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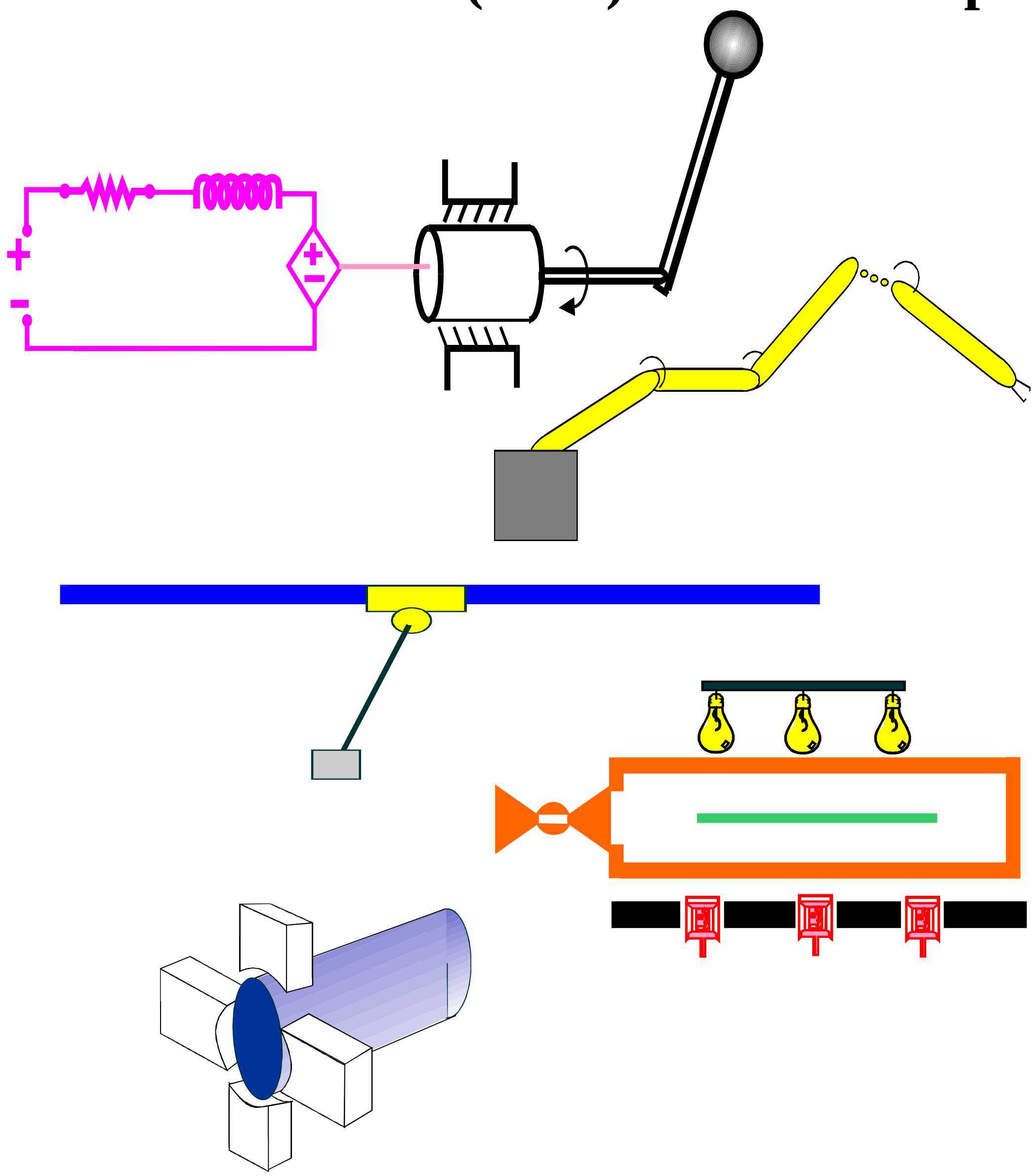

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Robust and Adaptive Output Feedback Control Strategies for a Class of Uncertain MIMO Nonlinear Systems

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Abstract: In this paper, a continuous output feedback (OFB) tracking controller is developed for a class of high-order multi-input multi-output (MIMO) nonlinear systems with an input gain matrix that has nonzero leading principal minors but can be non-symmetric. Under the mild assumption that the signs of the leading minors of the control input gain matrix are known, the controller yields semiglobal uniformly ultimately bounded (SGUUB) tracking while compensating for unstructured uncertainty in both the drift vector and the input matrix. First, a full-state feedback controller is designed based on limited assumptions on the structure of the system nonlinearities, and the controller is proven to yield SGUUB tracking through a Lyapunov-based analysis. Then, an output feedback control design based on a high gain observer is proposed. A comprehensive stability analysis of the closed-loop system under output feedback is carried out and a recovery of the state feedback SGUUB result is demonstrated for the output feedback control system. Next, an adaptive extension is provided for a sub-class of MIMO systems for which both the drift vector and input gain matrix are affine in the unknown constant parameters of the system. Two novelties of the results obtained here are the singularity free procedure for control design as the well as the fact that the estimation of uncertainty does not lead to uncontrollability.

1 Introduction

Over the years, a lot of progress has been reported in nonlinear systems – specifically, research has moved from analysis into systematic techniques for controller synthesis for classes of nonlinear systems. As reported in a survey paper by [11], many researchers either assume no uncertainty in the plant model or assume that the uncertainty is a product of unknown parameters with known nonlinearities – the latter case leading into adaptive control. For multi-input nonlinear systems that are representable in the parametric strict feedback form, Krstic *et. al.* [13] were able to formalize the adaptive backstepping design procedure; however, the gain matrix premultiplying the control is assumed to be known. In [12], a general procedure was presented for designing switching adaptive controllers for multi-input nonlinear systems which includes feedback linearizable systems, parametric-pure feedback systems, and systems with a control Lyapunov function that is linear in the parameters. Other examples of adaptive results can be found in [6], [7], and many others (See references in [11]). Even where unstructured uncertainties are dealt with, classes of single-input single-output (SISO) systems make up a large amount of the total effort. For SISO nonlinear systems in the strict feedback form that are non-affine in the unknown parameters, a global uniform ultimate bounded result due to [27] employs a Nussbaum gain and a smooth parameter projection algorithm. In [5], Ding *et. al.* obtain an ultimately bounded output feedback result for uncertain SISO systems via a modification of the backstepping technique using a Nussbaum gain and a Lyapunov function that is flat in a specifiable region around the origin. It is to be noted that

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both these results are robust to disturbances and do not require any knowledge of the sign of the high-frequency gain.

As far as multi-input uncertain nonlinear systems are concerned, there are very few results available. In [29], a neural network-based state feedback adaptive controller was formulated for the following class of systems

$$\begin{aligned} x^{(n)} &= h(x, \dot{x}, \cdots, x^{(n-1)}, \phi(t), t) + \\ & \quad G(x, \dot{x}, \cdots, x^{(n-1)}, \phi(t), t)u \end{aligned} \tag{1}$$

with the only restriction that $G(\cdot)$ be uniformly positive or negative definite. The proposed controller was shown to guarantee the semi-global convergence of the tracking error to a residual set. The drawback of the control strategy is that the estimation strategy utilized can lead to loss of controllability in which case the control input tends to zero.

In this paper, our goal is to design a continuous output feedback (OFB) tracking controller based on the limited assumptions that the nonlinearities are second-order differentiable and the input gain matrix has nonzero leading principal minors but is allowed to be non-symmetric. The $\mathcal{C}^2$ restriction on nonlinearities is required in order to ensure that the Lyapunov analysis based synthesis procedure yields a continuous controller. Since state controllability requires that the smallest singular value of the input gain matrix be lowerbounded by a positive constant, a sufficient condition is to reuire the input gain matrix to have nonzero leading principal minors. However, we drop the requirement that the input gain matrix be symmetric since many practical nonlinear control systems do not possess such a property (See examples in [30, 24]). Motivated by a matrix decomposition introduced in [18] and subsequently utilized in [3], we decompose $G(\cdot)$ into the product of a symmetric p.d. matrix, a known diagonal matrix with +1 or -1 on the diagonal (comprised of the ratios of the signs of the leading minors of the control input gain matrix), and a unity upper triangular matrix. The symmetric p.d. matrix is exploited in the Lyapunov based stability analysis while the unity upper triangular matrix allows for an algebraic loop free sequential synthesis of control signals $u_i(t)\ \forall\, i = 1, 2, ...m$. Next, we adapt a high-gain observer (HGO) result in [2] for full-state asymptotically stable systems to uniformly ultimately bounded systems in order to design a continuous OFB controller that drives the closed-loop trajectories for the tracking errors into an arbitrarily small residual set. In order to broaden the applicability of the approach, we introduce a modular feedforward scheme which is shown to be achievable via neural network compensation.

In addition to the robust output feedback control strategy discussed above, this paper also deals with the design of an adaptive strategy for the following sub-class of the system introduced in (1) [31]

$$\begin{aligned} x^{(n)} = \ & h(x, \dot{x}, \cdots, x^{(n-1)}, \theta_1) \\ & + G(x, \dot{x}, \cdots, x^{(n-2)}, \theta_2)u \end{aligned} \tag{2}$$

where $(\cdot)^{(i)}$ denotes the i^{th} derivative with respect to time, the uncertain $\mathcal{C}^0$ functions $h(\cdot) \in \mathbb{R}^m$ and the uncertain matrix $G(\cdot) \in \mathbb{R}^{m \times m}$ were assumed to be affine in the unknown constant parameter vectors θ_1 and θ_2. Here, we extend the state feedback adaptive control design in [31] to achieve practical convergence of the tracking error to the origin under output feedback for the class of systems in (2). Again, the design mechanism relies on the matrix decomposition of [18] and an adaptation of the high-gain observer of [2]. The final stability result obtained is semi-global uniformly ultimately boundedness of the tracking errors.

The rest of the paper is organized as follows. In Section 2, we present the system dynamics and the input gain matrix decomposition. Section 3 details the error system development for the open-loop system. In Section 4, we propose full-state feedback (FSFB) and output feedback (OFB) controllers for the general MIMO system; this is followed by a Lyapunov based stability analysis. In Section 5, we present the development of an adaptive OFB controller for the sub-class of systems shown in (2). In Section 6, the theoretical development of the paper is complemented with a numerical simulation for a two degrees-of-freedom (DOF) mechanical system. Section 7 draws the conclusions.

2 Problem Formulation and Preliminaries

We consider a class of MIMO nonlinear systems having the form

$$x^{(n)} = h(\mathbf{x}) + G(\mathbf{x})u \tag{3}$$

where $x^{(i)}(t) \in \mathbb{R}^m$, $i = 0, 1, ..., n-1$ are the system states, $\mathbf{x} \triangleq [x^T\ \dot{x}^T\ ...\ (x^{(n-1)})^T]^T \in \mathbb{R}^{mn}$, $u(t) \in \mathbb{R}^m$ represents the control input, and $h(\mathbf{x}) \in \mathbb{R}^m$ and $G(\mathbf{x}) \in \mathbb{R}^{m\times m}$ are uncertain $\mathcal{C}^2$ nonlinearities. We assume $G(\mathbf{x})$ is a real matrix with nonzero leading principal minors.

The following matrix decomposition will be a key factor in the proposed control design.

Lemma 1 *Any real matrix $G \in \mathbb{R}^{m\times m}$ with nonzero leading principal minors can be decomposed as [3]*

$$G = SDU \tag{4}$$

where $S \in \mathbb{R}^{m\times m}$ is symmetric positive definite, $D \in \mathbb{R}^{m\times m}$ is a diagonal matrix with diagonal entries $+1$ or -1, $U \in \mathbb{R}^{m\times m}$ is unity upper triangular.

Proof. Since the leading principal minors of are nonzero, there exists a unique factorization [26]

$$G = L_1 D_p L_2^T \tag{5}$$

where $L_1, L_2 \in \mathbb{R}^{m\times m}$ are unity lower triangular and $D_p \in \mathbb{R}^{m\times m}$ defined as follows

$$D_p \triangleq diag\{d_1, d_2, \cdots, d_m\} \tag{6}$$

and

$$d_1 \triangleq \triangle_1 \qquad d_i \triangleq \tfrac{\triangle_i}{\triangle_{i-1}} \quad i = 2, 3, ..., m. \tag{7}$$

In (7), $\triangle_1, \triangle_2, \cdots, \triangle_m$ are leading principal minors of G. Factoring D_p as

$$D_p = D_+ D \tag{8}$$

where D_+, $D \in \mathbb{R}^{m\times m}$ are defined as follows

$$D_+ \triangleq diag\{|d_1|, |d_2|, \cdots, |d_m|\} \tag{9}$$

$$D \triangleq diag\{\operatorname{sgn}(d_1), \operatorname{sgn}(d_2), \cdots, \operatorname{sgn}(d_m)\}. \tag{10}$$

After substituting (6)-(10) into (5), (4) is satisfied by selecting S and U as follows

$$S \triangleq L_1 D_+ L_1^T \qquad U \triangleq D^{-1} L_1^{-T} D L_2^T\ .$$

Remark 1 *For control purposes, we assume that D is known. For the case of a positive definite G, the factorization of G can be simplified as $G = SU$.*

After time differentiating (3), the following expression can be obtained

$$x^{(n+1)} = \varphi(\mathbf{x}, x^{(n)}) + G(\mathbf{x})\dot{u} \tag{11}$$

where $\varphi(\mathbf{x}, x^{(n)}) \in \mathbb{R}^m$ is defined as follows

$$\varphi(\mathbf{x}, x^{(n)}) \triangleq \dot{h}(\mathbf{x}) + \dot{G}(\mathbf{x})G^{-1}(\mathbf{x})\left(x^{(n)} - h(\mathbf{x})\right).$$

Invoking Lemma 1, (11) can be rewritten as

$$M(\mathbf{x})x^{(n+1)} = f(\mathbf{x}, x^{(n)}) + DU(\mathbf{x})\dot{u} \tag{12}$$

where $M(\mathbf{x}) \triangleq S^{-1}(\mathbf{x}) \in \mathbb{R}^{m\times m}$ is symmetric positive definite, $f(\mathbf{x}, x^{(n)}) \triangleq S^{-1}(\mathbf{x})\varphi(\mathbf{x}, x^{(n)}) \in \mathbb{R}^m$, and $S(\mathbf{x}), D, U(\mathbf{x})$ are defined as in Lemma 1. We will assume that the matrix $M(\cdot)$ is bounded by

$$\underline{m}\,\|\xi\|^2 \leqslant \xi^T M(\cdot)\xi \leqslant \bar{m}(\cdot)\,\|\xi\|^2 \qquad \forall \xi \in \mathbb{R}^m \tag{13}$$

where $\underline{m} \in \mathbb{R}$ denotes a positive constant, and $\bar{m}(\cdot) \in \mathbb{R}$ denotes a positive, non-decreasing function. The $(n+1)^{th}$ order system in (12) will be used as a basis for the subsequent control design and stability analysis.

3 Error System Development

Let $x_d(t) \in \mathbb{R}^m$ be a $\mathcal{C}^{n+2}$ reference trajectory such that

$$x_d^{(i)}(t) \in \mathcal{L}_\infty, \quad i = 0, 1, ..., n+2 \tag{14}$$

and $\mathbf{x}_d = [x_d^T \ \dot{x}_d^T \ ... \ (x_d^{(n-1)})^T]^T \in \mathbb{R}^{mn}$. The output tracking error $e_1(t) \in \mathbb{R}^m$ is defined as follows

$$e_1 = x_d - x. \tag{15}$$

The control objective is to ensure that $e_1(t) \to 0$ as $t \to \infty$ as well as guarantee the boundedness of all closed-loop signals. To simplify the subsequent control design, we introduce the following auxiliary error signals $e_i(t) \in \mathbb{R}^m$, $i = 2, 3, ..., (n+1)$

$$\begin{aligned} e_2 &= \dot{e}_1 + e_1 \\ e_3 &= \dot{e}_2 + e_2 + e_1 \\ e_4 &= \dot{e}_3 + e_3 + e_2 \\ &\vdots \\ e_{n+1} &= \dot{e}_n + e_n + e_{n-1}. \end{aligned} \tag{16}$$

Following [28], it is easy to obtain

$$e_i(t) = \sum_{j=0}^{i-1} a_{ij} e_1^{(j)}(t) \qquad \forall\, i = 2, 3, ..., (n+1) \tag{17}$$

where the known constant coefficients a_{ij} are generated via a Fibonacci number series [28]. We define the error signal $r(t) \in \mathbb{R}^m$ as follows

$$r = e_{n+1} + e_n. \tag{18}$$

After taking the time derivative of (18), multiplying both sides by $M(\mathbf{x})$, and then substituting from the derivative of (17) for $i = n+1$, we have

$$M\dot{r} = M \sum_{j=0}^{n} a_{n+1j} e_1^{(j+1)} + M\dot{e}_n. \tag{19}$$

After applying (15) to (19), we have

$$\begin{aligned} M\dot{r} &= M x_d^{(n+1)} - M x^{(n+1)} + M \left(\sum_{j=0}^{n-1} a_{n+1j} e_1^{(j+1)} + \dot{e}_n \right) \\ &= M \left(x_d^{(n+1)} + \sum_{j=0}^{n-1} a_{n+1j} e_1^{(j+1)} + \dot{e}_n \right) - f - DU(\mathbf{x})\dot{u} \end{aligned} \tag{20}$$

upon the use of (12). We now arrange (20) into the following form

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_{n+1} - \bar{U}(\mathbf{x})\dot{u} - D\dot{u} + N \tag{21}$$

where $N(\mathbf{x}, x^{(n)}, t) \in \mathbb{R}^m$ is defined as follows

$$\begin{aligned} N = \ & M \left(x_d^{(n+1)} + \sum_{j=0}^{n-1} a_{n+1j} e_1^{(j+1)} + \dot{e}_n \right) - f \\ & + \frac{1}{2}\dot{M}r + e_{n+1} \end{aligned} \tag{22}$$

and $\bar{U}(\mathbf{x}) \triangleq DU(\mathbf{x}) - D$. Note that $\bar{U}(\mathbf{x})$ is a strictly upper triangular matrix. We now define

$$N_d(t) \triangleq N(\mathbf{x}_d, x_d^{(n)}, x_d^{(n+1)}) \tag{23}$$

and rewrite (21) as

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_{n+1} - \bar{U}\dot{u} - D\dot{u} + N_d + \tilde{N} \tag{24}$$

where $\tilde{N}(\cdot) \triangleq N(\cdot) - N_d(t)$. Note that $N_d(t), \dot{N}_d(t) \in \mathcal{L}_\infty$ due to (14) and the $\mathcal{C}^2$ condition on $G(\mathbf{x})$ and $h(\mathbf{x})$. Since $N(\cdot)$ defined in (22) is $\mathcal{C}^1$, it can be shown that $\tilde{N}(\cdot)$ can be upper bounded as follows

$$\left\|\tilde{N}\right\| \leqslant \rho_N(\|z\|)\,\|z\| \tag{25}$$

where $z(t) \in \mathbb{R}^{mn+m}$ is defined as follows

$$z \triangleq \begin{bmatrix} e_1^T & e_2^T & ... & e_n^T & r^T \end{bmatrix}^T \tag{26}$$

and $\rho_N(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a globally invertible, nondecreasing function. We can write a compact form of the open-loop dynamics for $z(t)$ from (16) and (26) as follows

$$\dot{z}(t) = Az + B\dot{r} \tag{27}$$

where $A \in \mathbb{R}^{(mn+m)\times(mn+m)}, B \in \mathbb{R}^{(mn+m)\times m}$ are defined as follows [1]

$$A \triangleq \begin{bmatrix} -I_m & I_m & 0_m & \cdots & 0_m & 0_m \\ -I_m & -I_m & I_m & \cdots & 0_m & 0_m \\ 0_m & -I_m & -I_m & \cdots & 0_m & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_m & 0_m & 0_m & \cdots & -2I_m & I_m \\ 0_m & 0_m & 0_m & \cdots & 0_m & 0_m \end{bmatrix} \tag{28}$$

$$B = \begin{bmatrix} 0_m & 0_m & \cdots & 0_m & I_m \end{bmatrix}^T. \tag{29}$$

4 Robust Control Development

4.1 State Feedback Control Law

In this section, we assume full state feedback (*i.e.*, $x^{(i)}(t)$, $i = 0, ..., n-1$ in (3) are measurable). We propose the following state feedback control law

$$u = \int_{t_0}^{t} D\left(K + I_m\right)\left(e_{n+1}(\tau) + e_n(\tau)\right)d\tau + \int_{t_0}^{t} D\hat{f}(\tau)d\tau \tag{30}$$

where $K \in \mathbb{R}^{m\times m}$ is a positive-definite, diagonal matrix, D is defined in (10), and $\hat{f}(t) \in \mathbb{R}^m$ denotes a yet to be designed feedforward component included to compensate for the term $f(\mathbf{x}, x^{(n)})$ in (12). We assume $\hat{f}(t)$ is designed such that $\hat{f}(t) \in \mathcal{L}_\infty$. Note that $e_n(t)$ and $\int e_{n+1}(t)dt$ in (30) are measurable since they are a function of the system states and the reference trajectory. Note that $\int e_{n+1}(t)dt$ is measurable via integrating the right hand side of the last definition of (16).

[1] The notation I_m and 0_m denotes, respectively, $m \times m$ identity and null matrices for any positive integer m.

By employing the time derivative of the control input vector of (30), the vector $\bar{U}\dot{u}$ in (21) can be written as

$$\bar{U}\dot{u} = \begin{bmatrix} \operatorname{sgn}(d_1)\displaystyle\sum_{j=2}^{m} U_{1j}\dot{u}_j \\ \operatorname{sgn}(d_2)\displaystyle\sum_{j=3}^{m} U_{2j}\dot{u}_j \\ \vdots \\ \operatorname{sgn}(d_{m-1})\, U_{(m-1)m}\dot{u}_m \\ 0 \end{bmatrix} = \begin{bmatrix} \Lambda + \Phi_d \\ 0 \end{bmatrix}, \tag{31}$$

where d_i, $i = 1, ..., m-1$ is defined in (7), the auxiliary signals are defined as: $\Lambda \triangleq [\Lambda_1\ \Lambda_2 \cdots \Lambda_{m-1}]^T$, $\Phi_d \triangleq \left[\Phi_{d1}\ \Phi_{d2} \cdots \Phi_{d(m-1)}\right]^T$, and the individual elements $\Lambda_i, \Phi_{di} \in \mathbb{R}, i = 1, ..., m-1$ are defined as

$$\begin{aligned} \Lambda_i &= \operatorname{sgn}(d_i) \sum_{j=i+1}^{m} \left[\tilde{U}_{ij}(K_j+1)r_j + \tilde{U}_{ij}\hat{f}_j + U_{ij}(\mathbf{x}_d)(K_j+1)r_j\right] \quad &(32) \\ \Phi_{di} &= \operatorname{sgn}(d_i) \sum_{j=i+1}^{m} U_{ij}(\mathbf{x}_d)\hat{f}_j. &(33) \end{aligned}$$

In (32) above, $\tilde{U}_{ij} \triangleq U_{ij}(\mathbf{x}) - U_{ij}(\mathbf{x}_d)$ while the subscript ij denotes the ij^{th} element of the corresponding matrix, $K_j \in \mathbb{R}$ denotes the j^{th} diagonal element of K, and $r_j(t), \hat{f}_j(t) \in \mathbb{R}$ denote the j^{th} element of $r(t)$ and $\hat{f}(t)$ respectively. It can be shown that $\Lambda_i(t)$ is upper bounded as follows

$$\|\Lambda_i\| \leqslant \rho_{\Lambda i}(\|z\|)\,\|z\| \tag{34}$$

where $z(t)$ was defined in (26), and $\rho_{\Lambda i}(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a globally invertible, nondecreasing function containing only the diagonal elements $i+1$ to $m-1$ of K due to the strictly upper triangular nature of $\bar{U}(\mathbf{x})$. After taking the derivative of (30) and substituting into (24), one can arrive at the following closed-loop dynamics for $r(t)$

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_{n+1} - (K + I_m)r + \Pi + \Psi_d \tag{35}$$

where

$$\Pi = \tilde{N} - \begin{bmatrix} \Lambda \\ 0 \end{bmatrix}, \quad \Psi_d = N_d - \begin{bmatrix} \Phi_d \\ 0 \end{bmatrix} - \hat{f}. \tag{36}$$

It is not difficult to show that $\Psi_d(t), \dot{\Psi}_d(t), U(\mathbf{x}_d) \in \mathcal{L}_\infty$. Also note that $\Psi_d(t)$ is not a function of any control gains. The stability of the closed-loop system is stated by the following theorem:

Theorem 1 *Provided the elements of K are selected sufficiently large, the control law of (30) ensures that: (a) all closed-loop signals stay bounded for all time, and (b) tracking is locally uniformly ultimately bounded in the sense that*

$$\|e_i(t)\| \leqslant \epsilon, \qquad i = 1, 2, \cdots, n+1 \quad \forall\ t \in [t_0 + T, \infty) \tag{37}$$

where ϵ, T are some positive constants.

Proof. We begin the proof of Theorem 1 by defining the function $V(t, z) : \mathbb{R}_{\geq 0} \times \mathbb{R}^{m(n+1)} \to \mathbb{R}_{\geq 0}$ as follows

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2}r^T M r. \tag{38}$$

Note that (38) can be bounded as

$$\lambda_1 \|z\|^2 \leqslant V \leqslant \lambda_2(\|z\|) \|z\|^2 \tag{39}$$

where $z(t)$ was defined in (26), and λ_1 and $\lambda_2(\|z\|)$ are defined as follows

$$\lambda_1 = \frac{1}{2}\min(1, \underline{m}) \quad \lambda_2(\|z\|) \geqslant \frac{1}{2}\max(1, \bar{m}(\mathbf{x})) \tag{40}$$

and we have taken advantage of (13). After taking the time derivative of (38) and substituting from (35), we have

$$\begin{aligned} \dot{V} &= -\sum_{i=1}^{n+1} e_i^T e_i - r^T r - r^T K r + r^T \Pi + r^T \Psi_d \\ &\leqslant -\|z\|^2 + \left[\|r\| \left\|\tilde{N}\right\| - k_p \|r\|^2\right] + \left[\sum_{i=1}^{m-1} (|r_i| \|\Lambda_i\| - K_{di} |r_i|^2)\right] \\ &\quad + \left[\left\|N_d - \Phi_d - \hat{f}\right\| \|r\| - k_q \|r\|^2\right] \end{aligned} \tag{41}$$

where $K = (k_p + k_q) I_m + K_d$; here, k_p, k_q are positive scalars while K_d is a diagonal gain matrix defined as $K_d \triangleq \text{diag}\{K_{d1}, K_{d2}, \cdot\cdot, K_{d(m-1)}, 0\}$. After making use of (25), (34) and completing the squares on the last bracketed term of (41), we obtain

$$\begin{aligned} \dot{V} &\leqslant -\|z\|^2 + \left[\|r\| \rho_N(\|z\|) \|z\| - k_p \|r\|^2\right] \\ &\quad + \sum_{i=1}^{m-1} \left[|r_i| \rho_{\Lambda i}(\|z\|) \|z\| - K_{di} |r_i|^2\right] + \nu_0 \end{aligned} \tag{42}$$

where ν_0 is some positive constant such that $\frac{1}{k_q}\left\|N_d - [\Phi_d \ 0]^T - \hat{f}\right\|^2 \leqslant \nu_0$; note that we have taken advantage of the *a priori* boundedness of $N_d(\cdot), \Phi_d(\cdot)$, and $\hat{f}(\cdot)$. Again, completing the squares on the bracketed terms in (42), we obtain

$$\dot{V} \leqslant -\left(1 - \frac{1}{4k_p}\rho_N^2(\|z\|) - \sum_{i=1}^{m-1} \frac{1}{4K_{di}}\rho_{\Lambda i}^2(\|z\|)\right) \|z\|^2 + \nu_0. \tag{43}$$

Since $\rho_N(\|z\|)$ is independent of control gains and $\rho_{\Lambda i}(\|z\|)$ contains only control gains k_p, k_q, $K_{d(i+1)}$, $K_{d(i+2)}$,$\cdot\cdot\cdot$,$K_{d(m-1)}$, we can design K_{di} $\forall\, i = 1, 2, \cdot\cdot\cdot (m-1)$ in the above equation without an algebraic loop. We can state from (43) that

$$\dot{V} \leqslant -\lambda_3 \|z\|^2 + \nu_0 \tag{44}$$

provided that

$$\begin{aligned} k_p &> \frac{1}{2}\rho_N^2(\|z\|) \quad (\text{or } \|z\| < \rho_N^{-1}(\sqrt{2k_p})), \\ K_{di} &> \frac{(m-1)}{2}\rho_{\Lambda i}^2(\|z\|) \quad (\text{or } \|z\| < \rho_{\Lambda i}^{-1}(\sqrt{\frac{2K_{di}}{m-1}})) \end{aligned} \tag{45}$$

where $i = 1, 2, \cdot\cdot\cdot, m-1$. With the gains chosen as specified above, one can ensure that

$$\lambda_3 = \left\{1 - \frac{1}{4k_p}\rho_N^2(\|z\|) - \sum_{i=1}^{m-1} \frac{1}{4K_{di}}\rho_{\Lambda i}^2(\|z\|)\right\}$$

is a positive constant.

Since our aim is to prove Theorem 1 via an application of Lemma 2 in Appendix A, we define the relationships between the variables defined in Lemma 2 and those in the analysis above in (38)-(45). From (39), the lower and upper bounds for (38) can be expressed as follows

$$\gamma_1(\|z\|) = \lambda_1 \|z\|^2 \qquad \gamma_2(\|z\|) = \lambda_2(\|z\|) \|z\|^2 \ . \tag{46}$$

It is easy to check that $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are class $\mathcal{K}_\infty$ functions. Define the region $\mathcal{D}$ as

$$\mathcal{D} = \{ z \in \mathbb{R}^{m(n+1)} \big| \ \xi_1 < \|z\| < \ \xi_2\} \tag{47}$$

where ξ_1, ξ_2 are positive constants defined as

$$\xi_1 = \sqrt{\nu_0} \qquad \xi_2 = \min_{i=1,2,\cdots,(m-1)} (\rho_N^{-1}(\sqrt{2k_p}),\ \rho_{\Lambda i}^{-1}(\sqrt{\tfrac{2K_{di}}{m-1}})) \ . \tag{48}$$

We remark here that ξ_1 can be made as small as possible by picking k_q large enough while ξ_2 can be made as large as possible by choosing k_p and K_{di} large enough such that it is always possible to satisfy $(\gamma_1^{-1} \circ \gamma_2)(\xi_1) < \xi_2$ (See also Remark 2 below). From (44), we have

$$\dot{V} \leqslant -\lambda_4 \|z\|^2 \qquad \forall \ z \in \mathcal{D} \tag{49}$$

where λ_4 is some positive constant. The upper bound for the derivative of (49) can be expressed as

$$\gamma_3(\|z\|) = \lambda_4 \|z\|^2 \tag{50}$$

where $\gamma_3(\cdot)$ is a class $\mathcal{K}_\infty$ function. We are now in a position to apply Lemma 2 to prove boundedness and ultimate boundedness in the sense of Theorem 1. The region of attraction (to the ultimate bound) for the closed-loop system of (35) is given by $\mathcal{D}_z \triangleq \{ z \in \mathbb{R}^{m(n+1)} \big| \ \|z\| < (\gamma_2^{-1} \circ \gamma_1)(\xi_2)\}$. ■

Remark 2 *Though the stability result presented is local and only ultimately bounded, the boundaries of the bounded region $\mathcal{D}$ in (47) can be expanded by increasing the control gains. For $\xi_1 = \sqrt{\nu_0}$ where $\frac{1}{k_q} \left\| N_d - [\Phi_d \ 0]^T - \hat{f} \right\|^2 \leqslant \nu_0$, the inner boundary ξ_1 will shrink around the origin by increasing control gain k_q which also implies that the tracking error $e(t)$ can be driven to an arbitrarily small region around the origin. For $\xi_2 = \min(\rho_N^{-1}(\sqrt{2k_p}),\ \rho_{\Lambda i}^{-1}(\sqrt{\frac{2K_{di}}{m-1}}))$, the outer boundary can be expanded by increasing the control gain k_p and K_{di} $\forall\ i = 1, 2, \cdots, m-1$. Hence, the attraction region can be made arbitrarily large to include nearly any initial condition by increasing the control gains $k_p, k_q, K_{d1}, K_{d2}, \cdots, K_{d(m-1)}$ (i.e., a semi-global stability result).*

4.2 High Gain Observer

When $x(t)$ is the output of the system and is the only measurable state, we can only measure $e_1(t)$ given $x(t)$ and $x_d(t)$. Motivated by the result in [2], an estimate $\hat{z}(t) = \begin{bmatrix} \hat{e}_1^T & \hat{e}_2^T & \cdots & \hat{e}_n^T & \hat{r}^T \end{bmatrix}^T \in \mathbb{R}^{m(n+1)}$ for the auxiliary error signals $z(t) \in \mathbb{R}^{m(n+1)}$ can be obtained via the following HGO

$$\begin{aligned}
\dot{\hat{e}}_1 &= \hat{e}_2 - \hat{e}_1 + \frac{\alpha_1}{\varepsilon}(e_1 - \hat{e}_1) \\
\dot{\hat{e}}_2 &= \hat{e}_3 - \hat{e}_2 - \hat{e}_1 + \frac{\alpha_2}{\varepsilon^2}(e_1 - \hat{e}_1) \\
&\ \vdots \\
\dot{\hat{e}}_n &= \hat{r} - 2\hat{e}_n - \hat{e}_{n-1} + \frac{\alpha_n}{\varepsilon^n}(e_1 - \hat{e}_1) \\
\dot{\hat{r}} &= \frac{\alpha_{n+1}}{\varepsilon^{n+1}}(e_1 - \hat{e}_1)
\end{aligned} \tag{51}$$

where $\alpha_i \in \mathbb{R}, \forall\ i = 1, 2, \cdots, n+1$ are yet to be designed gain matrices, and ε is a positive scalar. To make for facile analysis in the singularly perturbed form, we further define scaled observer error $\eta(t) \triangleq \left[\begin{array}{cccc} \eta_1^T(t) & \eta_2^T(t) & \cdot\cdot & \eta_{n+1}^T(t) \end{array} \right]^T \in \mathbb{R}^{m(n+1)}$ as follows

$$\begin{aligned} \eta_i(t) &= \frac{1}{\varepsilon^{n+1-i}}(e_i - \hat{e}_i) \qquad i = 1, 2, \cdots, n \\ \eta_{n+1}(t) &= r - \hat{r}. \end{aligned} \tag{52}$$

Based on (52) and the definition of $\hat{z}(t)$ and $z(t)$, it is clear that

$$\hat{z} = z - D_\varepsilon \eta \tag{53}$$

where $D_\varepsilon \in \mathbb{R}^{(mn+m)\times(mn+m)}$ is defined as follows

$$D_\varepsilon \triangleq diag\left\{ \begin{array}{cccccc} \varepsilon^n I_m & \varepsilon^{n-1} I_m & \cdots & \varepsilon I_m & I_m \end{array} \right\}.$$

After differentiating (52) and taking advantage of the definition of (16) and the design of (51), we obtain the following dynamic observer error system:

$$\varepsilon \dot{\eta}(t) \quad = \quad A_0 \eta(t) + \varepsilon g \tag{54}$$

where $A_0 \in \mathbb{R}^{(mn+m)\times(mn+m)}$ and $g(t) \in \mathbb{R}^{mn+m}$ are defined as follows

$$A_0 \triangleq \begin{bmatrix} -\alpha_1 I_m & I_m & 0_m & \cdots & 0_m \\ -\alpha_2 I_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n I_m & 0_m & 0_m & \cdots & I_m \\ -\alpha_{n+1} I_m & 0_m & 0_m & \cdots & 0_m \end{bmatrix} \tag{55}$$

$$g = -\left[\begin{array}{ccccc} \eta_1^T & \varepsilon\eta_1^T + \eta_2^T & \cdot\cdot & \varepsilon\eta_{n-1}^T + 2\eta_n^T & -\dot{r}^T \end{array} \right]^T. \tag{56}$$

In (55), α_i, $\forall\ i = 1, 2, \cdots, n+1$ are chosen such that $A_0(\cdot)$ is Hurwitz. The boundary-layer system $\dfrac{d\eta(\tau)}{d\tau} = A_0\eta(\tau)$ (obtained by applying a change of variable $\tau = t/\varepsilon$ and then setting $\varepsilon = 0$) induces a Lyapunov function $W(\eta) = \eta^T P_0 \eta$ that satisfies the following properties

$$\left\{ \begin{array}{c} \lambda_{\min}(P_0)\left\|\eta\right\|^2 \leqslant W(\eta) \leqslant \lambda_{\max}(P_0)\left\|\eta\right\|^2, \\ \dot{W} = \frac{\partial W}{\partial \eta}\dot{\eta} \leqslant -\left\|\eta\right\|^2, \\ \left\|\frac{\partial W}{\partial \eta}\right\| \leqslant 2\left\|P_0\right\|\left\|\eta\right\|, \quad \left\|P_0\right\| \triangleq \lambda_{\max}(P_0). \end{array} \right. \tag{57}$$

In the above equation, $P_0 \in \mathbb{R}^{(mn+m)\times(mn+m)}$ is a p.d. matrix that satisfies $P_0 A_0 + A_0^T P_0 = -I_{m(n+1)}$. From (57), it is clear that $\eta(t) = 0$ is a globally exponentially stable equilibrium of the boundary-layer system.

4.3 OFB Control Law

From (54), it is clear that the solution of $\eta(t)$ contains terms like $\frac{1}{\varepsilon}e^{-\omega t/\varepsilon}$ for some $\omega > 0$ (*i.e.*, the amplitude of $\eta(t)$ is $O(\frac{1}{\varepsilon})$). This so-called peaking phenomenon can possibly drive an output feedback controller (that is based on a naive application of the separation principle) out of its region of attraction, thereby causing instability. To suppress the amplitude peaking of $\eta(t)$, the full-state control design of (30) is modified to an output feedback saturated control as follows [2]

$$u(t) = \int_{t_0}^{t} D^{-1} sat\{(K + I_m)(\hat{r}(\tau))\} d\tau + \int_{t_0}^{t} D^{-1}\hat{f}(\tau) d\tau \tag{58}$$

where the time derivative of (58) is saturated in the variable $\hat{r}(t)$ outside a compact set $\mathcal{D}_c \triangleq \{z(t) \in \mathbb{R}^{m(n+1)} \mid V(t) \leqslant c\}$ of the region of attraction $\mathcal{D}_z$ for the full-state feedback system. After substituting (58) into (24), the following expression can be obtained

$$\begin{aligned} \dot{r} &\triangleq \phi(z, D_\varepsilon\eta, t) \\ &= M^{-1}\left[-\frac{1}{2}\dot{M}r - e_{n+1} - sat\{(K+I_m)\hat{r}\} - \hat{f} \right. \\ &\quad \left. -\bar{U}D^{-1}(sat\{(K+I_m)\hat{r}\} + \hat{f}) + N_d + \tilde{N}\right]. \end{aligned} \tag{59}$$

After combining (27), (54), (56), and (59), we obtain the following closed-loop error dynamics for (3)

$$\begin{aligned} \dot{z}(t) &\triangleq f_r(z(t), D_\varepsilon\eta(t), t) \\ &= Az + B\phi(z, D_\varepsilon\eta, t) \\ \varepsilon\dot{\eta}(t) &= A_0\eta(t) + \varepsilon g(z(t), D_\varepsilon\eta(t)). \end{aligned} \tag{60}$$

From the global boundedness of $\hat{f}(t)$, the global saturation of the estimated states, and the boundedness of $z(t)$ inside $\mathcal{D}_c$, it is easy to see that $||f_r(z(t), \eta(t))|| \leqslant k_1 \; \forall \, z(t) \in \mathcal{D}_c$ and $\forall \, \eta(t) \in \mathbb{R}^{m(n+1)}$. Here, k_1 is a positive constant independent of ε.

4.4 OFB Stability Results

To directly construct a Lyapunov function to prove the stability of (60) is non-trivial owing to the augmented set of dynamics as well as saturation introduced in the OFB design. Instead, the proof is split into multiple steps (as similarly done in [2]) to reduce the complexity at each step. First, we prove the existence of a positively invariant set for the solutions of (60). In the second step, we regain the boundedness of solutions of (60) provided the trajectory $(z(t), \hat{z}(t))$ starts inside a compact subset of $\mathcal{D}_z \times \mathbb{R}^{m(n+1)}$. We are then able to show that the HGO constant ε can be chosen small enough to ensure that any trajectory $(z(t), \hat{z}(t))$ starting in the aforementioned compact subset results in $\eta(t)$ entering the invariant set (from step 1) before $z(t)$ can escape. In the third step, we recover semiglobal ultimate boundedness for (60).

We define $\mathcal{Z}$ to be any compact set in the interior of $\mathcal{D}_z$ such that $\mathcal{Z} \subset \mathcal{D}_c \subset \mathcal{D}_z$. We further define $\mathcal{H}$ to be any compact set in the interior of $\mathbb{R}^{m(n+1)}$. Let $\mathcal{D}_\varepsilon \triangleq \{\eta(t) \in \mathbb{R}^{m(n+1)} \mid W(\eta(t)) \leqslant \varrho\varepsilon^2\}$ be a compact set where $W(t)$ was defined in (57), ϱ is a positive constant that is yet to be selected, while ε is the HGO constant. In our analysis of the stability of (60), we will consider trajectories for closed-loop solutions $(z(t), \hat{z}(t))$ that start inside $\mathcal{Z} \times \mathcal{H}$. With these definitions, we present the stability analysis for the system of (60) via the following theorems:

Theorem 2 ***(Invariant Set Theorem)*** *Given $\Sigma \triangleq \mathcal{D}_c \times \mathcal{D}_\varepsilon$, there exists an $\bar{\varepsilon}_1 > 0$ such that $\forall\varepsilon \in (0, \bar{\varepsilon}_1]$, Σ is a positively invariant set for the trajectory $(z(t), \eta(t))$.*

Proof. Define a composite Lyapunov function $V_c(z, \eta)$ as follows

$$V_c = V(z) + W(\eta) \tag{61}$$

where $V(z)$ and $W(\eta)$ have been previously defined in (38) and (57). The derivative of (61) along the trajectory of (60) can be obtained as follows

$$\dot{V}_c = \dot{V} + \dot{W} \tag{62}$$

where $\dot{V}(t)$ and $\dot{W}(t)$ can be expressed as follows

$$\dot{V} = \frac{\partial V(z)}{\partial z} f_r(z(t), D_\epsilon\eta(t), t) \tag{63}$$

$$\dot{W} = \frac{\partial W(\eta)}{\partial \eta}\left(A_0\eta(t)/\varepsilon + g(z(t), D_\epsilon\eta(t), t)\right). \tag{64}$$

Our aim here is to show that $\dot{V}_c|_{\partial\Sigma} \leqslant 0$ where the notation $\partial\Sigma$ denotes the boundary of the compact set Σ. From the analysis in Appendix B, the term $\dot{V}(t)$ of (62) can be upperbounded as follows

$$\dot{V} \leqslant -\lambda_3 \|z\|^2 + \nu_0 + \kappa_1\kappa_2 \|\eta\|. \tag{65}$$

After defining $\varsigma_1 \triangleq \kappa_1\kappa_2\sqrt{\frac{\varrho}{\lambda_{\min}(P_0)}}$ and $\varsigma_2(c) \triangleq \frac{1}{2}\lambda_3\left[\gamma_2^{-1}(c)\right]^2$, we can utilize the definition of $\mathcal{D}_c$ to upperbound $\dot{V}(t)$ on the boundary $\partial\Sigma$ as follows

$$\dot{V}(t)|_{\partial\Sigma} \leqslant -\left[\varsigma_2(c) - \nu_0\right] - \left[\varsigma_2(c) - \varsigma_1\varepsilon\right] \tag{66}$$

where we have utilized the definition of $\mathcal{D}_\varepsilon$ to upperbound $\|\eta(t)\|$ as well as the fact that $V(z)$ is upperbounded by $\gamma_2(\|z\|)$ (See (46) and (106) in Appendix A). If one selects $c > \gamma_2\left(\sqrt{2\nu_0\lambda_3^{-1}}\right)$, then a selection of $\varepsilon_1 = \varsigma_1^{-1}\varsigma_2$ ensures that $\dot{V}(t)|_{\partial\Sigma} \leqslant 0 \ \forall\ 0 < \varepsilon \leqslant \varepsilon_1$. The term $\dot{W}(t)$ of (62) can be upperbounded as follows

$$\dot{W} \leqslant -\frac{1}{\varepsilon}\|\eta\|^2 + 2\|P_0\|\|\eta\|\|g\| \tag{67}$$

where we have utilized the fact that $P_0A_0 + A_0^TP_0 = -I_{m(n+1)}$. Given the boundedness of $z(t)$ inside $\mathcal{D}_c$, the linear dependence of $g(t)$ on $\eta(t)$, and the fact that ε is strictly less than 1, the definition of (56) suggests that $g(t)$ can be upperbounded as $\|g\| \leqslant \sigma_1\|\eta\| + \sigma_2 \ \forall\ z(t) \in \mathcal{D}_c$ and $\forall\ \eta(t) \in \mathbb{R}^{m(n+1)}$; here, $\sigma_1, \sigma_2 > 0$ are constants independent of ε (See Appendix B for explicit definitions for σ_1 and σ_2). Utilization of this upperbound given in (67) allows us to formulate the following upperbound on $\dot{W}$

$$\dot{W} \leqslant -\left(\frac{1}{\varepsilon} - 2\|P_0\|\sigma_1\right)\|\eta\|^2 + 2\|P_0\|\|\eta\|\sigma_2. \tag{68}$$

If one chooses $\varepsilon_2 < \frac{1}{2\|P_0\|\sigma_1}$, then $\forall\ 0 < \varepsilon \leqslant \varepsilon_2$, a choice of $\varrho = \frac{16\sigma_2^2\|P_0\|^3}{(1-2\|P_0\|\sigma_1\varepsilon)^2}$ ensures that $\dot{W}|_{\partial\Sigma} = \dot{W}|_{\partial\mathcal{D}_\varepsilon} \leqslant 0$. Finally, if one defines $\bar{\varepsilon}_1 = \min\{1, \varepsilon_1, \varepsilon_2\}$, then (66) and (68) imply that $\Sigma = \mathcal{D}_c \times \mathcal{D}_\varepsilon$ is an invariant set for $\forall\varepsilon \in (0, \bar{\varepsilon}_1]$. ■

Theorem 3 ***(Boundedness Theorem)*** *There exists an $\bar{\varepsilon}_2 \leqslant \bar{\varepsilon}_1$ such that $\forall\varepsilon \in (0, \bar{\varepsilon}_2]$, any trajectory $(z(t), \hat{z}(t))$ that starts inside $\mathcal{Z} \times \mathcal{H}$ is bounded for all time.*

Proof. Given the boundedness of $e(0)$ and $\hat{z}(0)$, the definition of (52) implies that $\|\eta(0)\| \leqslant k\varepsilon^{-n}$ where $k > 0$ is a positive constant of analysis while n is the order of the system of (3). From the boundedness assertion on $f_r(z(t), \eta(t))$ of (60) in the set $\mathcal{D}_c\times\ \mathbb{R}^{m(n+1)}$ made in Section 4.3, and the closed-loop system definition of (60), it is easy to see that $z(t)$ satisfies the following linear time growth upperbound in the compact set $\mathcal{D}_c$

$$\|z(t) - z(0)\| \leqslant k_1 t \tag{69}$$

where $k_1 > 0$ is a positive constant of analysis. Thus, there is a time T_c independent of ε such that $z(t) \in \mathcal{D}_c\ \forall\ t \in [0, T_c]$. Our aim now is to show that one can pick an ε such that if $\eta_i(t)$ starts outside the invariant set Σ, it can be made to enter the invariant set before $z(t)$ can exit $\mathcal{D}_c$ – the key idea to be exploited here is the growth bound established in (69). Proving this previous assertion would imply that the solution $(z(t), \eta(t))$ is in the invariant set Σ at some time T_ε which means that it will stay there $\forall\ t \in [T_\varepsilon, \infty)$. Outside the invariant set, $W(\eta) \geq \varrho\varepsilon^2 = \frac{16\sigma_2^2\|P_0\|^3\varepsilon^2}{(1-2\|P_0\|\sigma_1\varepsilon)^2}$ which implies from (57) that $\|\eta\| \geq \frac{4\sigma_2\|P_0\|\varepsilon}{(1-2\|P_0\|\sigma_1\varepsilon)}$. From (68), one can upperbound $\dot{W}(\eta)$ as follows

$$\begin{aligned}\dot{W} \ &\leqslant\ -\frac{1}{2}\left(\tfrac{1}{\varepsilon} - 2\|P_0\|\sigma_1\right)\|\eta\|^2 \\ &\quad + \left[-\tfrac{1}{2}\left(\tfrac{1}{\varepsilon} - 2\|P_0\|\sigma_1\right)\|\eta\| + 2\|P_0\|\sigma_2\right]\|\eta\|.\end{aligned}$$

The bracketed term in the above equation can be upperbounded by zero owing to the lower bound established on $\|\eta\|$ above. If one defines ($\forall\ 0 < \varepsilon \leqslant \bar{\varepsilon}_1$) a strictly increasing function $\varsigma_3(\varepsilon) = \varepsilon\left(1 - 2\|P_0\|\sigma_1\varepsilon\right)^{-1}$, then $\dot{W}(\eta) \leqslant -\frac{W(\eta)}{2\|P_0\|\varsigma_3(\varepsilon)} \leqslant 0\ \forall\ 0 < \varepsilon \leqslant \bar{\varepsilon}_1$ – by solving this differential inequality, an upperbound for $W(\eta(t))$ can be obtained as follows

$$W(\eta(t)) \leqslant \frac{k^2\|P_0\|}{\varepsilon^{2n}} \exp\left(-\frac{t}{2\|P_0\|\varsigma_3(\varepsilon)}\right) \tag{70}$$

where k is defined above in the $\|\eta(0)\|$ upperbound. Since $\lim_{t\to\infty} W(\eta(t)) = 0$ from above, there exists a time T_ε for $W(\eta(t))$ to reach inside the region defined by $W(\eta(t)) < \varrho\varepsilon^2$. So, we can find an $0 < \bar{\varepsilon}_2 < \bar{\varepsilon}_1$ small enough so that $W(\eta(t))$ enters $\mathcal{D}_\varepsilon$ at a time $T_\varepsilon \triangleq 2\|P_0\|\varsigma_3(\varepsilon)\ln\left(\frac{k^2\|P_0\|}{\varrho\varepsilon^{2(n+1)}}\right) \leqslant T_c/2$ $\forall\ \varepsilon \in (0, \bar{\varepsilon}_2]$. Since $\eta(t)$ enters the invariant set Σ in less than half the time it takes for $z(t)$ to exit Σ, this implies that $(z(t), \eta(t))$ enters Σ during $[0, T_\varepsilon]$ and hence $z(t), \eta(t) \in \mathcal{L}_\infty$ for all times $t \geq T_\varepsilon$. For $t \in [0, T_\varepsilon]$, the trajectory $(z(t), \eta(t))$ is bounded by virtue of (69) and (70). Thus, we have proved that all closed-loop trajectories $(z(t), \hat{z}(t))$ starting in $\mathcal{Z} \times \mathcal{H}$ are bounded for all time. ■

Theorem 4 *(**Ultimately Boundedness Theorem**) Given any solution $(z(t), \hat{z}(t))$ that starts in $\mathcal{Z} \times \mathcal{H}$ and given any small $\delta > \sqrt{2\lambda_3^{-1}\nu_0}$, there exists an $0 < \bar{\varepsilon}_3(\delta) \leqslant \bar{\varepsilon}_2$ and a $T_1(\delta) > 0$ such that $\|z(t)\| \leqslant \delta$ and $\|\hat{z}(t)\| \leqslant 2\delta\ \forall\ t \geq T_1$ and $\forall\ \varepsilon \in (0, \bar{\varepsilon}_3]$.*

Proof. From (70), $\lim_{\varepsilon\to 0} W(\eta(t)) = 0\ \forall\ 0 < \varepsilon \leqslant \bar{\varepsilon}_2$; hence, for any given small value δ, we can find $\varepsilon_3 = \varepsilon_3(\delta) \leqslant \bar{\varepsilon}_2$ such that for $\forall\varepsilon \in (0, \varepsilon_3]$, the following upperbound can be defined

$$\|\eta(t)\| \leqslant \delta \quad \forall\ t \geq T_{\varepsilon 3} = T_{\varepsilon 3}(\delta). \tag{71}$$

Inside the invariant set Σ, one can upperbound the time derivative of $V(t)$ of (38) as follows

$$\dot{V} \leqslant -\lambda_3\|z\|^2 + \nu_0 + k_3\varepsilon \tag{72}$$

where we have utilized the upperbound of (65) and the definition of the compact set $\mathcal{D}_\varepsilon$. Define a compact set $\mathcal{D}_\mu \triangleq \left\{z \in \mathbb{R}^{m(n+1)} :\ \|z(t)\| \leqslant \mu(\varepsilon) \triangleq \sqrt{\frac{2(\nu_0 + k_3\varepsilon)}{\lambda_3}}\right\}$; then $\|z(t)\| \notin \mathcal{D}_\mu$ implies that $\dot{V}(t)$ can be upperbounded as

$$\dot{V}(t) \leqslant -\frac{\lambda_3}{2}\|z\|^2 - \frac{\lambda_3}{2}\left(\|z\|^2 - \frac{2(\nu_0 + L_1\varepsilon)}{\lambda_3}\right) \leqslant -\frac{\lambda_3}{2}\|z\|^2 \leqslant 0 \tag{73}$$

which implies that $V(z(t))$ is decreasing outside $\mathcal{D}_\mu$. Define a compact set $\mathcal{D}_V \triangleq \{V(z) \leqslant \upsilon_0(\varepsilon) \triangleq \max_{\mathcal{D}_\mu} V(z)\}$; clearly, $\mathcal{D}_\mu \subset \mathcal{D}_V$ since $\upsilon_0(\varepsilon)$ is defined to be a non-decreasing scalar function. We know that $\mathcal{D}_\mu$ and $\mathcal{D}_V$ lie inside $\mathcal{D}_c\ \forall\ \varepsilon \in (0, \bar{\varepsilon}_2]$. Given any $\delta > \sqrt{2\lambda_3^{-1}\nu_0}$, one can pick an $\varepsilon_4 = \varepsilon_4(\delta) \leqslant \bar{\varepsilon}_2$ such that $\mathcal{D}_V \subset \mathcal{D}_\delta \triangleq \left\{z \in \mathbb{R}^{m(n+1)} : \|z(t)\| \leqslant \delta\right\}$. From these assertions and the upperbound of (73), it is obvious that the set $\Sigma_{ub} \triangleq \mathcal{D}_V \times \mathcal{D}_\varepsilon$ is positively invariant. Moreover, any trajectory in Σ will enter Σ_{ub} in a finite time $T_{\varepsilon 4} = T_{\varepsilon 4}(\delta)$ for $\forall\varepsilon \in (0, \varepsilon_4]$. Defining $\bar{\varepsilon}_3 = \min\{\varepsilon_3, \varepsilon_4\}$ and $T_u = \max\{T_{\varepsilon 3}, T_{\varepsilon 4}\}$, we obtain

$$\|\hat{z}(t)\| \leqslant \|z(t)\| + \|\eta(t)\| \leqslant 2\delta \qquad \text{for } \forall\varepsilon \in (0, \bar{\varepsilon}_3] \text{ and } \forall t \geq T_u \tag{74}$$

where we have utilized the definition of (52) and the fact that $\varepsilon^k < 1\ \forall\ k = 0, 1, \cdot\cdot, n$. Thus $(z(t), \hat{z}(t))$ starting in $\mathcal{Z} \times \mathcal{H}$ are ultimately bounded. ■

Remark 3 *Since $\lim_{k_q\to\infty} \nu_0 = 0$ (See Remark 2), ν_0 can be made arbitrarily small with k_q large enough. This allows us to drive δ to any arbitrarily small value thus ensuring that the ultimate bound on $(z(t), \eta(t))$ can be made arbitrarily small.*

5 Adaptive Output Feedback Control

Now we consider a sub-class of MIMO nonlinear systems having the form

$$x^{(n)} = h(x, \dot{x}, \cdots, x^{(n-1)}, \theta_1) + G(x, \dot{x}, \cdots, x^{(n-2)}, \theta_2)u. \tag{75}$$

In this section, we redefine $\mathbf{x} \triangleq [x^T \ \dot{x}^T \ ... \ (x^{(n-2)})^T]^T \in \mathbb{R}^{mn-m}$. We assume $G(\mathbf{x}, \theta_2)$ is a real matrix with nonzero leading principal minors. Invoking Lemma 1, (75) can be rewritten as

$$M(\mathbf{x},\theta_2)x^{(n)} = f(\mathbf{x},x^{(n-1)}, \theta_1, \theta_2) + DU(\mathbf{x}, \theta_2)u \tag{76}$$

where $M(\mathbf{x},\theta_2) \triangleq S^{-1}(\mathbf{x},\theta_2) \in \mathbb{R}^{m\times m}$ is symmetric and positive definite, $f(\mathbf{x},x^{(n-1)}, \theta_1, \theta_2) \in \mathbb{R}^m$

$$f(\mathbf{x},x^{(n-1)}, \theta_1, \theta_2) \triangleq S^{-1}(\mathbf{x},\theta_2)h(\mathbf{x},x^{(n-1)}, \theta_1)$$

and $S(\mathbf{x},\theta_2), U(\mathbf{x}, \theta_2), D$ are defined as in Lemma 1.

We define the error signal $r(t) \in \mathbb{R}^m$ as follows

$$r = e_n + e_{n-1}. \tag{77}$$

where $e_i(t) \in \mathbb{R}^m \ \forall \ i = 1, 2, \cdots, n$ are defined in (16). After taking the time derivative of (77), and then substituting from the derivative of (17) for $i = n$, we have

$$\dot{r} = M^{-1}\left(-\frac{1}{2}\dot{M}r + Y\theta - Du - e_{n-1}\right) \tag{78}$$

where (76) was utilized and $Y(\cdot)\theta \in \mathbb{R}^m$ is a linear parameterization that is defined as follows

$$\begin{aligned} Y(\cdot)\theta \quad = \quad & M\left(x_d^{(n)} + \sum_{j=0}^{n-2} a_{n,j}e_1^{(j+1)} + \dot{e}_{n-1}\right) \\ & -f(\mathbf{x},x^{(n-1)}, \theta_1, \theta_2) + \frac{1}{2}\dot{M}r \\ & -D\left(U(\mathbf{x}, \theta_2) - I_m\right)u + e_{n-1}. \end{aligned} \tag{79}$$

In (79), θ is the unknown constant vector of system parameters, and $Y(\cdot)$ is a measurable regression matrix. By defining $z(t) \triangleq [\ e_1^T \quad e_2^T \quad ... \quad e_{n-1}^T \quad r^T \ ]^T \in \mathbb{R}^{mn}$, we can write a compact form of the open-loop dynamics for $z(t)$ from (16) as follows

$$\dot{z}(t) = Az + B\dot{r} \tag{80}$$

where $A \in \mathbb{R}^{mn\times mn}$, $B \in \mathbb{R}^{mn\times m}$ are defined similar to (28) and (29).

5.1 State Feedback Control Law

In this section, we assume full state feedback, *i.e.*, $x^{(i)}(t)$, $i = 0, ..., n-1$ in (3) are measurable. We propose the following state feedback adaptive control law

$$u = D^{-1}\left(Y\hat{\theta} + Kr + k\left\|Y\right\|^2 r\right) \tag{81}$$

where $K \in \mathbb{R}^{m\times m}$ is a positive-definite, diagonal matrix, $k \in \mathbb{R}$ is a control gain to be designed, and the parameter adaptation law for $\hat{\theta}$ is designed as follows

$$\dot{\hat{\theta}} = \text{Proj}\left\{\Gamma Y^T r, \hat{\theta}\right\} \tag{82}$$

where $\Gamma \in \mathbb{R}^{m \times m}$ is a constant diagonal, positive definite matrix and $\operatorname{Proj}\{\cdot\}$ is a parameter projection operator [22]. The projection algorithm is employed in order to bound $\hat{\theta}(t)$ to a known compact set Ω_ε in the sense that

$$\hat{\theta}(t) \in \Omega_\varepsilon \qquad \forall t > 0 \quad \text{if } \hat{\theta}(0) \in \Omega_\varepsilon.$$

After applying (81) to (78), and then multiplying both sides by $M(\mathbf{x},\theta_2)$, we have

$$M\dot{r} = -\frac{1}{2}\dot{M}r + Y\tilde{\theta} - e_{n-1} - Kr - k\|Y\|^2 r \tag{83}$$

where $\tilde{\theta}(t)$ is defined as follows

$$\tilde{\theta}(t) \triangleq \theta - \hat{\theta}. \tag{84}$$

To analyze the stability of the proposed control law, a non-negative function $V_1(t) \in \mathbb{R}$ is defined as follows

$$V_1 = \frac{1}{2}\sum_{i=1}^{n-1} e_i^T e_i + \frac{1}{2} r^T M r. \tag{85}$$

Note that (85) can be bounded as

$$\lambda_1 \|z\|^2 \leqslant V_1 \leqslant \lambda_2(\|z\|) \|z\|^2 \tag{86}$$

where $z(t)$ was defined in (80), and λ_1 and $\lambda_2(\|z\|)$ are defined as follows

$$\lambda_1 \triangleq \frac{1}{2}\min(1, \underline{m}) \quad \lambda_2(\|z\|) \triangleq \frac{1}{2}\max(1, \bar{m}(\cdot)) \tag{87}$$

and we have taken advantage of (13). After taking the time derivative of (85) and substituting from (20), we have

$$\dot{V}_1 \leqslant -\sum_{i=1}^{n-1} e_i^T e_i - r^T K r + \|r\| \cdot \|Y\| \cdot \left\|\tilde{\theta}\right\| - k\|Y\|^2 \cdot \|r\|^2 \tag{88}$$

where (16) and (18) were utilized. After applying the nonlinear damping argument in [13], (88) can be rewritten as follows

$$\dot{V}_1 \leqslant -\lambda_3 \|z\|^2 + \varepsilon/k \tag{89}$$

where $\lambda_3 \triangleq \min\{1, \lambda_{\min}(K)\}$, $\lambda_{\min}(K)$ denotes the minimum eigenvalue of K, and $\varepsilon \in \mathbb{R}$ is a positive constant defined as follows

$$\varepsilon = \sup_{\hat{\theta} \in \Omega_\varepsilon} \left\|\tilde{\theta}\right\|^2. \tag{90}$$

Similar to the proof of Theorem 1, Lemma 2 can be applied to (85) and (89) to prove the uniformly ultimately bounded tracking result.

5.2 High Gain Observer

Similar to Section 4.2, an estimate $\hat{z}(t) = \begin{bmatrix} \hat{e}_1^T & \hat{e}_2^T & \cdots & \hat{e}_{n-1}^T & \hat{r}^T \end{bmatrix}^T \in \mathbb{R}^{mn}$ for the auxiliary error signals $z(t)$ can be obtained via the following HGO

$$\begin{aligned}
\dot{\hat{e}}_1 &= \hat{e}_2 - \hat{e}_1 + \frac{\alpha_1}{\varepsilon}(e_1 - \hat{e}_1) \\
\dot{\hat{e}}_2 &= \hat{e}_3 - \hat{e}_2 - \hat{e}_1 + \frac{\alpha_2}{\varepsilon^2}(e_1 - \hat{e}_1) \\
&\vdots \\
\dot{\hat{e}}_{n-1} &= \hat{r} - 2\hat{e}_{n-1} - \hat{e}_{n-2} + \frac{\alpha_{n-1}}{\varepsilon^{n-1}}(e_1 - \hat{e}_1) \\
\dot{\hat{r}} &= \frac{\alpha_n}{\varepsilon^n}(e_1 - \hat{e}_1).
\end{aligned} \tag{91}$$

A scaled observer error $\eta(t) \triangleq \begin{bmatrix} \eta_1^T & \eta_2^T & \cdots & \eta_n^T \end{bmatrix} \in \mathbb{R}^{mn}$ is defined as follows

$$\begin{aligned} \eta_i(t) &= \frac{1}{\varepsilon^{n-i}}(e_i - \hat{e}_i) \qquad i = 1, 2, \cdots, n-1 \\ \eta_n(t) &= r - \hat{r}. \end{aligned} \tag{92}$$

After differentiating (92) and taking advantage of the definition of (16) and the design of (91), we obtain the compact form for observer error system of $\eta(t)$ as follows

$$\varepsilon\dot{\eta}(t) = A_0\eta(t) + \varepsilon g \tag{93}$$

where $A_0 \in \mathbb{R}^{mn\times mn}$ and $g(\cdot) \in \mathbb{R}^{mn}$ defined as follows

$$A_0 \triangleq \begin{bmatrix} -\alpha_1 I_m & I_m & 0_m & \cdots & 0_m \\ -\alpha_2 I_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n-1} I_m & 0_m & 0_m & \cdots & I_m \\ -\alpha_n I_m & 0_m & 0_m & \cdots & 0_m \end{bmatrix} \tag{94}$$

$$g \triangleq -\begin{bmatrix} \eta_1^T & \varepsilon\eta_1^T + \eta_2^T & \cdot\cdot & \varepsilon\eta_{n-2}^T + 2\eta_{n-1}^T & -\dot{r}^T \end{bmatrix}^T. \tag{95}$$

5.3 OFB Control Law

Similar to Section 4.3, the full-state control design of (81) is modified to an output feedback saturated control as follows

$$u(t) = D^{-1}\left[\hat{Y}\hat{\theta} + K\text{sat}\{\hat{r}\} + k\left\|\hat{Y}\right\|^2 \text{sat}\{\hat{r}\}\right] \tag{96}$$

where $\hat{Y}(\cdot)$ is the same measurable regression matrix defined in (79) with respect to sat$\{\hat{z}(t)\}$ instead of $z(t)$, and K, k are defined in (81). In (96), $\hat{\theta}(t)$ is generated by the projection algorithm in (82) with respect to $\hat{Y}(t)$ and $\hat{r}(t)$ as follows

$$\dot{\hat{\theta}} = \text{Proj}\left\{\Gamma\hat{Y}^T\hat{r}, \hat{\theta}\right\}. \tag{97}$$

Since we use the same projection algorithm as in (82), it is clear that $\hat{\theta}(t) \in \Omega_\varepsilon$ if $\hat{\theta}(0) \in \Omega_\varepsilon$. The saturation of input is employed outside a compact set $\mathcal{D}_c \triangleq \{z(t) \in \mathbb{R}^{mn} \mid V(t) \leqslant c\}$ of the region of attraction domain $\mathcal{D}_z$. After substituting (96) into (19),

$$\begin{aligned} \dot{r} &\triangleq \phi(z, D_\epsilon\eta, t) \\ &= M^{-1}\left[-\frac{1}{2}\dot{M}r + Y\theta - e_{n-1} + \hat{Y}(\text{sat}\{\hat{z}\})\hat{\theta}\right. \\ &\qquad \left. + K\text{sat}\{\hat{r}\} + k\left\|\hat{Y}\right\|^2 \text{sat}\{\hat{r}\}\right]. \end{aligned} \tag{98}$$

After combining (80), (93-95), and (98), we can obtain the same compact form as (60) for closed-loop error dynamics for (3). Since we obtain the same compact form for the closed-loop system as in (60), it is easy to follow the steps in Section 4.4 to prove the uniformly ultimately bounded tracking result.

6 Simulation

In this section, the results of a numerical simulation are presented in order to demonstrate the performance of the proposed control law. The following two DOF system was considered [25]

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{q}_2 & -h\left(\dot{q}_1 + \dot{q}_2\right) \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{99}$$

where $q_i(t)$ denotes the i^{th} DOF position,

$$\begin{cases} H_{11} = a_1 + 2a_3 \cos q_2 + 2a_4 \sin q_2, \\ H_{12} = H_{21} = a_2 + a_3 \cos q_2 + a_4 \sin q_2, \\ H_{22} = a_2, h = a_3 \sin q_2 - a_4 \sin q_2, \end{cases} \tag{100}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \alpha(q_1, q_2) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{101}$$

$a_1 = 4.42$, $a_2 = 0.97$, $a_3 = 1.04$, and $a_4 = 0.6$. In (101), $u_1(t), u_2(t)$ are the control inputs, and $\alpha = H_{11}H_{22} - H_{12}H_{21} \in \mathbb{R}$ can be taken as some sort of environment related factors such as shown in [1, 4] or items shown in the input-output module [8, 9, 16].

The control objective is to make $q(t) = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}^T$ track the following reference trajectory

$$q_d(t) = \begin{bmatrix} 30\sin(t)\left(1 - \exp(-0.3t^3)\right) \\ 45\sin(t)\left(1 - \exp(-0.3t^3)\right) \end{bmatrix} \text{deg}; \tag{102}$$

hence, the system tracking error was defined as $e_1(t) = q_d(t) - q(t)$. The initial conditions were set to $q(0) = 0.06$ [deg] and $\dot{q}\,(0) = 0$ [deg.s^{-1}].

Toward designing the feedforward function $\hat{f}\,(t)$, neural-network (NN) based compensation was utilized. We employed a radial basis neural network (RBNN) [15] to approximate the unknown nonlinear function $N_d(\cdot)$ in (23). The RBNN is comprised of a layer of radial basis activation functions with $p = 10$ number of neurons and the output of the neural network system is designed as follows

$$\hat{f}(t) = \hat{W}(t)^T \sigma(\bar{V}^T \chi_1 + V_0) \tag{103}$$

where $\chi_1 = [\ q_d^T \ \ \dot{q}_d^T \ \ \ddot{q}_d^T \ \ \dddot{q}_d^T\ ]^T \in \mathbb{R}^8$ denotes a bounded vector of the desired trajectory and its derivatives, $\bar{V} \in \mathbb{R}^{10\times 8}$ is set to constant uniformly distributed random values between -1 and 1, $\hat{W}(t) \in \mathbb{R}^{10\times 2}$ is an estimate of the ideal weight gain matrix W, while $V_0 \in \mathbb{R}^{10}$ is set to constant random base values uniformly distributed between $V_{\max}$ and $V_{\min}$; in the simulation, we pick $V_{\max} = 20$ and $V_{\min} = -20$ [15]. The sigmoid function $\sigma(s) = \dfrac{1}{1 + \exp(-s)}$ is utilized as activation function. We proposed the following weight tuning law for $\hat{W}(t)$

$$\begin{cases} \dot{\hat{W}} = -\varkappa_1 \hat{W} + \Gamma_1 \sigma(\bar{V}^T \chi_1) \text{sat}(\hat{e}_2 + \zeta_1) \\ \dot{\zeta}_1 = \frac{1}{\varkappa_2}(-\zeta_2 + \hat{e}_2), \dot{\zeta}_2 = \frac{1}{\varkappa_2}(-\zeta_2 + \hat{e}_2), \end{cases} \tag{104}$$

where $\varkappa_1$, $\varkappa_2$ are some small positive constants, $\Gamma_1 \in \mathbb{R}^{10\times 10}$ is a diagonal, positive definite matrix, $\text{sat}(\cdot) \in \mathbb{R}^2$ is the standard saturation function, while $\zeta_1, \zeta_2 \in \mathbb{R}^2$ are auxiliary filter signals. It is not difficult to check that $\hat{W}(t) \in \mathcal{L}_\infty$. The weight tuning gains in (104) were tuned by trial-and-error until a good tracking performance was achieved. This resulted in the following gain values

$$\begin{cases} \varkappa_1 = 1e - 2, \quad \varkappa_2 = 1e - 4, \\ \Gamma_1 = \text{diag}(100, 220, 200, 120, 150, 150, 240, 200, 160, 280). \end{cases} \tag{105}$$

The control gains in (30) were chosen to be $K = diag\{5, 3\}$. In the OFB scenario, we select the HGO as defined in (51) with parameters setting: $n = 2, \alpha_1 = 9.1e-1$, $\alpha_2 = 1.5e-1$, and $\alpha_3 = 1.5e-2$. The simulation was carried out both with and without the feedforward term $\hat{f}(t)$ in the control input. No serious peaking phenomenon was observed during the simulation with the HGO constant chosen between $\varepsilon = 1e-3$ and $\varepsilon = 1e-4$ and the maximum and minimum values for the saturation $\hat{z}(t)$ set at ± 100. Figures 1-6 show the comparisons of the tracking errors, control torques, and HGO estimation errors under four different scenarios. The simulation results demonstrate that the OFB controller recovers the performance of the FSFB controller. It is to be noted here that the magnitude of the tracking error $e_1(t)$ can be further reduced by increasing control gain matrix K. As seen in the figures, a substantial reduction in the tracking errors is observed when feedforward compensation is introduced.

7 Conclusion

This paper considered the output feedback tracking control problem for a class of MIMO nonlinear systems for which the input gain matrix is full rank but not necessarily symmetric. By utilizing the decomposition property of the input gain matrix, a continuous FSFB control strategy was proposed to compensate for uncertainty in the nonlinear functions associated with the system dynamic model and to ensure nearly semi-global uniformly ultimately bounded tracking under limited smoothness restrictions on uncertain system nonlinearities. Furthermore, a high-gain observer was introduced in order to estimate the derivatives of the system output to realize OFB tracking control of the system. Rigorous analysis was provided to prove the near SGUUB stability of the OFB control design. NN based approximation techniques were employed in the control design to reduce the amount of control effort requirement. Simulation results were presented to provide a clear, quantitative view of both the FSFB and OFB control performance. An extension was also provided to show the design of an adaptive output feedback controller for a subclass of systems.

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Appendix A

Lemma 2 *Assume that for any bounded set $\mathcal{D} \subset \mathbb{R}^n$, the function F maps $\mathcal{D} \times \mathbb{R}_{\geq 0}$ into bounded sets in $\mathbb{R}^n$, and $\gamma_1, \gamma_2, \gamma_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are class $\mathcal{K}_\infty$ functions. If there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying*

$$\gamma_1(\|x(t)\|) \leqslant V(x(t), t)) \leqslant \gamma_2(\|x(t)\|) \tag{106}$$

$\forall\ (x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, and positive constants ξ_1, ξ_2 satisfying $\xi_2 > (\gamma_1^{-1} \circ \gamma_2)(\xi_1)$ such that, along the trajectory of $\dot{x} = F(x, t)$,

$$\dot{V}(x(t), t) < -\gamma_3(\|x(t)\|) \tag{107}$$

$\forall\ t \geq 0$ and $\forall\ x \in \{x \in \mathbb{R}^n : \xi_1 < \|x(t)\| < \xi_2\}$, then the solution $x(t) : [t_0, \infty) \to \mathbb{R}^n$ of $\dot{x}(t) = F(x, t)$, with initial state x_0, has the following properties:

(i) Locally uniformly bounded. If $\|x_0\| \leqslant s$, then $\|x(t)\| \leqslant d(s)\ \forall\ t \in [t_0, \infty)$ where

$$d(s) = \begin{cases} (\gamma_1^{-1} \circ \gamma_2)(s) & \text{if} \quad \xi_1 \leqslant s \leqslant (\gamma_2^{-1} \circ \gamma_1)(\xi_2) \\ (\gamma_1^{-1} \circ \gamma_2)(\xi_1) & \text{if} \quad s \leqslant \xi_1. \end{cases} \tag{108}$$

(ii) Locally uniformly ultimately bounded. Given ξ with $d(\xi_1) < \xi \leqslant \xi_2$, if $\|x_0\| \leqslant s$, then $\|x(t)\| \leqslant \xi\ \forall\ t \in [t_0 + T(\xi, s), \infty)$ where

$$T = \begin{cases} 0 \quad \text{if} \quad s \leqslant (\gamma_2^{-1} \circ \gamma_1)(\xi) \\ \begin{cases} \dfrac{\gamma_2(s) - (\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1)(\xi)}{\min\limits_{\{\gamma_2^{-1} \circ \gamma_1(\xi) \leqslant r \leqslant d(s)\}} \{\gamma_3(r)\}} \\ \text{if } (\gamma_2^{-1} \circ \gamma_1)(\xi) < s \leqslant (\gamma_2^{-1} \circ \gamma_1)(\xi_2). \end{cases} \end{cases} \tag{109}$$

Proof. Direct application of Theorem 2.15 in [23].

Appendix B

Inside the set $\Sigma = \mathcal{D}_c \times \mathcal{D}_\varepsilon$, saturation does not apply and the term $\dot{V}$ in (62) can be obtained as follows

$$\begin{aligned} \dot{V} = \ & -\sum_{i=1}^{n} e_i^T e_i + e_n^T e_{n+1} + \tfrac{1}{2} r^T \dot{M} r \\ & + r^T \left[-\tfrac{1}{2} \dot{M} r - e_{n+1} - \bar{U} D^{-1} ((K + I_m) r + \hat{f}) \right. \\ & \left. - (K + I_m) r - \hat{f} + N_d + \tilde{N} \right] \\ & - r^T \left(\bar{U} D^{-1} + I_m \right) (K + I_m) \eta_{n+1} \end{aligned}$$

where we have utilized the definition of (38), the dynamics of (16) and (21), the control input design of (96), and the error definition of (52). By utilizing the upperbound of (44) as well as the fact that $\|r\| \leqslant \|z\| \leqslant \kappa_1 \triangleq \sqrt{c \lambda_1^{-1}}$ inside the compact set $\mathcal{D}_c$, we can upperbound $\dot{V}(t)$ as follows

$$\dot{V} \leqslant \ -\lambda_3 \|z\|^2 + \nu_0 + \kappa_1 \kappa_2 \|\eta\|$$

where we select $0 < \varepsilon < 1$ and $\kappa_2 \triangleq 2(\max_{x\in\Sigma_x}\left\{\left\|\bar{U}(x)\,D^{-1}(x)\right\|\right\}+1)\left\|K+I_m\right\|$, $\kappa_3 \triangleq \max\limits_{x\in\Sigma_x}\left\{\left\|\bar{U}(x)\,D^{-1}(x)\right\|\right\}$, and $\Sigma_x \triangleq \{x \in \mathbb{R}^m : x = x_d - e \ \forall\ e \in \Sigma\}$. We note here that the upperbound of (44) is applicable here because $\mathcal{D}_c \subset \mathcal{D}_z$.

By defining $\kappa_4 \triangleq \max\limits_{x\in\Sigma_x}\left\{\left\|M^{-1}(x)\right\|\right\}$ and $\kappa_5 \triangleq \max\limits_{x\in\Sigma_x}\left\{\left\|\frac{\partial M(x)}{\partial x}\right\|\right\}$, it is possible to upperbound $g(\cdot)$ of (56) as follows

$$\begin{aligned} \|g\| \leqslant\ & (3+\kappa_2\kappa_4)\left\|\eta\right\| + \left[\kappa_4\left\|N_d\right\| + \kappa_4(1+\kappa_3)\left\|\hat{f}\right\|\right] \\ & + \left[(3+\kappa_4+\kappa_4(1+\kappa_3)\left\|K+I_m\right\|)\left\|z\right\|\right. \\ & \left. + \tfrac{\kappa_3\kappa_4}{2}\left(\left\|\dot{x}_d\right\| + \left\|\dot{e}\right\|\right)\left\|z\right\| + \kappa_4\rho_N(\left\|z\right\|)\left\|z\right\|\right] \end{aligned} \tag{110}$$

where we have utilized the bound of (25). Since the first bracketed term is *a priori* bounded and the second bracketed term contains smooth functions of $z(t)$ that is bounded inside $\mathcal{D}_c$, it is easy to see that the upperbound on $g(\cdot)$ can be expressed as follows

$$\|g\| \leqslant \sigma_1\left\|\eta\right\| + \sigma_2 \tag{111}$$

where the definitions of σ_1 and σ_2 can be obtained by comparing (110) and (111).

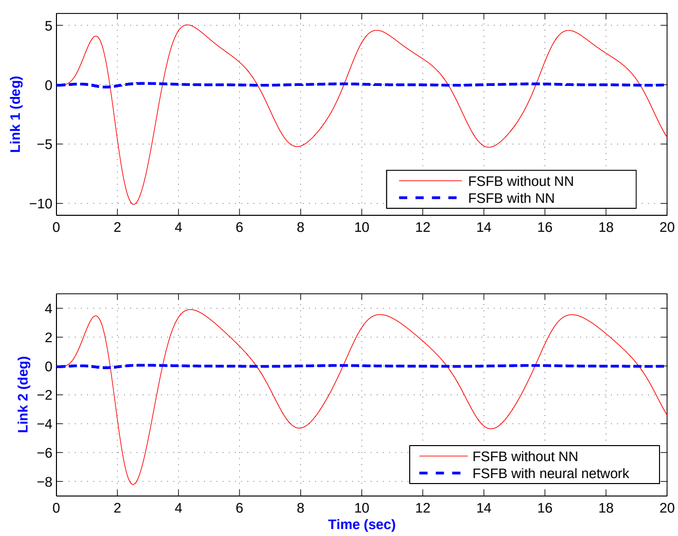

Figure 1: Tracking Error Comparison w/ and w/o NN Feedforward Compensation

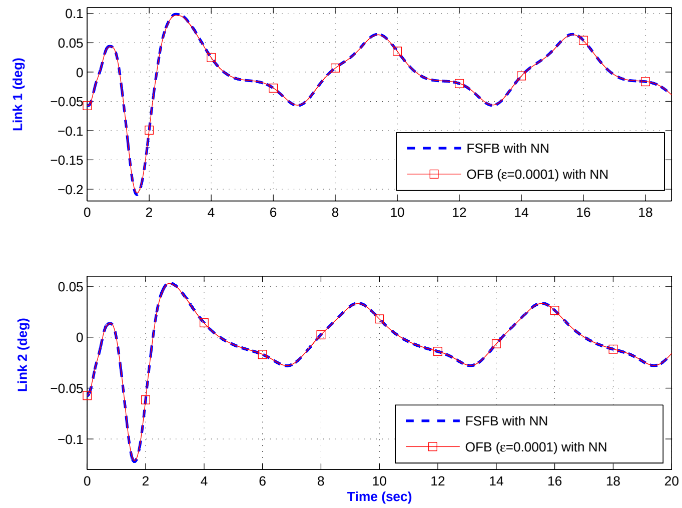

Figure 2: Tracking Error Comparison: FSFB vs OFB ($\varepsilon = 1e-4$)

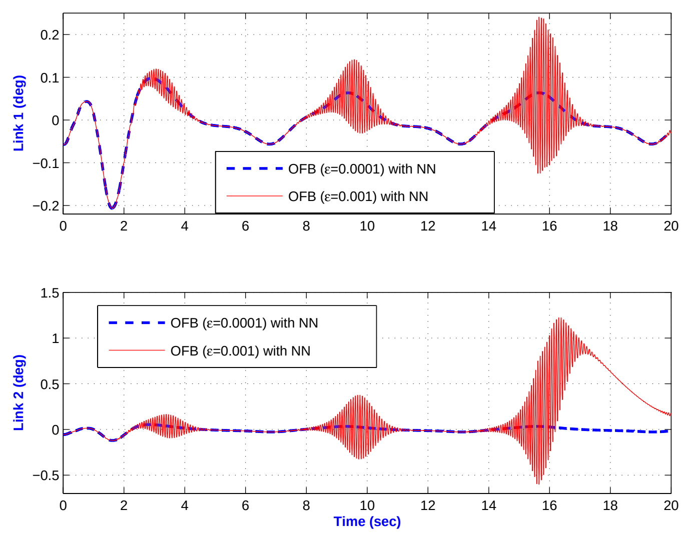

Figure 3: OFB Tracking Error Comparison for HGO Constant $\varepsilon = 1e-4$ and $\varepsilon = 1e-3$

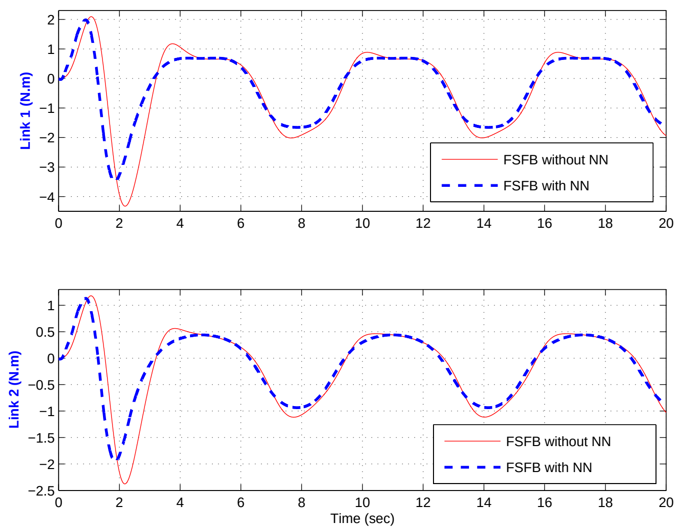

Figure 4: Control Input Comparison w/ and w/o NN Feedforward Compensation

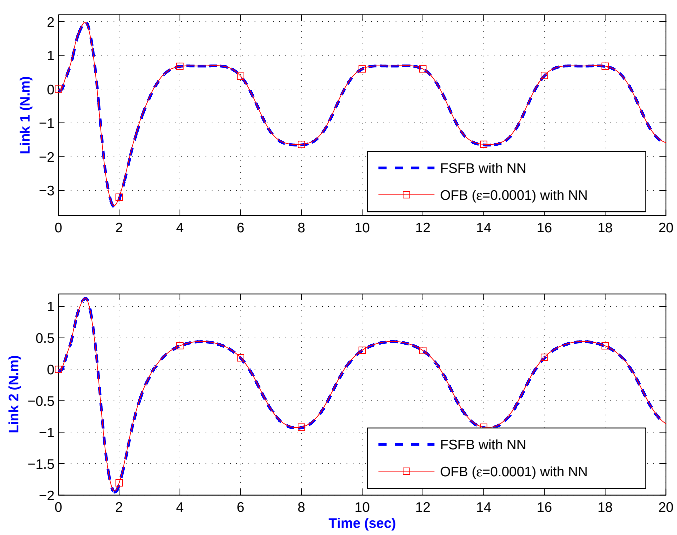

Figure 5: Control Input Comparison: FSFB vs OFB ($\varepsilon = 1e - 4$)

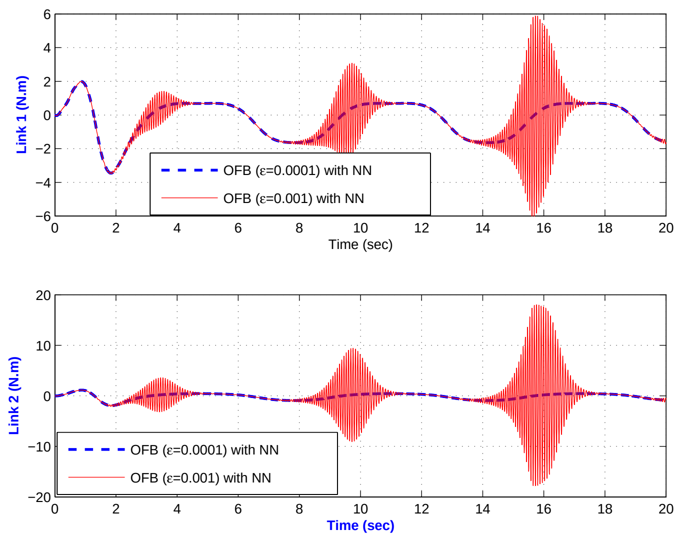

Figure 6: OFB Control Input Comparison for HGO Constant $\varepsilon = 1e - 4$ and $\varepsilon = 1e - 3$