bmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 1 & -3 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \right| = (\lambda - 1)^2 \cdot (\lambda - 2) = 0$$

} Gives Eigenvalues

$\lambda = 1, m = 2$ $\qquad \lambda = 2, m = 1$ } Eigenvalues

Non Repeated Root

$$\lambda = 2 \quad (A - \lambda I) \cdot v = 0 \quad \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

} First Eigenvector

Repeated Root $\qquad n = 3$ and $m = 2 \qquad \lambda = 1$

Step 1: Find smallest "p" so the $\quad \text{Rank}\left[(A - \lambda I)^p\right] = n - m = 3 - 2 = 1 \qquad \lambda = 1$

$$p = 1 \; (A - \lambda I)^1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Rank} = 2$$

$$(A - \lambda I)^2 = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Rank} = 1 \qquad \boxed{p = 2}$$

} answer

Step 2: $N_k = Rank\cdot(A-\lambda I)^{k-1} - Rank\cdot(A-\lambda I)^{k}$ for $1 \le k \le p = 2$ $\lambda = 1$

$$N_1 = Rank\cdot(A-\lambda I)^{0} - Rank\cdot(A-\lambda I)^{1} = 3 - 2 = 1$$

$$N_2 = Rank\cdot(A-\lambda I)^{1} - Rank\cdot(A-\lambda I)^{2} = 2 - 1 = 1$$

Step 3: Find a x_p such that $(A-\lambda I)^{p}\cdot x_p = 0$ but $(A-\lambda I)^{p-1}\cdot x_p \ne 0$

$\lambda = 1, p = 2$

$$(A-\lambda I)^{2}\cdot x_2 = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{but} \quad (A-\lambda I)^{1}\cdot x_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \\ z \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so a possible solution is $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ } First Eigenvector

Now find $x_j = (A-\lambda I)\cdot x_{j+1}$ for $j = p-1, p-2, \ldots, 1$ $p = 2$

$x_1 = (A-\lambda I)\ x_2$ $j = 1$

$$x_1 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ } Second Eigenvector

Step 4: After reducing all N_k by one, all the results become zero. Hence we stop at this stage

x_2, x_1 are the eigen vectors

Motivation of State-Space Solution

Scalar System

$$\frac{d}{dt}x = ax + bu$$ } "a" and "b" are scalar constants

Solve the equation for x(t)

$$s\cdot X(s) - x(0) - a\cdot X(s) = b\cdot U(s)$$ } Use the Laplace Transform

$$X(s) = \frac{b\cdot U(s)}{s-a} + \frac{x(0)}{s-a}$$ } Simplify

After taking the Inverse Laplace Transform

$$x(t) = e^{at}x(0) + \mathcal{L}^{-1}\left\{\frac{b}{s-a}U(s)\right\}$$ } Inverse Laplace Transform

Let $H(s) = \dfrac{b}{s-a} \Rightarrow h(t) = e^{at}b$

$$x(t) = e^{at}x(0) + h(t) * u(t)$$ } Convolution Property

$$x(t) = e^{at}x(0) + \int_0^t h(t-\tau)u(\tau)d\tau$$ } Convolution Integral

$$x(t) = e^{at}\,x(0) + \int_0^t \exp(a(t-\tau))\cdot b\cdot u(\tau)\,d\tau$$ } Substitute for h(t) to obtain the solution for x(t)

Initial Condition Contribution — Input Contribution

Solution to the Matrix Equation

$$\frac{d}{dt}x = Ax + Bu$$ } x - n x 1, A - n x n, B - n x 1

Claim that $x(t) = e^{At}\,x(0) + \int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$ } Looks same as the solution to the scalar system

is the solution of the equation

where $\exp(A\cdot t)$ is a matrix

To Prove the above claim, we need three pieces of mathematical machinery

1) What is exp (A t) when "A" is a matrix ?

$$\exp(a\cdot t) = 1 + a\cdot t + \frac{a^2\cdot t^2}{2!} + \frac{a^3\cdot t^3}{3!} + \ldots\ldots \quad \}$$ Use Taylor Series

$$\exp(A\cdot t) = I + A\cdot t + \frac{A^2\cdot t^2}{2!} + \frac{A^3\cdot t^3}{3!} + \ldots\ldots \quad \}$$ Use Taylor Series

2) Note $\frac{d}{dt}\exp(A\cdot t) = A\cdot\exp(A\cdot t) = \exp(A\cdot t)\cdot A$

$$\frac{d}{dt}\exp(A\cdot t) = 0 + A + \frac{2\cdot A^2\cdot t}{(1)\cdot(2)} + \frac{3\cdot A^3\cdot t^2}{(1)\cdot(2)\cdot(3)} + \frac{4\cdot A^4\cdot t^3}{(1)\cdot(2)\cdot(3)\cdot(4)} + \ldots\ldots\ldots \quad \}$$ Take time derrivative

$$\frac{d}{dt}\exp(A\cdot t) = A\left(I + A\cdot t + \frac{A^2\cdot t^2}{2!} + \frac{A^3\cdot t^3}{3!} + \ldots\right) = A\cdot\exp(A\cdot t) \quad \}$$ Factor out "A"

Note: $\frac{d}{dt}\exp(At) = \exp(A\cdot t)\cdot A$ } Why ? because AB = BA

if B = A

3) We also need Leibnitz Rule

$$\frac{d}{dx}\left[\int_{a_1(x)}^{a_2(x)} F(x,u)du\right] = \int_{a_1(x)}^{a_2(x)} \frac{\partial F(x,u)}{\partial x}du + F(x,a_2(x))\frac{d}{dx}a_2(x) - F(x,a_1(x))\frac{d}{dx}a_1(x)$$

$\frac{\partial}{\partial x}$ is the partial derivative with respect to x

Solution to the Matrix Equation

$$\frac{d}{dt}x = A\cdot x + B\cdot u$$ } Matrix equation

$$x(t) = e^{At}\cdot x(0) + \int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$$ } Claimed Solution

$$\frac{d}{dt}x = \frac{d}{dt}e^{At}\cdot x(0) + \frac{d}{dt}\int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$$ } Insert the solution into LHS

$$\frac{d}{dt}x = Ae^{At}\,x(0) + \int_0^t A\cdot\exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau + Bu(t)$$ } Take time derivative and use Leibnitz Rule

$$A\cdot x + B\cdot u = Ae^{At}\,x(0) + \int_0^t A\cdot\exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau + Bu(t)$$ } Insert solution into RHS

The right sides of both the above equations are equal ; hence, we have obtained a solution to

$$\dot{x} = Ax + Bu$$

Note since $y = C\cdot x$ the solution for y(t) is given by

$$y(t) = C\cdot e^{At}\,x(0) + C\cdot\int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$$ } Output Solution

So if we have a system, given u(t), we can find y(t) with the above formula

All we need to find is $\exp(A\cdot t)$ which is called the **State Transition Matrix**

Calculating the State Transition Matrix

Note $\exp(A\cdot t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$ } By Definition

In the solution for x(t) we require $\exp(A\cdot t)$ hence we need a method for calculating $\exp(A\cdot t)$

$\mathcal{L}\{e^{at}\} = (s-a)^{-1}$ } If "a" is a scalar

$\mathcal{L}\{e^{At}\} = (sI-A)^{-1}$ } "A" is a n x n matrix and "I" is the n x n identity matrix

Example Find $\exp(A\cdot t)$ with $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$

$$\exp(At) = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1}$$ } Definition

$$\exp(A\cdot t) = \mathcal{L}^{-1}\begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix}\cdot\frac{1}{\Delta(s)}$$ where $\Delta(s) = s^2 + 4s + 3 = (s+1)\cdot(s+3)$ } Take Inverse

$$\exp(A\cdot t) = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+4}{(s+1)\cdot(s+3)} & \dfrac{1}{(s+1)\cdot(s+3)} \\ \dfrac{-3}{(s+1)\cdot(s+3)} & \dfrac{s}{(s+1)\cdot(s+3)} \end{bmatrix}$$ } Set up for PFE tool

$$\exp(A\cdot t) = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{(s+1)} - \dfrac{0.5}{(s+3)} & \dfrac{0.5}{(s+1)} - \dfrac{0.5}{(s+3)} \\ \dfrac{-1.5}{(s+1)} + \dfrac{1.5}{(s+3)} & \dfrac{-0.5}{(s+1)} + \dfrac{1.5}{(s+3)} \end{bmatrix}$$ } Use PFE tool

$$\exp(A\cdot t) = \begin{bmatrix} \dfrac{3}{2}\cdot\exp(-t) - \dfrac{1}{2}\cdot\exp(-3\cdot t) & \dfrac{1}{2}\cdot\exp(-t) - \dfrac{1}{2}\cdot\exp(-3\cdot t) \\ \dfrac{-3}{2}\cdot\exp(-t) + \dfrac{3}{2}\cdot\exp(-3\cdot t) & \dfrac{-1}{2}\cdot\exp(-t) + \dfrac{3}{2}\cdot\exp(-3\cdot t) \end{bmatrix}$$ } Take Inverse Laplace Transform

Example : Finding the output using the state space solution

Find y(t) if u(t) = unit step

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

} State space form

$$y(t) = C \cdot e^{At}\, x(0) + C \cdot \int_0^t \exp(A \cdot (t - \tau)) \cdot B \cdot u(\tau)\, d\tau$$

} Definition

$$y(t) = C \cdot \int_0^t \exp(A \cdot (t - \tau)) \cdot B \cdot u(\tau)\, d\tau$$

} $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

} From Previous Problem

$$y(t) = C\int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (1)\, d\tau$$

} Substitute "B" and u(t)

$$y(t) = C\int_0^t \begin{bmatrix} \frac{1}{2}e^{-(t-\tau)} - \frac{1}{2}e^{-3(t-\tau)} \\ -\frac{1}{2}e^{-(t-\tau)} + \frac{3}{2}e^{-3(t-\tau)} \end{bmatrix} d\tau$$

} Substitute exp(At) and multiply matrices

$$y(t) = C \left. \begin{bmatrix} \frac{1}{2}e^{-(t-\tau)} - \frac{1}{6}e^{-3(t-\tau)} \\ -\frac{1}{2}e^{-(t-\tau)} + \frac{3}{6}e^{-3(t-\tau)} \end{bmatrix} \right|_0^t$$

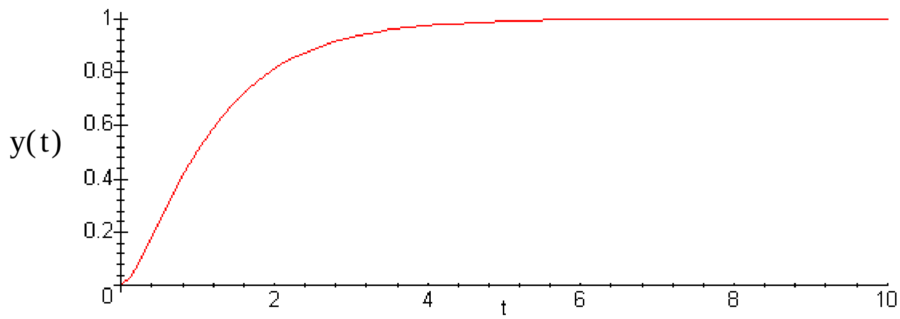
} Integrate term by term

$$y(t) = \begin{bmatrix} 3 & 0.2 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}\right) \\ \left(-\frac{1}{2} + \frac{3}{6}\right) - \left(-\frac{1}{2}e^{-t} + \frac{3}{6}e^{-3t}\right) \end{bmatrix}$$

} Plug limits of integration and "C"

$$y(t) = 1 - \frac{3}{2}\exp(-t) + \frac{1}{2}\exp(-3\,t) + \frac{1}{10}\exp(-t) - \frac{3}{30}\exp(-3\,t)$$ } Multiply matrices

$$y(t) = 1 - \frac{7}{5}\exp(-t) + \frac{2}{5}\exp(-3\,t)$$ } Simplify

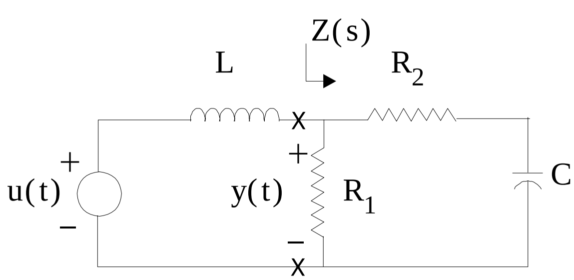


} Does this make sense in the circuit ? Why?

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x$$

} State space realization for this problem and the circuit below



} Circuit that we calculated a SSR for

$$H(s) = \frac{.2\,s + 3}{s^2 + 4\,s + 3}$$ } Transfer function for circuit

Definitions

Time Domain Solution

$$\frac{d}{dt}x = A\cdot x + B\cdot u \qquad y = Cx$$ } State Space equation

$$x(t) = e^{At}\,x(0) + \int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$$ } State Solution

$$y(t) = C\cdot e^{At}\,x(0) + C\cdot\int_0^t \exp(A\cdot(t-\tau))\cdot B\cdot u(\tau)\,d\tau$$ } Output solution

$$\exp(A\cdot t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$ } State transition matrix

Frequency Domain Solution

$$\frac{d}{dt}x - A\cdot x = B\cdot u$$ } State Space equation

$$s\cdot I\cdot X(s) - x(0) - A\cdot X(s) = B\cdot U(s)$$ } Take Laplace Transform

$$(s\cdot I - A)\cdot X(s) = x(0) + B\cdot U(s)$$ } Simplify

$$X(s) = (sI - A)^{-1}\cdot x(0) + (sI - A)^{-1}\cdot B\cdot U(s)$$ } Simplify to obtain State Solution

$$Y(s) = C(sI - A)^{-1}\,x(0) + C(sI - A)^{-1}\cdot B\cdot U(s)$$ } Multiply by "c" to obtain the output solution

"Open Loop" Transfer Function $\frac{Y(s)}{U(s)} = H(s)$

$$\frac{Y(s)}{U(s)} = H(s) = C(sI - A)^{-1}\cdot B$$ } From above output solution with x(0)=0

So given a state space representaion, we can use the above definition to determine the poles and zeros of the system from

$$H(s) = \frac{Cadj(sI - A)B}{|sI - A|}$$ } Since $M^{-1} = \frac{adj(M)}{|M|}$

Poles are given by

$$|sI - A| = 0$$

Zeros are given by $C\cdot adj(sI - A)\,B = 0$

More Definitions

A state variable realization (A,B,C) can be found from the differential equation or a transfer function

The transfer function of the realization satisfies

1. $H(s)$ has real coefficients
2. Rational in s $\}$ $H(s) = \frac{\text{poly}}{\text{poly}}$
3. Strictly proper $\}$ for example $H(s) = \frac{s^2 + 1}{s^3 + 3}$

A realization is minimal if it has the smallest number of states possible

A realization is minimal if H(s) has no pole-zero cancellation

A realization is controllable if the n x n matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$

has full rank (i.e., a non zero determinant)

If a realization is controllable then the modes or poles of the system can be arbitrarily changed

A realization is observable if the n x n $\begin{bmatrix} C \cdot A^{n-1} \\ \cdots\cdots \\ CA \\ C \end{bmatrix}$ has full rank (i.e., a non zero determinant)

If a realization is observable then the modes or poles can be observed at the output y(t)

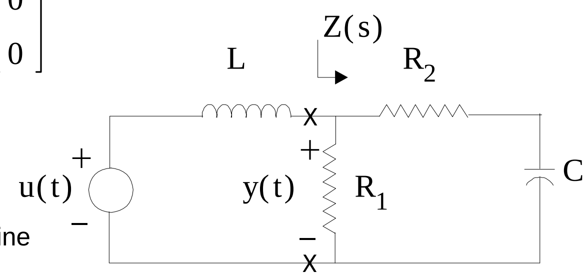
If a realization is controllable and observable, then it is minimal

If a realization is minimal, then it is controllable and observable

Circuit Example

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = (3 \quad 0.2) \cdot x$$



Find the open loop transfer function and determine if the system is controllable and observable

$$H(s) = C \cdot (sI - A)^{-1} \cdot B = C \cdot \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} \cdot B$$ } By definition

$$H(s) = C \cdot \begin{bmatrix} s+4 & 1 \\ -3 & s \end{bmatrix} \cdot \frac{B}{\Delta(s)} = C \begin{bmatrix} 1 \\ s \end{bmatrix} \cdot \frac{1}{\Delta(s)} = \frac{0.2\,s + 3}{s^2 + 4\,s + 3} = \frac{0.2\,s + 3}{(s+1)\cdot(s+3)}$$ } Poles of H(s) are at -1 and -3

$$\Delta(s) = s^2 + 4\,s + 3$$ } open-loop characteristic equation

Calculate the controllability and observability matrices with n=2

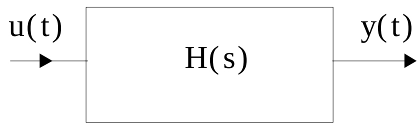
$$\left[B, \; AB \right] = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$ } non zero determinant means the matrix has full rank; hence the system is controllable

$$\begin{bmatrix} CA \\ C \end{bmatrix} = \begin{bmatrix} -0.6 & 2.2 \\ 3 & 0.2 \end{bmatrix}$$ } non zero determinant means the matrix has full rank, hence the system is observable

$$H(s) = \frac{0.2 \cdot (s + 15)}{(s+1)\cdot(s+3)}$$ } The system is clearly **minimal -** no pole / zero cancellation

If we have H(s) the minimality test can use to determine controllability and observability as opposed to the above matrix check

Control Motivation from our Previous Work

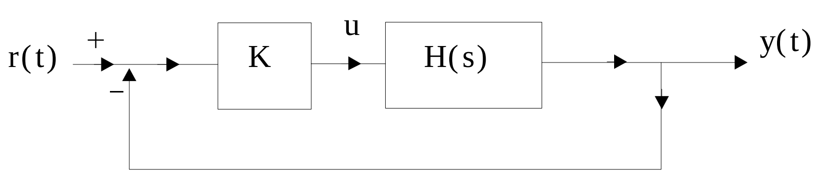


} Open loop system

Given an input u(t), y(t) will have a shape that is dependent on u(t) and H(s). The shape of y(t) may not be the desired shape so we change H(s) by using "output" feedback

$u(t) = K \cdot (r(t) - y(t))$ } Control Input (Classical Control)

$r(t)$ is the reference input



} Closed loop system

$\Delta(s) = 1 + K \cdot H(s)$ } Closed loop denominator

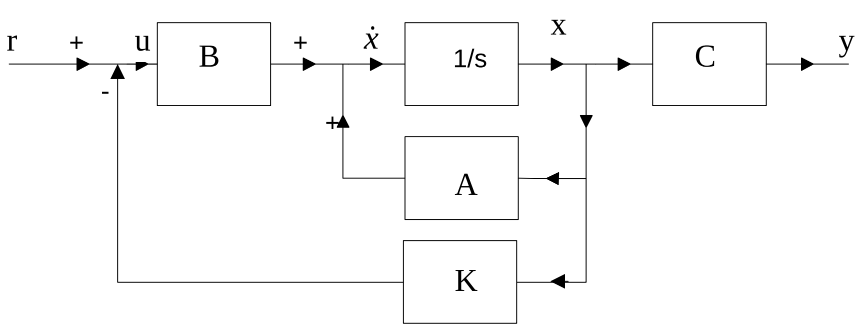
Poles can be changed by adjusting K (see Root Locus Design)

As we have seen using frequency domain techniques, we are limited in how we can change the closed-loop system by using output feedback. That is we cannot place poles arbitrarily. That is, the desired closed loop poles may not lie on the root locus, or we may not be able to obtain the exact response that is desired.

Given a state-space formulation for H(s), that is controllable, we can arbitrarily place the poles using "state" feedback; hence, we can achieve any desired shape for the output response

$$\frac{d}{dt}x = Ax + Bu$$
$$y = Cx$$
} State Space Equation

$$u = -Kx + r$$
} State Feedback Law - K is 1 x n



} State Space Block Diagram

Pole Placement

State Space Representation of $H(s) = C\cdot(sI - A)^{-1}\cdot B$

$$\frac{d}{dt}x = A\cdot x + B\cdot u \qquad y = Cx$$ } State Space Form

Let $u(t) = -Kx + r(t)$ } State Feedback Law

K is a 1 x n vector of feedback gains used to place the closed loop poles

$r(t)$ is a reference input

$$\frac{d}{dt}x = A\cdot x + B(-K\cdot x + r)$$ } substitute u(t)

$$\boxed{\begin{aligned}\frac{d}{dt}x &= (A - BK)\cdot x + Br \\ y &= C\cdot x\end{aligned}}$$ } Closed loop state space system

$$sI\cdot X(s) - x(0) = (A - BK)\cdot X(s) + B\cdot R(s)$$ } Take the Laplace Transform

Extra Step

$$X(s) = (sI - (A - BK))^{-1}\cdot x(0) + (sI - (A - B\cdot K))^{-1}\cdot B\cdot R(s)$$ } Solve for X(s)

$$\frac{Y(s)}{R(s)} = C\cdot(sI - (A - B\cdot K))^{-1}\cdot B$$ } Closed loop TF x(0) = 0

$$\frac{Y(s)}{R(s)} = \frac{C\cdot adj(sI - (A - B\cdot K))\cdot B}{|sI - (A - BK)|}$$ } Definition of Inverse

$$C\cdot adj(sI - (A - BK))\cdot B = 0$$ } Gives zeros of the closed loop TF

$$|sI - (A - BK)| = 0$$ } Gives poles of the closed-loop TF

If the system is controllable, the gain vector "K" can be calculated to place the poles by using the following formula

$$K = (0, 0, ..., 0, 1)\ \mathcal{C}^{-1}(A, B)\ \Delta_d(A)$$ } Ackerman's formula

$$\Delta_d(A) = \Delta_d(s)\Big|_{s = A}$$ } $\Delta_d(s)$ specifies the desired location of the closed loop poles

The open loop system will be controllable as long as the matrix

$\mathcal{C}(A, B) = \left(B, AB, ..., A^{n-1}\cdot B\right)$ has full rank i.e., $\det\left(\mathcal{C}(A, B)\right) \neq 0$

Example - State Feedback Control

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u \qquad y = (3 \quad 0.2) \cdot x \qquad H(s) = \frac{0.2 \cdot s + 3}{(s+1)\cdot(s+3)}$$

Open loop poles are at $s = -1, -3$

Let $u = -Kx + r$ } State Feedback Controller

Find K to place the closed-loop poles at s = -5

$$\Delta_d(s) = (s+5)^2 = s^2 + 10 \cdot s + 25$$ } $n = 2$ desired closed loop poles at s = -5

$$\Delta_d(A) = A^2 + 10 \cdot A + 25 \cdot I$$ } Find $\Delta_d(A)$

$$\Delta_d(A) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} + 10 \cdot \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} 22 & 6 \\ -18 & -2 \end{bmatrix}$$ } $\Delta_d(A)$

$$\mathcal{C}(A,B) = (B \quad AB) = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$ } $n = 2$ Calculate the Controllability Matrix

$$K = (0 \quad 1)\, \mathcal{C}^{-1}(A,B) \cdot \Delta_d(A) = (22 \quad 6)$$ } Ackerman's Formula

Check the answer

$$\frac{Y(s)}{R(s)} = C(sI - (A - BK))^{-1} B$$ } Closed-loop TF formula

$$A - BK = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot (22 \quad 6) = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix}$$ } Find $A - BK$

$$\frac{Y(s)}{R(s)} = C \cdot \begin{bmatrix} s & -1 \\ 25 & s+10 \end{bmatrix}^{-1} \cdot B$$ } Substitute for $A - BK$

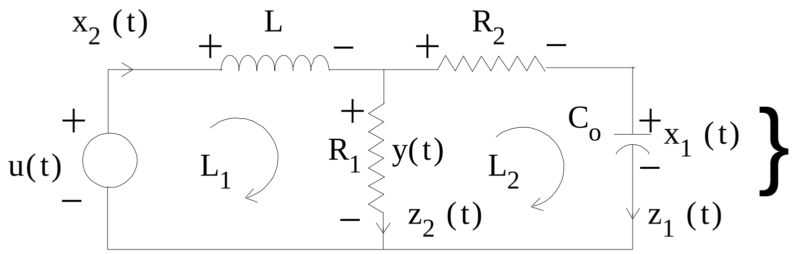
$$\frac{Y(s)}{R(s)} = C \cdot \begin{bmatrix} s+10 & 1 \\ -25 & s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\Delta(s)}$$ } Take the inverse of $A - BK$ and substitute for B

$$\Delta(s) = s^2 + 10\, s + 25 = (s+5)^2$$

$$\frac{Y(s)}{R(s)} = (3 \quad 0.2) \cdot \begin{bmatrix} 1 \\ s \end{bmatrix} \cdot \frac{1}{\Delta(s)}$$ } Simplify

$$\frac{Y(s)}{R(s)} = \frac{0.2\, s + 3}{(s+5)^2}$$ } answer is correct Why ?

A New State Space Realization for the Circuit



Find $\dot{x} = Ax + Bu$

$y = Cx$

for our circuit

Define the current throught the inductor as a state
Define the voltage across the capacitor as a state
z_1, z_2 are auxiliary variables that will be eliminated

$z_1 = C_o \cdot \frac{d}{dt} x_1$ } Capacitive Law

$x_2 = z_1 + z_2 \quad \Rightarrow \quad z_2 = x_2 - C_o \cdot \frac{d}{dt} x_1$ } Kirchoff's current law use z_1

$u = L \cdot \frac{d}{dt} x_2 + R_1 \cdot z_2 = L \cdot \frac{d}{dt} x_2 + R_1 \cdot \left(x_2 - C_o \cdot \frac{d}{dt} x_1 \right)$ } Write equation for voltage loop L_1

$x_1 = -R_2 \cdot z_1 + R_1 \cdot z_2 = -R_2 \left(C_o \cdot \frac{d}{dt} x_1 \right) + R_1 \cdot \left(x_2 - C_o \cdot \frac{d}{dt} x_1 \right)$ } Write equation for voltage loop L_2

$L \cdot \frac{d}{dt} x_2 + R_1 \cdot x_2 - R_1 \cdot C_o \cdot \frac{d}{dt} x_1 = u \qquad -C_o \cdot (R_1 + R_2) \cdot \frac{d}{dt} x_1 + R_1 \cdot x_2 - x_1 = 0$ } Simplify

$$\begin{bmatrix} -R_1 C & L \\ -C_o \cdot (R_1 + R_2) & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{d}{dt} x_1 \\ \frac{d}{dt} x_2 \end{bmatrix} = \begin{bmatrix} 0 & -R_1 \\ 1 & -R_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u$$

} Put in Matrix Form $\qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} \frac{d}{dt} x_1 \\ \frac{d}{dt} x_2 \end{bmatrix} = \begin{bmatrix} -R_1 C_o & L \\ -C_o \cdot (R_1 + R_2) & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -R_1 \\ 1 & -R_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -R_1 C_o & L \\ -C_o \cdot (R_1 + R_2) & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u$$

..... Matrix Algebra

$$\begin{bmatrix} -R_1 C & L \\ -C_o \cdot (R_1 + R_2) & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -L \\ C_o \cdot (R_1 + R_2) & -R_1 C_o \end{bmatrix} \cdot \frac{1}{\Delta}$$

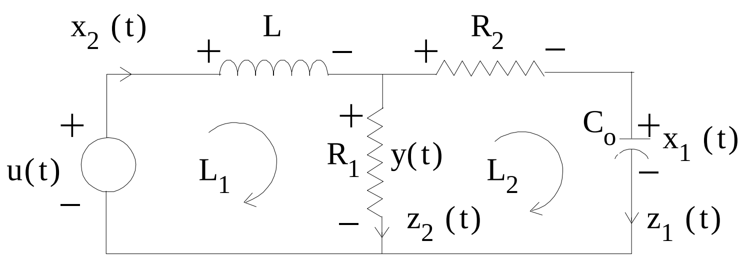
} Calculate Matrix Inverse

$\Delta = LC_o \cdot (R_1 + R_2)$

$$\frac{d}{dt}x = \begin{bmatrix} \frac{-1}{C_o\cdot(R_1+R_2)} & \frac{R_1}{C_o\cdot(R_1+R_2)} \\ \frac{-R_1}{L\cdot(R_1+R_2)} & \frac{-R_1\cdot R_2}{L\cdot(R_1+R_2)} \end{bmatrix}\cdot x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\cdot u$$

} Simplify to obtain $\dot{x} = Ax + Bu$ form

Find the output of the system



$$y = R_1\cdot z_2 = R_1\cdot\left(x_2 - C_o\cdot\frac{d}{dt}x_1\right)$$

} From the circuit

$$y = R_1\cdot x_2 - R_1\cdot C_o\cdot\left[\frac{-1}{C_o\cdot(R_1+R_2)}\cdot x_1 + \frac{R_1}{C_o\cdot(R_1+R_2)}\cdot x_2\right]$$

} Substitute for $\frac{d}{dt}x_1$ from above matrix equation

$$y = \left[\left(\frac{R_1}{R_1+R_2}\right) \quad \left(\frac{R_1\cdot R_2}{R_1+R_2}\right)\right]\cdot x$$

} $y = C\,x$ form

Let $R_1 = 3\Omega$, $L = 3.8\ H$, $R_2 = 1\Omega$, $C_o = .066\ F$ } same numbers as before

$$\frac{d}{dt}x = \begin{bmatrix} -3.79 & 11.36 \\ -0.2 & -0.2 \end{bmatrix}\cdot x + \begin{bmatrix} 0 \\ 0.26 \end{bmatrix}\cdot u$$

$$y = (0.75 \quad 0.75)\cdot x$$

} Final State Space form for the circuit

Calculate the Open Loop Transfer Function for the Circuit

$$\frac{d}{dt}x = \begin{bmatrix} -3.79 & 11.36 \\ -0.2 & -0.2 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0.26 \end{bmatrix} \cdot u$$

$$y = (0.75 \quad 0.75) \cdot x$$

} State Space Equation

$$H(s) = C \cdot (sI - A)^{-1} \cdot B$$

} Open loop TF formula

$$(sI - A)^{-1} = \begin{bmatrix} s + 3.79 & -11.36 \\ 0.2 & s + 0.2 \end{bmatrix}^{-1} = \begin{bmatrix} s + 0.2 & 11.36 \\ -0.2 & s + 3.79 \end{bmatrix} \cdot \frac{1}{\Delta}$$

} Matrix Algebra

$$\Delta = (s + 3.79) \cdot (s + 0.2) + 0.2 \cdot (11.36) = s^2 + 4s + 3$$

} open loop characteristic equation

$$H(s) = C \cdot \begin{bmatrix} s + 0.2 & 11.36 \\ -0.2 & s + 3.79 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0.26 \end{bmatrix} \cdot \frac{1}{\Delta} = (0.75 \quad 0.75) \cdot \begin{bmatrix} 2.95 \\ 0.26 \cdot s + 0.9854 \end{bmatrix} \cdot \frac{1}{\Delta}$$

} **Matrix Algebra**

$$H(s) = \frac{0.2 \cdot s + 3}{s^2 + 4 \cdot s + 3}$$

} same TF we obtained before

The same circuit gives two different state space forms but retains the same TF

<u>RCF SSF</u>

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x$$

State x does not have any physical meaning, it is just a mathematical creation

<u>Physics-based SSF</u>

$$\dot{x} = \begin{bmatrix} -3.79 & 11.36 \\ -0.2 & -0.2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.26 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.75 & 0.75 \end{bmatrix} x$$

State x has physical meaning (i.e., x_1 is the voltage across the capacitor and x_2 is the current through the inductor)

These physics-based states facilitates the construction of the feedback controller

Example - State Feedback Control Design

$$\dot{x} = \begin{bmatrix} -3.79 & 11.36 \\ -0.2 & -0.2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.26 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.75 & 0.75 \end{bmatrix} x$$

} Physics Based SSF for circuit

For the circuit, let $u = -Kx + r$ } State Feedback law

Find K so that the closed loop poles are at s = -5

$\Delta_d(s) = (s + 5)^2$ } $n = 2$ and desired closed loop poles at s = -5

$\Delta_d(A) = (A + 5\,I)^2 = A^2 + 10 \cdot A + 25 \cdot I$ } Find $\Delta_d(A)$

$\mathcal{C}(A, B) = (B \quad AB)$ } $n = 2$ controllability matrix

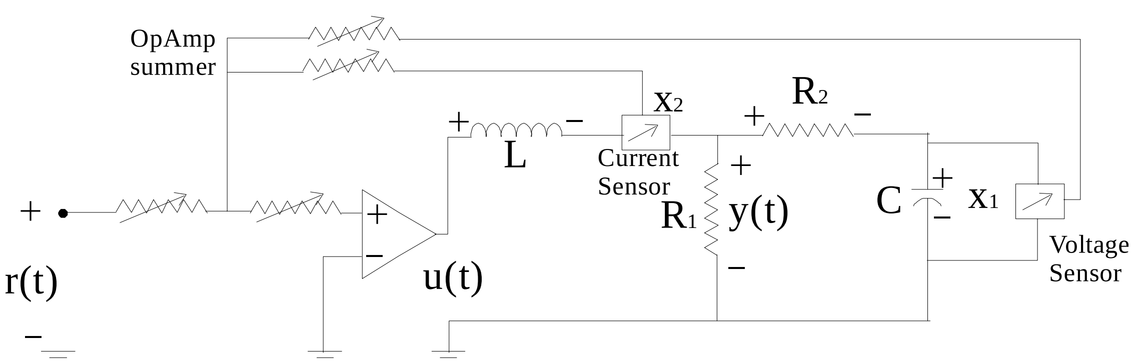
$$K = (0 \quad 1) \cdot \begin{bmatrix} 0 & 2.85 \\ 0.26 & -0.052 \end{bmatrix}^{-1} \cdot \Delta_d(A)$$

} Find K by Ackerman's Formula

$K = (-2.83 \quad 23.95)$ } $k_1 = -2.83$ $k_2 = 23.95$ } answer

$u = -Kx + r = -k_1 \cdot x_1 - k_2 \cdot x_2 + r$ } Controller

Control Ciruit Construction



Variable resistors can be adjusted to achieve the values required for k_1 and k_2

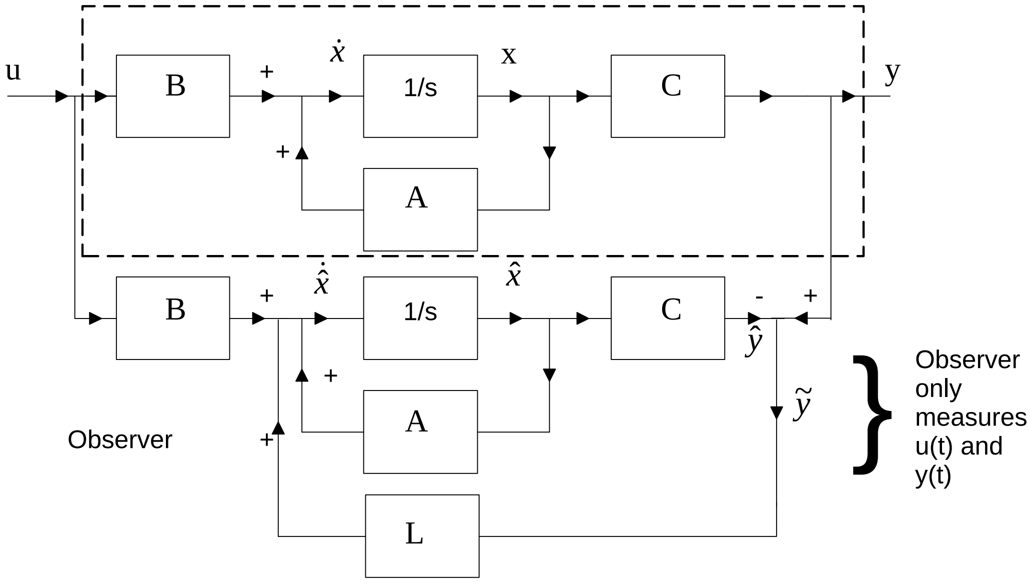
Time for a Movie

Click on image to start movie

Observers

Problem: ***What if the state x is not measurable ?***

Given A,B,C, y(t), u(t), and build an observer for estimating x



$$\dot{x} = Ax + Bu \qquad \Rightarrow \qquad \frac{Y(s)}{U(s)} = H(s)$$
$$y = Cx$$
} Plant

The observer is a device, we build to manufacture $\hat{x}$ (an estimate for x)

The observer gain L is used to make $\hat{x} \to x$ quickly; L is an n x 1 vector

$$\dot{\hat{x}} = A\hat{x} + L\tilde{y} + Bu$$
$$\tilde{y} = y - \hat{y} = Cx - C\hat{x} = C\tilde{x}$$
} From the Block Diagram $\tilde{x} = x - \hat{x}$ } observation error

$$\dot{\hat{x}} = A\hat{x} + LC\tilde{x} + Bu$$
} Substitute for $\tilde{y}$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + LC\tilde{x} + Bu)$$
} Find the error dynamics

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$
} Observation error dynamics - tells us how fast $\tilde{x}$ goes to zero

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$ } Error Dynamics

$$s\tilde{X}(s) - \tilde{x}(0) = (A - LC)\tilde{X}(s)$$ } Take Laplace Transform

$$(sI - (A - LC))\tilde{X}(s) = \tilde{x}(0)$$ } Rearrange

$$\tilde{X}(s) = (sI - (A - LC))^{-1}\tilde{x}(0)$$ } Solve for $\tilde{X}(s)$

$$\tilde{X}(s) = \frac{adj(sI - (A - LC))}{|sI - (A - LC)|}\tilde{x}(0)$$ } Definition of Inverse

Poles of the observation error system are given

$$\Delta_0(s) = |sI - (A - LC)| = 0$$

If the poles of $\Delta_0(s)$ are far in the left half plane, then $\tilde{x} \to 0$

quickly i.e., $\hat{x} \to x$ quickly

The observer gain "L" is utilized to place the observer poles in any location as long as the open loop is observable.

The open-loop system will be observable as long as the matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} CA^{n-1} \\ \cdots\cdots \\ \cdots\cdots \\ C \end{bmatrix}$$ is full-rank i.e., $\det(\mathcal{O}(A, C)) \neq 0$

If the system is observable, the gain matrix "L" is calculated as follows

$$L = \Delta_o(A)\, \mathcal{O}^{-1}(A, C) \begin{bmatrix} 1 \\ 0 \\ \cdot\cdot \\ 0 \end{bmatrix}$$ } Ackerman’s formula

$$\Delta_o(A) = \Delta_o(s)\big|_{s=A}$$ where $\Delta_o(s)$ specifies the desired location of the observer poles.

Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \left.\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx \end{matrix}\right\}$$

$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x$$

Find an observer so $\hat{x} \to x$ in 0.5 seconds

$$\left.\begin{matrix} \dot{\hat{x}} = A\hat{x} + L\tilde{y} + Bu \\ \hat{y} = C\hat{x} \end{matrix}\right\}$$ observer structure

$$\Delta_o(s) = (s + 10)^2 = s^2 + 20 \cdot s + 100$$ } construct $\Delta_0(s)$ for n = 2 and s = -10.

$$\mathcal{O}(A, C) = \begin{bmatrix} CA \\ C \end{bmatrix} = \begin{bmatrix} -0.8 & 2.4 \\ 3 & 0.2 \end{bmatrix}$$ } Calculate observability matrix with n = 2

$$L = \Delta_o(A) \cdot \mathcal{O}^{-1}(A, C) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.321 \\ 20.19 \end{bmatrix}$$ } Find L via Ackerman's formula

check answer

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$ } error dynamics tells us how fast $\tilde{x} = x - \hat{x} \to 0$

$$\left| sI - (A - LC) \right| = \left| \begin{bmatrix} s + 12.962 & -.136 \\ 64.571 & s + 7.038 \end{bmatrix} \right|$$ } Observer poles are determined by this formula

$$\Delta_o(s) = (s + 12.962) \cdot (s + 7.038) + 8.72$$ } Take Determinant

$$\Delta_o(s) = s^2 + 20 \cdot s + 100 = (s + 10)^2$$ } Simplify

Therefore if we build an observer then $\hat{x} \to x$ in 0.5 seconds

State Space Regulator

With only measurements of y(t), we can place the closed-loop poles anywhere as long as the open-loop plant H(s) is controllable and observable with the following regulator

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L\tilde{y} + Bu \\ \hat{y} &= C\hat{x} \\ u &= -K\hat{x} + r \end{aligned}\right\}$$

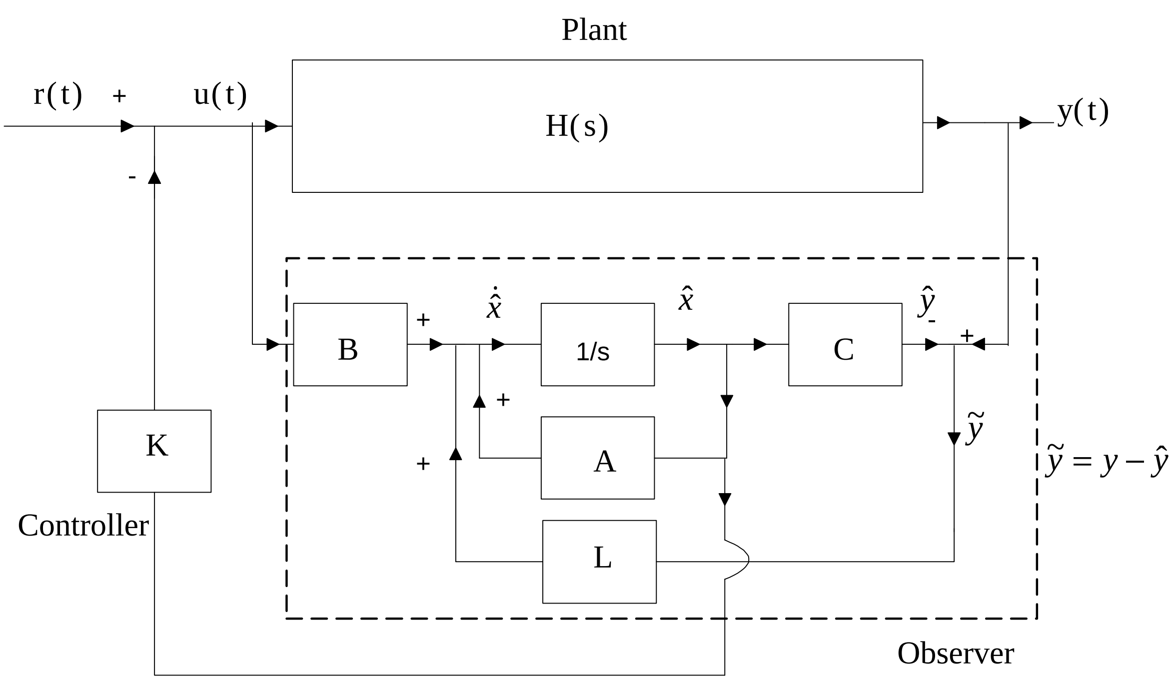
State-space regulator only measures r(t) and y(t)

$$\tilde{y} = y - \hat{y}$$

r(t) is the reference input

A, B, C are found by determining a state-space realization for the plant H(s).
The vectors K and L are found by using the given design specifications and Ackerman's formula.

State Space Regulator Block Diagram



Examining the Regulator Dynamics

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
} State-space realization for the open loop plant

$$\dot{x} = Ax + B(r - K\hat{x})$$
$$y = Cx$$
} substitute for u(t) to obtain closed loop plant $u = -K\hat{x} + r$

$$\dot{\hat{x}} = A\hat{x} + L\tilde{y} + Bu$$
$$\hat{y} = C\hat{x}$$
} standard observer structure

$$\dot{\hat{x}} = A\hat{x} + B(r - K\hat{x}) + LC(x - \hat{x})$$
$$\hat{y} = C\hat{x}$$
} substitute for u(t) and for $\tilde{y} = C\tilde{x}$

Find the Error Dynamics by using $\tilde{x} = x - \hat{x}$ $\Rightarrow \dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$

$$\dot{\tilde{x}} = Ax + B(r - K\hat{x}) - (A\hat{x} + B(r - K\hat{x}) + LC(x - \hat{x}))$$
} Subtract observer from closed-loop plant

$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$
} simplified error dynamics

$$\dot{x} = Ax + Br - BK(x - \tilde{x})$$
} Rewrite closed-loop plant in terms of $\tilde{x}$ $(\hat{x} = x - \tilde{x})$

$$\dot{x} = (A - BK)x + BK\tilde{x} + Br$$
} Simplified Plant

$$\dot{x} = (A - BK)x + BK\tilde{x} + Br$$
$$\dot{\tilde{x}} = (A - LC)\tilde{x}$$
$$y = Cx$$
} Composite Closed-loop System Dynamics } If $\tilde{x} = 0$

$$\dot{x} = (A - BK)x + Br$$
$$y = Cx$$

Define a new state variable system as follows

$$z = \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \qquad A_{cl} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \qquad B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad C_{cl} = (C \quad 0)$$

$$\dot{z} = A_{cl} z + B_{cl} r$$
$$y = C_{cl} z$$
} Compact Notation for Closed-loop System

$$\frac{Y(s)}{R(s)} = C_{cl} \cdot (sI - A_{cl})^{-1} \cdot B_{cl}$$
} Closed-loop TF $\Rightarrow$ How many poles does this system have ?

Calculate the transfer function

$$\frac{Y(s)}{R(s)} = C_{cl} \cdot \left(sI - A_{cl}\right)^{-1} \cdot B_{cl} \quad \Bigg\} \quad A_{cl} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \quad B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad C_{cl} = (C \quad 0)$$

$$\left(sI - A_{cl}\right)^{-1} = \begin{bmatrix} sI - (A - BK) & -BK \\ 0 & sI - (A - LC) \end{bmatrix}^{-1} \quad \Big\} \text{ Plug in } A_{cl}$$

$$(sI - A_{cl})^{-1} = \begin{bmatrix} (sI - (A - BK))^{-1} & BK(sI - (A - BK))^{-1}(sI - (A - LC))^{-1} \\ 0 & (sI - (A - LC))^{-1} \end{bmatrix} \quad \Big\} \text{ Matrix Identity}$$

Matrix Identity $\begin{bmatrix} M & N \\ 0 & P \end{bmatrix}^{-1} = \begin{bmatrix} M^{-1} & -M^{-1} NP^{-1} \\ 0 & P^{-1} \end{bmatrix}$

$$\frac{Y(s)}{R(s)} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} (sI - (A - BK))^{-1} & BK(sI - (A - BK))^{-1}(sI - (A - LC))^{-1} \\ 0 & (sI - (A - LC))^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \Big\} \text{ Plug in } B_{cl} \text{ and } C_{cl}\,.$$

$$\frac{Y(s)}{R(s)} = C(sI - (A - BK))^{-1} B \quad \Big\} \text{ Closed-loop TF looks exact like the full state feedback form}$$

$$\frac{Y(s)}{R(s)} = \frac{(C \cdot \mathrm{adj}(sI - (A - BK)))\, B}{\left| sI - (A - BK) \right|} \quad \Big\} \text{ Matrix Identity}$$

$(C \cdot \mathrm{adj}(sI - (A - BK)))\, B = 0$ gives zero of closed-loop TF.

$\left| sI - (A - BK) \right| = 0$ gives poles of closed-loop TF.

The closed-loop TF has "2n" poles; "n" poles come from the observer and "n" poles come from the closed-loop plant. How do you know this is true ?

L sets the location of the "n" observer poles.
K sets the location of the "n" closed-loop plant poles.

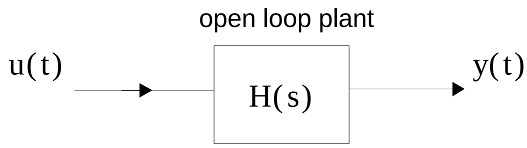
During the above calculation, the "n" observer poles given by
$\left| sI - (A - LC) \right| = 0$ canceled with the "n" observer zeros.

This cancellation of the observer poles is always true and leads to what is called the separation principle.

<u>Separation Principle</u> - the gains K and L can be designed separately to achieve the desired closed-loop plant and observer pole locations, respectively.

Example

$$H(s) = \frac{.2\,s + 3}{s^2 + 3 \cdot s + 4}$$



Design a regulator so the desired closed-loop poles are at s = -1, -2, and the observer poles are at s = -10, -12.

Find a SSR using RCF

$$\left.\begin{aligned} \frac{d}{dt}x &= \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= (3 \quad .2)\ x \end{aligned}\right\} \qquad A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad n = 2 \quad C = (3 \quad 0.2)$$

<u>Formulate Observer</u>

$$\left.\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L\tilde{y} \\ \hat{y} &= C\hat{x} \end{aligned}\right\} \text{ observer}$$

$$L = \Delta_o(A)\ \mathcal{O}^{-1}(A, C) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \} \text{ Ackerman's Formula}$$

$$\Delta_o(s) = (s + 10)\ (s + 12) = s^2 + 22\ s + 120 \quad \} \text{ desired observer pole locations.}$$

$$\mathcal{O}(A, C) = \begin{bmatrix} CA \\ C \end{bmatrix} = \begin{bmatrix} -0.8 & 2.4 \\ 3 & 0.2 \end{bmatrix} \quad \} \text{ observability matrix}$$

$$L = (A + 10\ I)\ (A + 12\ I) \begin{bmatrix} -.8 & 2.4 \\ 3 & 0.2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 26.1 \end{bmatrix} \quad \} \text{ answer}$$

Check $\quad |sI - (A - LC)| = \left| \begin{bmatrix} s + 13.78 & -0.8 \\ 82.34 & s + 8.22 \end{bmatrix} \right| \quad \}$ Use formula

$$\Delta_o(s) = (s + 13.78) \cdot (s + 8.22) + 82 \cdot (0.08) = s^2 + 22 \cdot s + 120 \quad \} \text{ checks}$$

Note: observer poles are placed s-10 times further out in the LHP

Formulate Controller

$u = -K\hat{x} + r$ } controller

$K = (0 \quad 1)\cdot \mathcal{C}^{-1}(A,B)\ \Delta_{cl}(A)$ } Ackerman's fromula

$\Delta_{cl}(s) = (s+1)\ (s+2) = s^2 + 3\cdot s + 2$ } desired closed-loop poles

$\mathcal{C}(A,B) = (B \quad AB) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$ } Controllability Matrix

$K = (0 \quad 1)\cdot \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}^{-1} (A + I)\cdot(A + 2\cdot I) = (-2 \quad 0)$ } answer

Check $\left| sI - (A - BK) \right| = \left| \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \right|$ } use formula

$\Delta_{cl}(s) = s\cdot(s+3) + 2 = s^2 + 3\ s + 2$ } checks

Review

$$\dot{\hat{x}} = A\hat{x} + L\tilde{y} + Bu$$
$$\hat{y} = C\hat{x}$$
$$u = -K\hat{x} + r$$

} State-space regulator
How do you find A,B,C,K and L for a given problem ?

State Transformation

$$\frac{d}{dt}x = Ax + Bu$$
$$y = Cx$$

} Standard Form for the State Equation

Let T be an n x n constant nonsingular matrix i.e. $|T| \neq 0$

$x_n = T^{-1}\ x$ } Define a new state x_n

$x = T \cdot x_n \qquad \frac{d}{dt}x = T \cdot \frac{d}{dt}x_n$ } From the above definition

$$T \cdot \frac{d}{dt}x_n = A \cdot T \cdot x_n + Bu$$
$$y = C \cdot T \cdot x_n$$

} State equation in terms of the new state

$$\frac{d}{dt}x_n = T^{-1} \cdot A \cdot T \cdot x_n + T^{-1} \cdot B \cdot u$$
$$y = C \cdot T \cdot x_n$$

} Premultiply by T^{-1}

$$\frac{d}{dt}x_n = A_n \cdot x_n + B_n \cdot u$$
$$y = C_n \cdot x_n$$

} Transformed State Equation

$$A_n = T^{-1} \cdot A \cdot T \qquad B_n = T^{-1} \cdot B \qquad C_n = C \cdot T$$

Definition

Two state space realizations are equivalent if

$$A_n = T^{-1} \cdot A \cdot T$$
$$B_n = T^{-1} \cdot B$$
$$C_n = C \cdot T$$

} For some constant nonsingular matrix T

Theorem

Equivalent systems have the same transfer function

Proof:

$$H(s) = C\cdot(sI - A)^{-1}\cdot B$$ } By Definition

$$H_n(s) = C_n\cdot(sI - A_n)^{-1}\cdot B_n$$ } By Definition

$$H_n(s) = C\cdot T\cdot(sI - T^{-1}\cdot A\cdot T)^{-1}\cdot T^{-1}\cdot B$$ } Substitute for A_n, B_n, C_n

$$H_n(s) = CT\cdot(sT^{-1}\ IT - T^{-1}\ AT)^{-1}\cdot T^{-1}\ B$$ } Property of the Identity Matrix

$$H_n(s) = CT\cdot\left[T^{-1}\cdot(sI - A)\ T\right]^{-1}\cdot T^{-1}\ B$$ } Rearrange

$$H_n(s) = C\cdot T\cdot T^{-1}\cdot(sI - A)^{-1}\cdot T\cdot T^{-1}\cdot B$$ } Use matrix identity $(MNP)^{-1} = P^{-1}\cdot N^{-1}\cdot M^{-1}$

$$H_n(s) = C\cdot(sI - A)^{-1}\cdot B = H(s)$$ } Proof is complete

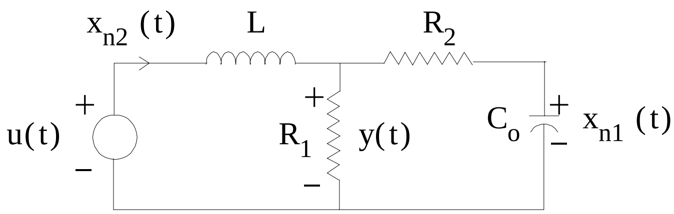
Theorem

Given two state space representations for the same system, the transformation T for $x_n = T^{-1}\ x$ can be calculated by either method

$$T = \mathcal{C}(A, B)\,\mathcal{C}^{-1}(A_n, B_n)$$ } Use Controllability Matrices

$$T = \mathcal{O}^{-1}(A, C)\,\mathcal{O}(A_n, C_n)$$ } Use Observability Matrices

Below are two state space realizations for the circuit



The same circuit gives two different state space forms but retains the same TF

RCF SSF

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 0.2 \end{bmatrix} x$$

Physics-based SSF

$$\dot{x}_n = \begin{bmatrix} -3.79 & 11.36 \\ -0.2 & 0.2 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 0.26 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0.75 & 0.75 \end{bmatrix} x_n$$

Find the matrix T which will satisfy $x_n = T^{-1}\, x$

$$\mathcal{C}(A,B) = (B \quad AB) = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$ } Controllability matrix for RCF

$$\mathcal{C}(A_n, B_n) = [B_n \quad A_n \cdot B_n] = \begin{bmatrix} 0 & 2.954 \\ 0.26 & 0.052 \end{bmatrix}$$ } Controllability matrix for Physics-based SSF

$$T = \mathcal{C}(A,B) \cdot \mathcal{C}^{-1}(A_n, B_n) = \begin{bmatrix} 0.339 & 0 \\ -1.286 & 3.846 \end{bmatrix}$$ } answer

Hence, the state that did not have physical meaning in the RCF form are actually linear combinations of the states in the Physics-based SSF

$$x = T \cdot x_n = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.339 & 0 \\ -1.286 & 3.846 \end{bmatrix} \cdot \begin{bmatrix} x_{n1} \\ x_{n2} \end{bmatrix}$$ } By Definition

$$x_1 = 0.339 \cdot x_{n1}$$
$$x_2 = -1.286 \cdot x_{n1} + 3.846 \cdot x_{n2}$$ } State Relationships

Transformation to Jordan Normal Form

The Jordan Normal Form is a very special SSR because the eigenvalues and their controllability and observability characteristics can be read off of the SSR by inspection.

$$\left.\begin{aligned}\frac{d}{dt}x &= A\cdot x + B\cdot u \\ y &= c\cdot x\end{aligned}\right\}\quad \text{State Space equation}$$

$$\left.\begin{aligned}\frac{d}{dt}x_n &= A_J\cdot x_n + B_J\cdot u \\ y_n &= C_J\cdot x_n\end{aligned}\right\}\quad A_J = T^{-1}\cdot A\cdot T \quad B_J = T^{-1}\cdot B \quad C_J = C\cdot T \quad x_n = T^{-1}\, x$$

Jordan Normal Form

" T " is the matrix which consists of " n " eigenvectors of the matrix "A". Put the generalized (i.e., for repeated roots) eigenvectors in " T " in the reverse order while keeping them
next to each other

Theorem

If the eigenvalues of "A" are distinct then "A" is called **simple**.
However, repeated eigenvalues does not imply that A is not simple

Theorem

"A" is simple if and only if A_J is a main diagonal matrix

(i.e., there must be only zeros appearing off the main diagonal)

Definition

The eigenvalues of A_J are always along the main diagonal

Theorem (for SISO system only)

If A is simple, then

a) eigenvalue in A_J is controllable if the corresponding element in B_J is non zero.

b) eigenvalue in A_J is observable if the corresponding element in C_J is non zero.

If A is not simple,

a) eigenvalue in A_J is controllable if the last element of the corresponding Jordan block in B_J is non zero.

b) eigenvalue in A_J is observable if the first element of the corresponding Jordan block in C_J is non zero.

Rules for drawing Jordan Blocks

1) Draw vertical and horizontal lines to isolate the eigenvalues which lie on the main diagonal of A_J
2) Never isolate a "one" on the superdiagonal - any time you have repeated eigenvalues you have the possibility of a "one" showing up on the super diagonal
3) Draw as many lines as possible
4) Extend lines through B_J and C_J

Example - Find the JNF and discuss Controllability and Observability

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

} State Space equation

$$\lambda_1 = -3 \ , \quad \lambda_2 = -2$$

} eigenvalues or poles of "A"

$$v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \ , \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

} eigenvectors of "A"

$$T = \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$$

} Formulate transformation matrix

$$A_J = T^{-1}AT = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$B_J = T^{-1}B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$C_J = CT = \begin{bmatrix} -2 & -1 \end{bmatrix}$$

} Apply JNF formulas

$$\dot{x}_n = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} x_n + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & -1 \end{bmatrix} x_n$$

} We can now mark the Jordan Blocks

Rules for drawing Jordan Blocks

1) Draw vertical and horizontal lines to isolate the eigenvalues which lie on the main diagonal of A_J

2) Never isolate a "one" on the superdiagonal - any time you have repeated eigenvalues you have the possibility of a "one" showing up on the super diagonal

3) Draw as many lines as possible

4) Extend lines through B_J and C_J

Jordan Block

$$\dot{x}_n = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} x_n + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & -1 \end{bmatrix} x_n$$

Jordan Block

} Jordan Normal Form eigenvalues are on the main diagonal

Analysis

Since "A" is a simple matrix (i.e., "A" is a main diagonal matrix)

a) eigenvalue in A_J is controllable if the corresponding element in B_J is non zero.

b) eigenvalue in A_J is observable if the corresponding element in C_J is non zero.

Eigenvalue	Controllable	Observable
-3	Yes	Yes
-2	Yes	Yes

Example - Find the JNF and discuss Controllability and Observability

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

} State Space equation

$$\lambda_1 = 1, m = 2 \ , \quad \lambda_2 = 2, m = 1$$

} eigenvalues or poles of "A"

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \quad v_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

} eigenvectors of "A"

$$T = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

} Formulate transformation matrix

$$A_J = T^{-1}AT = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
$$B_J = T^{-1}B = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$
$$C_J = CT = \begin{bmatrix} 1 & 1 & 8 \end{bmatrix}$$

} Apply JNF formulas

$$\dot{x}_n = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_n + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 8 \end{bmatrix} x_n$$

} We can now mark the Jordan Blocks

Rules for drawing Jordan Blocks

1) Draw vertical and horizontal lines to isolate the eigenvalues which lie on the main diagonal of A_J

2) Never isolate a "one" on the superdiagonal - any time you have repeated eigenvalues you have the possibility of a "one" showing up on the super diagonal

3) Draw as many lines as possible

4) Extend lines through B_J and C_J

Jordan Block

$$\dot{x}_n = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_n + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 8 \end{bmatrix} x_n$$

Jordan Block

} Jordan Normal Form with eigenvalues are on the main diagonal

<u>Analysis</u>

If "A" is not simple

a) eigenvalue in A_J is controllable if the last element of the corresponding block in B_J is non zero.

b) eigenvalue in A_J is observable if the first element of the corresponding block in C_J is non zero.

Eigenvalue	Controllable	Observable
1	No	Yes
2	Yes	Yes

Example - Find the JNF and discuss Controllability and Observability

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State Space equation

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$\lambda_1 = 1, m = 3$$

eigenvalues or poles of "A"

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ , \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \ , \quad v_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

eigenvectors of "A"

$$T = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 0.33 & -0.66 & 0 \\ 0 & 0.33 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Formulate transformation matrix

$$A_J = T^{-1}AT = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_J = T^{-1}B = \begin{bmatrix} 0.33 \\ 0 \\ 0 \end{bmatrix}$$

$$C_J = CT = \begin{bmatrix} 3 & 5 & 0 \end{bmatrix}$$

Apply JNF formulas

$$\dot{x}_n = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_n + \begin{bmatrix} 0.33 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 5 & 0 \end{bmatrix} x_n$$

We can now
mark the
Jordan Blocks

Rules for drawing Jordan Blocks

1) Draw vertical and horizontal lines to isolate the eigenvalues which lie on the main diagonal of A_J

2) Never isolate a "one" on the superdiagonal - any time you have repeated eigenvalues you have the possibility of a "one" showing up on the super diagonal

3) Draw as many lines as possible

4) Extend lines through B_J and C_J

Jordan Block

$$\dot{x}_n = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_n + \begin{bmatrix} 0.33 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 5 & 0 \end{bmatrix} x_n$$

} Jordan Normal Form with eigenvalues on the main diagonal

Jordan Block

<u>Analysis</u>

If "A" is not simple

a) eigenvalue in A_J is controllable if the last element of the corresponding block in B_J is non zero.

b) eigenvalue in A_J is observable if the first element of the corresponding block in C_J is non zero.

Eigenvalue	Controllable	Observable
1	No	Yes

Example - Discuss controllability and observability for the system below

$$\dot{x}_n = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} u \quad \Bigg\} \text{ Jordan Normal Form}$$

$$y = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \end{bmatrix} x_n$$

Rules for drawing Jordan Blocks

1) Draw vertical and horizontal lines to isolate the eigenvalues which lie on the main diagonal of A_J
2) Never isolate a "one" on the superdiagonal - any time you have repeated eigenvalues you have the possibility of a "one" showing up on the super diagonal
3) Draw as many lines as possible
4) Extend lines through B_J and C_J

$$\dot{x}_n = \left[\begin{array}{cc|cc|ccc|c} -1 & 1\textcircled{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1\textcircled{B} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right] x_n + \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 1 \\ \hline 0 \\ 0 \\ 2 \\ \hline 1 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|cc|ccc|c} 1 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \end{array}\right] x_n$$

Jordan Blocks

$$\dot{x}_n = \left[\begin{array}{cc|cc|ccc|c} -1 & 1 \; (A) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \; (B) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right] x_n + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = \left[\begin{array}{cc|cc|ccc|c} 1 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \end{array}\right] x_n$$

Jordan Blocks

<u>Analysis</u>

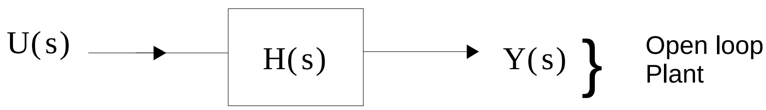
a) eigenvalue in A_J is controllable if the last element of the corresponding block in B_J is non zero.

b) eigenvalue in A_J is observable if the first element of the corresponding block in C_J is non zero.

Eigenvalue	Controllable	Observable
-1 (A)	No	Yes
-1 (B)	Yes	Yes
-2	Yes	Yes
-3	Yes	No

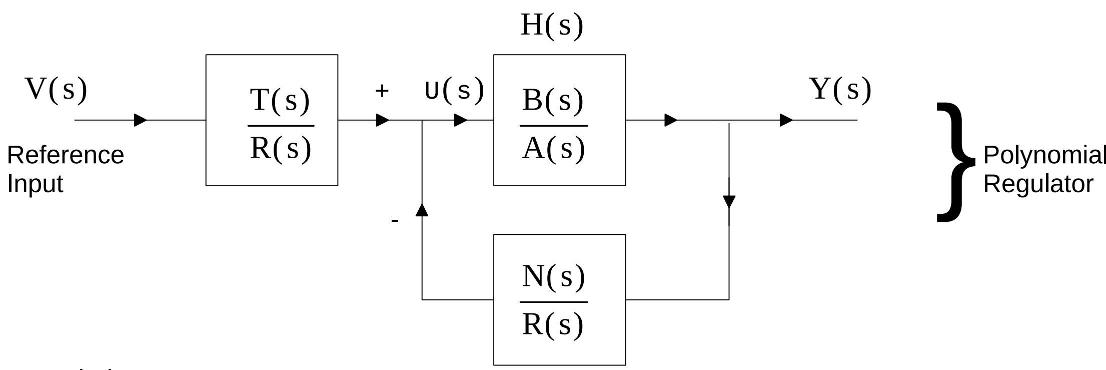
Polynomial Regulation

Given an open loop system



where $H(s) = \frac{B(s)}{A(s)}$ } open-loop plant numerator polynomial
} open-loop plant denominator polynomial

Find the polynomials T(s), N(s), R(s) in the control structure



such that

$$\frac{Y(s)}{V(s)} = \frac{T(s)}{R(s)} \cdot \frac{\frac{B(s)}{A(s)}}{\left(1 + \frac{N(s)}{R(s)} \cdot \frac{B(s)}{A(s)}\right)} = H_d(s)$$

} Desired closed-loop transfer function

$H_d(s) = \frac{Q(s)}{P(s)}$ } desired closed-loop numerator polynomial
} desired closed-loop denominator polynomial

Restrictions

1) $\text{degree}(P(s)) = \text{degree}(A(s))$
2) Q(s) must contain the roots found in $B^{-}(s)$

Steps for Calculating T(s), R(s) and N(s)

Step 1 Determine A(s), B(s), Q(s) and P(s) from the problem statement and check restriction 1

$$H(s) = \frac{B(s)}{A(s)} \quad \Big\} \quad \text{open-loop TF}$$

$$H_d(s) = \frac{Q(s)}{P(s)} \quad \Big\} \quad \text{desired closed-loop TF}$$

Restriction 1: $\text{degree}(P(s)) = \text{degree}(A(s))$

Step 2 Find $B^+(s)$ and $B^-(s)$ and check restriction 2

$$B(s) = B^+(s)B^-(s) \quad \Big\} \quad \text{By Definition}$$

$B^-(s)$ denotes the polynomial which contains the roots of B(s) that have positive real parts

$B^+(s)$ denotes the polynomial which contains the roots of B(s) that have nonpositive real parts

$$\left.\begin{aligned} B(s) &= s(s+1)(s-1) \\ B^+(s) &= s(s+1) \\ B^-(s) &= (s-1) \end{aligned}\right\} \quad \text{Example}$$

Restriction 2: Q(s) must contain the roots found in $B^-(s)$

Step 3 Define R(s) and T(s) in terms of subpolynomials

$$T(s) = \frac{T_1(s)Q(s)}{B^-(s)} \qquad R(s) = R_1(s)B^+(s) \quad \Big\} \quad \text{By Definition}$$

$R_1(s), T_1(s)$ are unknown at this point.

Step 4 Determine the degree of each polynomial

$$\deg\{T_1\} = \deg\{A\} - \deg\{B^+\} - 1 = L$$

$$\deg\{T_1\} = L \quad \Big\} \quad \text{By definition}$$

$$\deg\{R_1\} = \deg\{T_1\} = L$$

$$\deg\{N\} = \deg\{A\} - 1 = C$$

$$\deg\{N\} = C \quad \Big\} \quad \text{By definition}$$

Step 5 Define the structure of $T_1(s), R_1(s), N(s)$

$$T_1(s) = (s + a)^L$$ } "a" is far in the left-half plane and "L" is known

$$R_1(s) = s^L + r_{L-1} \cdot s^{L-1} + \ldots\ldots + r_0$$ } "L" is known

$$N(s) = n_c s^c + n_{c-1} \cdot s^{c-1} + \ldots\ldots + n_0$$ } "C" is known

At this point the coefficients denoted by r_i, n_i are unknown

Step 6 Solve the Diophantine Equation to determine r_i, n_i coefficients in $R_1(s)$ and $N(s)$

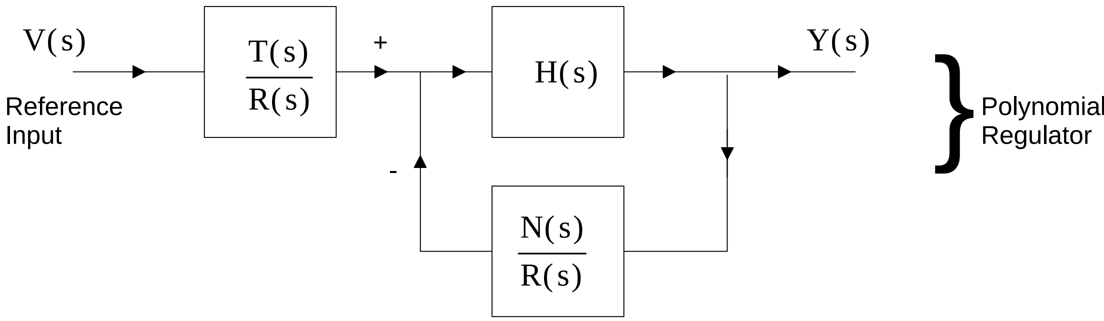
$$A(s)R_1(s) + B^-(s)N(s) = P(s)T_1(s)$$ } Diophantine Equation

After matching the coefficients in the above equation, we obtain the polynomials $R_1(s), N(s)$

Step 7 Construct R(s) and T(s)

$$T(s) = \frac{T_1(s)Q(s)}{B^-(s)} \qquad R(s) = R_1(s)B^+(s)$$

Example



Given $H(s) = \dfrac{s-1}{s^2}$ Find T(s), R(s) and N(s) such that $\dfrac{Y(s)}{V(s)} = \dfrac{s-1}{(s+1)^2}$

Step 1 Determine A(s), B(s), Q(s) and P(s) from the problem statement and check restriction 1

$$H(s) = \frac{B(s)}{A(s)} = \frac{s-1}{s^2} \quad \Big\} \text{ open-loop TF}$$

$$H_d(s) = \frac{Q(s)}{P(s)} = \frac{s-1}{(s+1)^2} \quad \Big\} \text{ desired closed-loop TF}$$

Restrictions

1) $\text{degree}(P(s)) = \text{degree}(A(s)) = 2$ } OK

Step 2 Find $B^+(s)$ and $B^-(s)$ and check restriction 2

$$H(s) = \frac{B(s)}{A(s)} = \frac{s-1}{s^2} \quad \Big\} \text{ Plant}$$

$B(s) = (s-1)$ } $B(s) = B^+(s)B^-(s)$

$B^-(s) = (s-1)$ } roots of B(s) with positive real parts

$B^+(s) = 1$ } roots of B(s) with nonpositive real parts

Restrictions

2) Q(s) must contain the roots found in $B^-(s)$

$Q(s) = (s-1)$ $B^-(s) = (s-1)$ } OK

Step 3 Define R(s) and T(s) in terms of subpolynomials

$$T(s) = \frac{T_1(s)Q(s)}{B^-(s)} = \frac{T_1(s)(s-1)}{(s-1)} \qquad R(s) = R_1(s)B^+(s) = R_1(s)(1)$$

$R_1(s), T_1(s)$ are unknown at this point.

Step 4 Determine the degree of each polynomial

$$\deg\{T_1\} = \deg\{A\} - \deg\{B^+\} - 1 = 2 - 0 - 1 = 1 \quad \Rightarrow \quad L = 1$$
$$\deg\{T_1\} = 1$$
$$\deg\{R_1\} = \deg\{T_1\} = 1$$
$$\deg\{N\} = \deg\{A\} - 1 = 2 - 1 = 1 \quad \Rightarrow \quad C = 1$$
$$\deg\{N\} = 1$$

Step 5 Define the structure of $T_1(s), R_1(s), N(s)$

Poles of T_1 should be 10 times bigger than the largest pole of P(s) } Rule of Thumb

Let a = 10

$$T_1(s) = (s + 10)^L = (s + 10) \quad \} \quad L = 1$$
$$R_1(s) = s + r_0 \quad \} \quad \deg\{R_1\} = 1$$
$$N(s) = n_1 s + n_0 \quad \} \quad \deg\{N\} = 1$$

Step 6 Solve the Diophantine Equation to determine r_i, n_i coefficients in $R_1(s)$ and $N(s)$

$$A(s)R_1(s) + B^-(s)N(s) = P(s)T_1(s) \quad \} \quad \text{Diophantine Equation}$$

$$s^2 (s + r_0) + (s - 1)(n_1 s + n_0) = (s^2 + 2s + 1)(s + 10) \quad \} \quad \text{Insert Polynomial}$$

$$s^3 + s^2 r_0 + n_1 s^2 + (-n_1 + n_0) s - n_0 = s^3 + 12 s^2 + 21 s + 10 \quad \} \quad \text{Simplify}$$

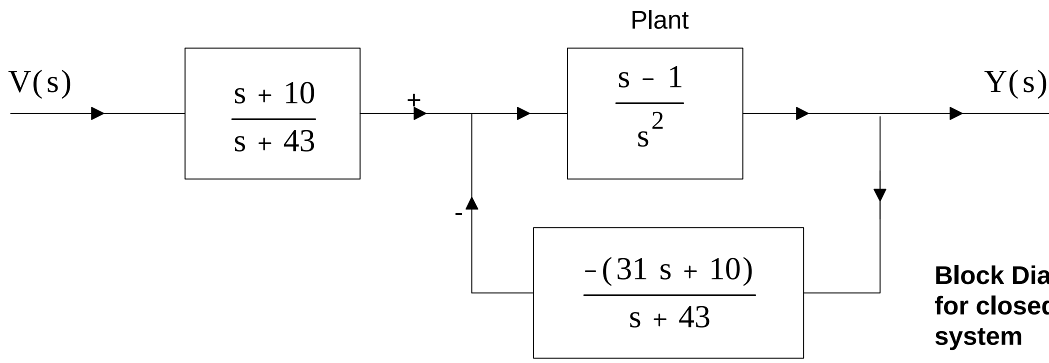
$$s^2 (r_0 + n_1) = s^2 (12)$$
$$s(-n_1 + n_0) = s(21) \qquad -n_0 = 10 \quad \} \quad \text{Simplify}$$

$$n_0 = -10 \qquad n_1 = -31 \qquad r_0 = 43 \quad \} \quad \text{Solve for coefficients}$$

$$N(s) = -(31 s + 10) \qquad R_1(s) = s + 43 \quad \} \quad \text{Gives } R_1(s) \text{ and } N(s)$$

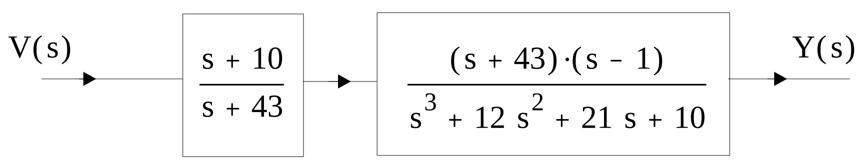
Step 7 Construct R(s) and T(s)

$$T(s) = \frac{T_1(s)Q(s)}{B^-(s)} = (s+10)\frac{(s-1)}{(s-1)} = s + 10$$
$$R(s) = R_1(s)B^+(s) = R_1(s)(1) = s + 43$$

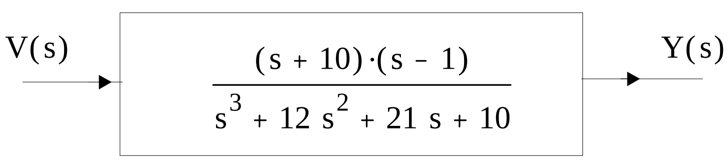


Block Diagram for closed-loop system

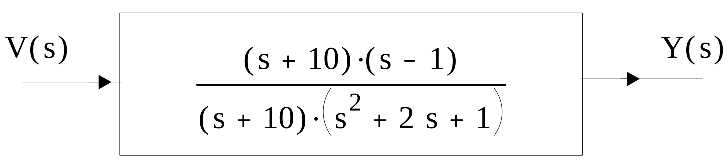
<u>Check your answer</u>



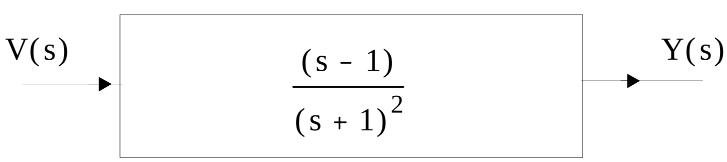
} Feedforward Feedback Combination



} Combine cascade blocks



} Factor denominator



} Cancel observer pole

checks since $\frac{Y(s)}{V(s)} = \frac{s - 1}{(s + 1)^2}$

Nonlinear Systems

$$\frac{d}{dt}x = f(x,u) \qquad f(x,u) \text{ is an n x 1 vector}$$ } General form for most nonlinear systems

$$\frac{d}{dt}x = \left(\frac{\partial f}{\partial x}\right)^T \Bigg|_{x=x_e} x + \left(\frac{\partial f}{\partial u}\right)^T \Bigg|_{x=x_e} u$$ } Linearized about an equlibrium point x_e .

$$\frac{d}{dt}x = Ax + Bu$$ } $A = \left(\frac{\partial f}{\partial x}\right)^T \Bigg|_{x=x_e}$ $B = \left(\frac{\partial f}{\partial u}\right)^T \Bigg|_{x=x_e}$ $\frac{d}{dt}x = Ax + Bu$

where

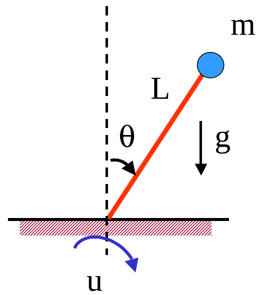
$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_1} ----------- \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_2} ----------- \frac{\partial f_n}{\partial x_2} \\ --------------- \\ --------------- \\ \frac{\partial f_1}{\partial x_n} \frac{\partial f_2}{\partial x_n} ----------- \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$ } Calculation for $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial u} = \left[\frac{\partial f_1}{\partial u} \frac{\partial f_2}{\partial u} ---------- \frac{\partial f_n}{\partial u}\right]$$ } Calculation for $\frac{\partial f}{\partial u}$

The equilibrium point x_e is usually arrived at by examining the "physics" of the problem or by setting f(x,u) = 0 and solving for x.

Note: x_e is an n x 1 vector; so youneed "n" values for the equlibrium point

Example



} One link robot

$$u = mL\ddot{\theta} - mgL\sin\theta$$ } Dynamic Equation

u - input control torque
m - mass of robot
L - link of robot
θ - link angle
g - gravitational coefficient

Linearize the system about the equilibrium point $\theta_e = 0$ $\dot{\theta}_e = 0$ and then obtain a state-space realization.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

} $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\quad y = \theta$

(i.e., find A, B, C)

Step 1 - Write the dynamics in the $\dot{x} = f(x,u)$ form.

$$u = mL\ddot{\theta} - mgL\sin\theta$$ } original dynamics

$$\ddot{\theta} = g\sin(\theta) + \frac{1}{mL}u$$ } rearrange

$x_1 = \theta \quad x_2 = \dot{\theta} \quad \Rightarrow \dot{x}_2 = \ddot{\theta}, \; \dot{x}_1 = x_2$ } From problem statement

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = g\sin(x_1) + \frac{1}{mL}u$$

} use above information to rewrite dynamics

$$\dot{x} = f(x,u) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g\sin(x_1) + \frac{1}{mL}u \end{bmatrix}$$

} $\dot{x} = f(x,u)$ form

n = 2

Step 2) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} \end{bmatrix}$$

} use formula with n = 2

$$\dot{x} = f(x,u) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ g(\sin(x_1)) + \frac{1}{mL} u \end{bmatrix}$$

} from above work

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & g \cdot \cos(x_1) \\ 1 & 0 \end{bmatrix}$$

} substitute for f_1 and f_2 into the formula

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 & \frac{1}{ml} \end{bmatrix}$$

} substitute for f_1 and f_2 into the formula

Step 3) Substitute the equilibrium point to find A and B

$$x_{1e} = \theta_e = 0$$

$$x_{2e} = \dot{\theta}_e = 0$$

$$x_e = \begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix} = \begin{bmatrix} \theta_e \\ \dot{\theta}_e \end{bmatrix}$$

} Equilibrium Point from problem statement

$$A = \left(\frac{\partial f}{\partial x} \right)^T \Bigg|_{x=x_e} = \begin{bmatrix} 0 & 1 \\ +g & 0 \end{bmatrix}$$

$$B = \left(\frac{\partial f}{\partial u} \right)^T \Bigg|_{x=x_e} = \begin{bmatrix} 0 \\ \frac{1}{mL} \end{bmatrix}$$

} Plug-in equilibrium point and transpose

Step 4) Find "C" from problem statement

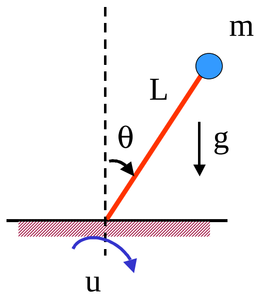
$$y = \theta \Rightarrow y = x_1$$

} Since $x_1 = \theta$ by definition

$$y = (1 \quad 0) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C = (1 \quad 0)$$

Linearized System for One Link Robot



$$\left.\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/mL \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}\right\}$$

Find the open-loop TF and the open loop impulse response. Then assuming $\theta, \dot{\theta}$ are measurable, design a state feedback controller which places the poles of the closed loop system at -1.

$$\frac{Y(s)}{U(s)} = H(s) = C(sI - A)^{-1} \cdot B$$ } Open Loop TF formula }

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -g & s \end{bmatrix}^{-1} = \begin{bmatrix} s & 1 \\ g & s \end{bmatrix} \cdot \frac{1}{\Delta(s)}$$ Matrix Algebra

$$\Delta(s) = s^2 - g$$

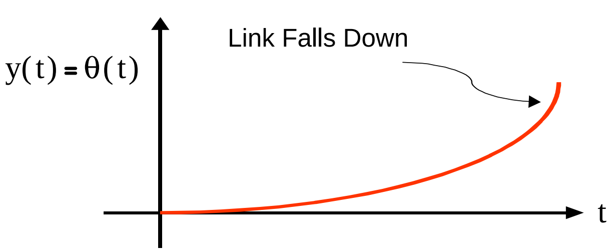
$$H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ g & s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1/mL \end{bmatrix} \frac{1}{\Delta(s)} = \frac{1/mL}{s^2 - g}$$ } Open-loop TF

$Y(s) = H(s)$ if $U(s) = 1$ } Impulse response

$$Y(s) = \frac{\frac{1}{mL}}{\left(s + \sqrt{g}\right) \cdot \left(s - \sqrt{g}\right)}$$ } Factor denominator

$$Y(s) = \frac{1}{mL} \left[\frac{\frac{-1}{2\sqrt{g}}}{s + \sqrt{g}} + \frac{\frac{1}{2\sqrt{g}}}{s - \sqrt{g}} \right]$$ } PFE Tool

$$y(t) = \frac{1}{2\sqrt{g}\, mL} \cdot \left(e^{\sqrt{g}\, t} - e^{-\sqrt{g}\, t} \right)$$ } Inverse Laplace Transform Impulse Response



} Unstable Impulse Response

Control Design

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/mL \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$
} State Equation

Find K in $u = -Kx + r$ so poles are at s = -1 } Objective

$$\Delta_d(s) = (s+1)^2$$ } $n = 2$

$$\Delta_d(A) = \left[\begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]^2 = \begin{bmatrix} g+1 & 2 \\ 2g & g+1 \end{bmatrix}$$ } Let $s = A$

$$K = (0 \quad 1) \cdot \begin{bmatrix} 0 & \frac{1}{mL} \\ \frac{1}{mL} & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} g+1 & 2 \\ 2g & g+1 \end{bmatrix}$$ } Ackerman's formula

$$K = \begin{bmatrix} mL(g+1) & 2mL \end{bmatrix}$$ } answer

Check the answer

$$\Delta(s) = \left| sI - (A - BK) \right|$$ } Formula for closed-loop denominator

$$A - BK = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/mL \end{bmatrix} \begin{bmatrix} mL(g+1) & 2mL \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$ } Find A - BK

$$\Delta(s) = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \right| = \left| \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \right|$$

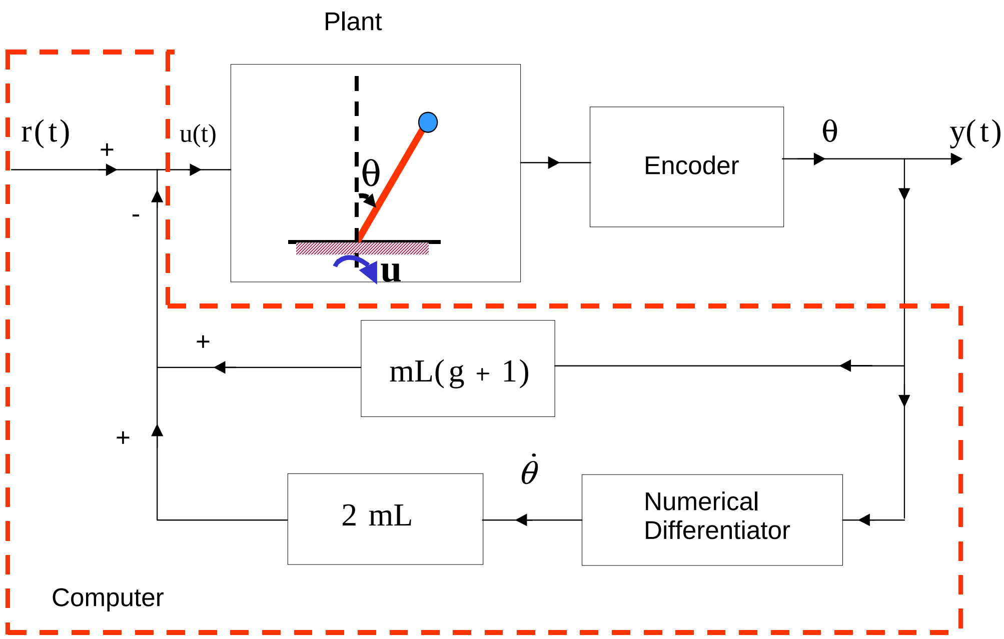
$$\Delta(s) = s^2 + 2s + 1 = (s+1)^2$$ } checks

Note the control is given by

$u = -Kx + r$ where $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ so

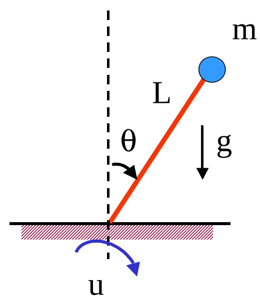
$$u = -mL(g+1)\theta - 2mL\dot{\theta} + r$$ } control which must be implemented

Computer Implementation



$$u = -mL(g+1)\theta - 2mL\dot{\theta} + r \quad \Big\} \text{ Control}$$

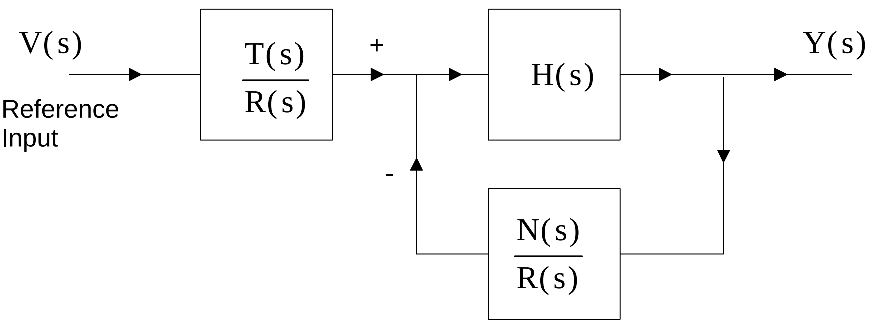
Same Example



What if $\dot{\theta}$ is not available for measurements ?

Solution: Use a polynomial regulator

Polynomial Regulator



} Polynomial Regulator

Given $H(s) = \dfrac{1}{mL(s^2 - g)}$ } Open-loop TF

Find T(s), R(s) and N(s) such that $\dfrac{Y(s)}{V(s)} = \dfrac{1}{(s+1)^2}$ } Objective: Place both closed-loop poles at s= -1

Step 1 Determine A(s), B(s), Q(s) and P(s) from the problem statement and check restriction 1

$$H(s) = \frac{B(s)}{A(s)} = \frac{1}{mL \cdot (s^2 - g)}$$ } open-loop TF

$$H_d(s) = \frac{Q(s)}{P(s)} = \frac{1}{(s+1)^2}$$ } desired closed-loop TF

Restrictions

1) $\text{degree}(P(s)) = \text{degree}(A(s)) = 2$ } OK

Step 2 Find $B^{+}(s)$ and $B^{-}(s)$ and check restriction 2

$$H(s) = \frac{B(s)}{A(s)} = \frac{1}{mL\cdot(s^2 - g)} \quad \} \quad \text{Plant}$$

$$B(s) = \frac{1}{mL}$$

$$B^{+}(s) = \frac{1}{mL}$$

$$B^{-}(s) = 1$$

Restrictions

2) Q(s) must contain the roots found in $B^{-}(s)$

$$Q(s) = 1 \qquad B^{-}(s) = 1 \quad \} \quad \text{OK}$$

Step 3 Define R(s) and T(s) in terms of subpolynomials

$$T(s) = \frac{T_1(s)Q(s)}{B^{-}(s)} = \frac{T_1(s).1}{1} \qquad R(s) = R_1(s)B^{+}(s) = \frac{R_1(s)}{mL}$$

$R_1(s), T_1(s)$ are unknown at this point.

Step 4 Determine the degree of each polynomial

$$\deg\{T_1\} = \deg\{A\} - \deg\{B^{+}\} - 1 = 2 - 0 - 1 = 1 \quad \Rightarrow \quad L = 1$$

$$\deg\{T_1\} = 1$$

$$\deg\{R_1\} = \deg\{T_1\} = 1$$

$$\deg\{N\} = \deg\{A\} - 1 = 2 - 1 = 1 \quad \Rightarrow \quad C = 1$$

$$\deg\{N\} = 1$$

Step 5 Define the structure of $T_1(s), R_1(s), N(s)$

Poles of T_1 should be 10 times bigger than the largest pole of P(s) } Rule of Thumb

Let a = 10

$$T_1(s) = (s + 10)^L = (s + 10) \quad \} \quad L = 1$$

$$R_1(s) = s + r_0 \quad \} \quad \deg\{R_1\} = 1$$

$$N(s) = n_1 s + n_0 \quad \} \quad \deg\{N\} = 1$$

r_0, n_1, n_0 are unknown at this point

Step 6 Solve the Diophantine Equation to determine r_i, n_i coefficients in $R_1(s)$ and $N(s)$

$$A(s)R_1(s) + B^-(s)N(s) = P(s)T_1(s)$$

$$(s^2 - g)\cdot(s + r_o) + (1)\cdot(n_1\cdot s + n_0) = (s^2 + 2s + 1)\cdot(s + 10)$$ } Insert Polynomial

$$s^3 + r_0\cdot s^2 - g\cdot s - g\cdot r_0 + n_1\cdot s + n_0 = s^3 + 12s^2 + 21s + 10$$ } Simplify

$$s^2\cdot(r_0) = s^2\cdot(12) \quad \Rightarrow \quad r_0 = 12$$

$$s\cdot(-g + n_1) = s\cdot(21) \quad \Rightarrow \quad n_1 = g + 21$$

$$-g\cdot r_0 + n_0 = 10 \quad \Rightarrow \quad n_0 = 10 + g\cdot r_0 = 10 + g\cdot 12$$

$r_0 = 12$ $\quad$ $n_1 = g + 21$ $\quad$ $n_0 = 12g + 10$ } Solve for coefficients

$$R_1(s) = s + 12$$

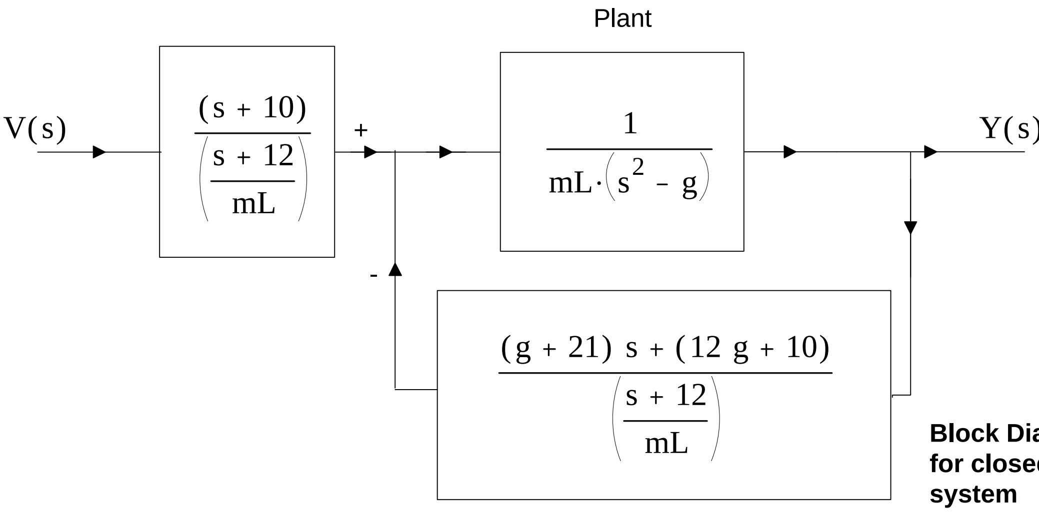
$$N(s) = (g + 21)s + (12g + 10)$$

} Gives $R_1(s)$ and $N(s)$

Step 7 Construct R(s) and T(s)

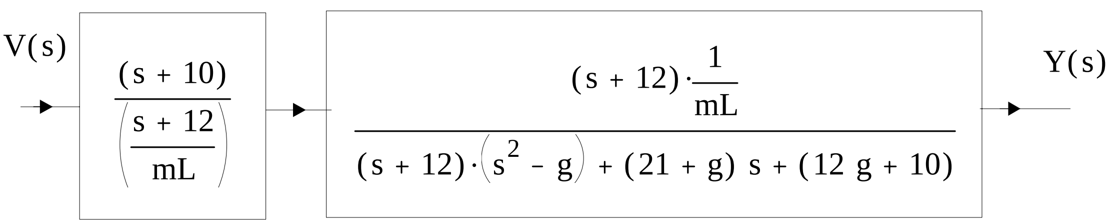
$$T(s) = \frac{T_1(s)Q(s)}{B^-(s)} = \frac{(s+10)(1)}{(1)} = s + 10$$

$$R(s) = R_1(s)B^+(s) = \frac{s+12}{mL}$$

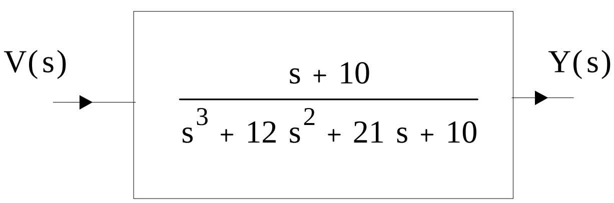


Block Diagram for closed-loop system

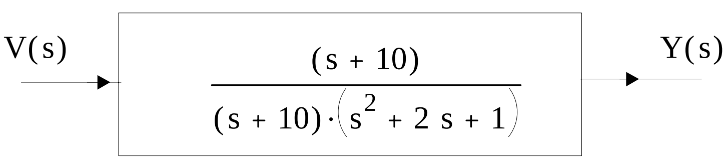
<u>**Check your answer**</u>



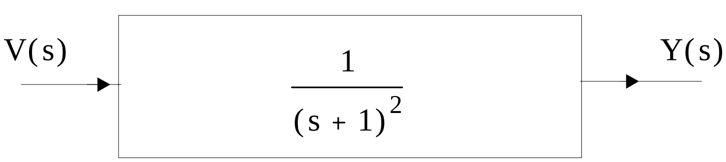
.... Feedforward Feedback Combination



} Combine cascade blocks



} Factor denominator



} Cancel observer pole

checks since $\frac{Y(s)}{V(s)} = \frac{1}{(s+1)^2}$

Appendix

Causality

We also note that even though only causal systems can be implemented in real time, noncausal systems are still of inetrest. For one thing, the independent variable may not always be time, as for example, in optics. Second, even when time is the independent variable, noncausal systems are useful in providing benchmarks of performance; the ideal band-pass filter is a classic example. Furthermore, as the reader may know, noncausal systems may often be satisfactorily approximated by causal systems plus a delay. FInally, noncausal systems may arise in the course of analysis, when decomposing or recombining causal systems.

The skeleton-in-the-closet of L_+ Transforms

$L_+[1(t)] = s^{-1} = L_-[1(t)]$ and $L_+[\dot{x}] = s(L_+ x) - x(0+)$

we obtain $L_+[d(t)] = s.s^{-1} - 1 = 0 = L_+[0]$

This may be a surprise, though it is consistent with the direct calculation

$$L_+[d(t)] = \int_{0+}^{\infty} d(t) e^{-st} dt = 0$$

The expected formula for the Laplace transforms of $d(t)$ is 1. But this actually follows from the use of the L_- formulas.

$$L_-[d(t)] = L_-[d1(t)/dt] = s.s^{-1} - 0 = 1$$

which also agrees with the direct evaluation.

More detailed discussions of the L transform can be found in many places, e.g., L.A. Zadeh and C. Desoer, Linear Systems --- A State-Space Approach, MGraw-Hill, New York, 1963.

"Derivatives" of Discontinuous Functions and Delta Functions. The introduction of the delta function enables us to give meaning to the derivative of a function at a point of discontinuity. We should note that Q δ(.) can be regarded as the derivative of a step function. This derivative relationship is certainly true (in the conventional sense) as long as $\varepsilon \neq 0$ and the use of the symbol δ(.) allows us to extend it to the case where ε = 0.

$$d(t) = \lim_{e \to 0} p_e(t) = \lim_{e \to 0} \frac{1(t) - 1(t-e)}{e} = \frac{d}{dt} 1(t)$$

Having gotten this far, we can go further and, using the same ideas, introduce a derivative for the delta function and in fact also second-and higher-order derivatives. To introduce the derivative, say $\delta^{(1)}(.)$, we write

$$d^{(1)}(t) = d\, d(t)/dt = \lim_{e \to 0} \frac{d(t) - d(t-e)}{e}$$

A pictorial interpretation using sequences is provided by the figure below. As $e \to 0$, the function in Fig a) tends towards the unit impulse $\delta(.)$, while the function in Fig b) approaches what can be interpreted as its derivative. Since for very small ε the functionin Fig b) approaches what can be interpreted as its derivative. Snce for very small ε the function in Fig b) consists of two narrow but strong pulses of opposite sign, the function in the limit is called the *unit doublet*. It is an idealization of a dipole in electromagnetic theory and a couple (torque) in mechanics.

This procedure can be similarly extended to define $\delta^{(2)}(.), \delta^{3}(.)$ and so on: We merely with smoother and smoother approximating sequences for $\delta(.)$. The derivatives obtained by the above procedures may be called *generalized derivatives* since the conventional derivatives are not defined.