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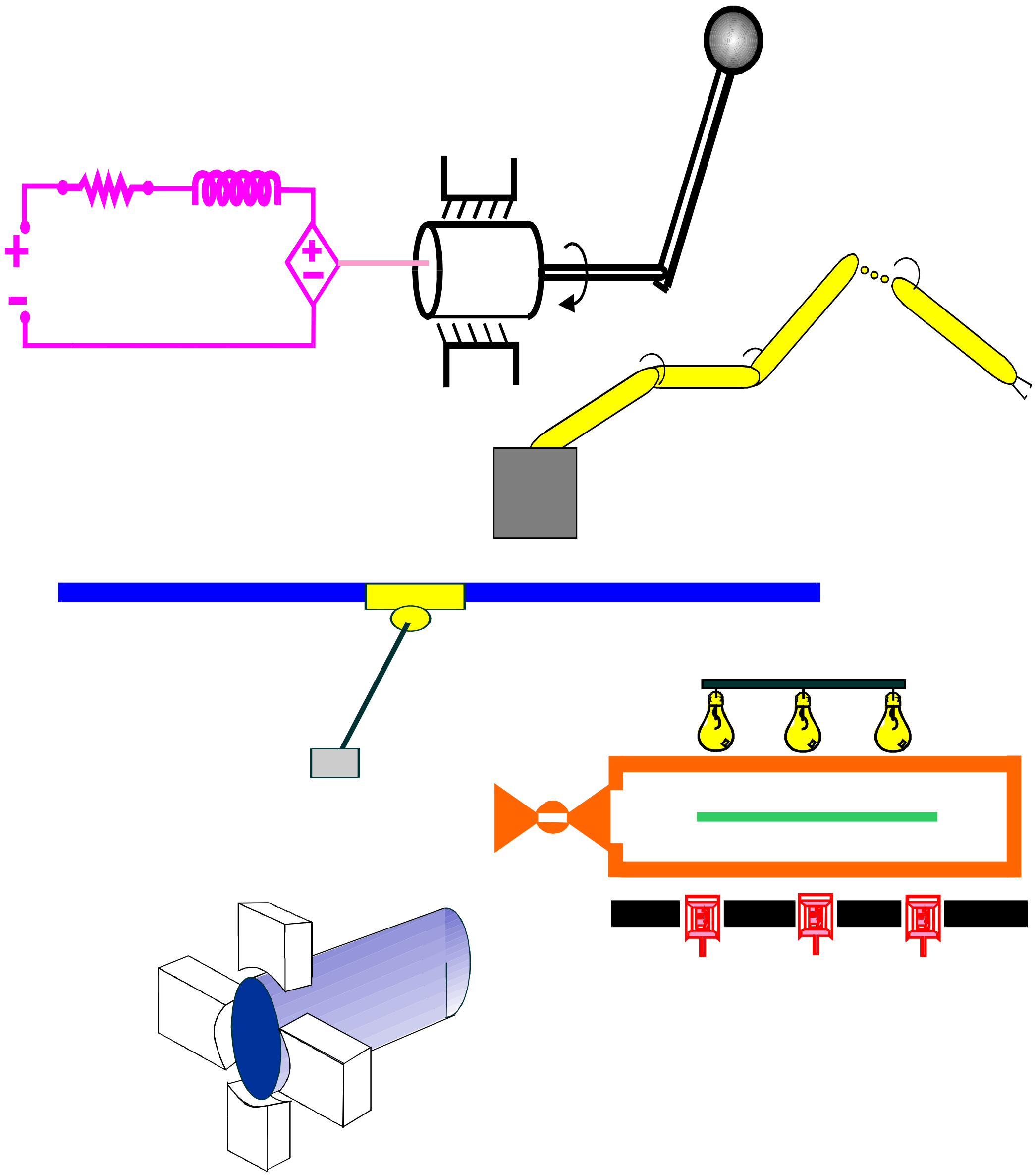

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A Modular Controller for a Class of Uncertain MIMO Nonlinear Systems with Non-Symmetric Input Gain Matrix

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Abstract

In this paper, we consider a general class of MIMO nonlinear systems with unstructured uncertainty in both the drift vector as the input matrix. With a positive definite restriction on the input matrix that is required for controllability, we are able to construct a continuous state feedback control mechanism that achieves semi-global ultimate bounded tracking. The Lyapunov based stability result is facilitated through a decomposition of the aforementioned input matrix into a symmetric positive definite matrix and a unity upper triangular matrix. Furthermore, a modular feed-forward compensation scheme is introduced in the form of neural network or fuzzy logic schemes.

1 Introduction

We consider the control design problem for the class of high-order MIMO nonlinear systems that are affine in the control input and represented by

$$\begin{aligned} \frac{d^{(n)}x}{dt} &= h(x, \dot{x}, \cdots, x^{(n-1)}, \phi(t), t) + \\ &\quad G(x, \dot{x}, \cdots, x^{(n-1)}, \phi(t), t)u \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^m$ is the output vector, $h(\cdot) \in \mathbb{R}^m$ represents an uncertain nonlinear function vector; $\phi(t) \in \mathbb{R}^l$ denotes an unknown, time-varying parameter vector; $G(\cdot) \in \mathbb{R}^{m\times m}$ is an uncertain nonlinear input gain matrix, and $u(t) \in \mathbb{R}^m$ denotes the control input vector.

Adaptive control schemes for linear time-invariant uncertain SISO systems (that are minimum phase) have been known for quite some time, *e.g.*, see [1]. However, generalization to the MIMO counterpart has not been easy; a multitude of authors have dealt with this problem by various assumptions on the high-frequency gain (HFG) matrix K_p. In [1], it is assumed that K_p is known; an upper bound on the norm of K_p is assumed to be known in [2]; the approach of Weller and Goodwin [3] partitions $K_p = LU$ while requiring *apriori* knowledge of the lower bounds of the diagonal entries of the upper triangular matrix U; finally, the method proposed in [4] assumes the existence of a matrix S_p such that K_pS_p is positive definite and symmetric. Most recently, Costa *et. al.* [5] solved the MIMO adaptive control problem for minimum-phase systems with relative degree 1 under the assumption that the signs of the leading principal minors of the HFG matrix are known.

A literature survey reveals that the results for uncertain nonlinear systems have been restricted to sub-classes of systems. For SISO nonlinear systems in the strict feedback form that are non-affine in the unknown parameters, [6] obtained a global uniform ultimate bounded result by employing a Nussbaum gain and a smooth parameter projection algorithm. In [7], Ding *et. al.* obtain an ultimately bounded output feedback result for uncertain SISO systems via a modification of the backstepping technique using a Nussbaum gain and a Lyapunov function that is flat in a specifiable region around the origin. The above results are robust to disturbances and do not require any knowledge of the sign of the high-frequency gain. For multi-input nonlinear systems that are representable in the parametric strict feedback form, Krstic *et. al.* [8] were able to formalize the adaptive backstepping design procedure; however, the gain matrix premultiplying the control is assumed to be known. In [9], a general procedure was presented for designing switching adaptive controllers for multi-input nonlinear systems which includes feedback linearizable systems, parametric-pure feedback systems, and systems with a control Lyapunov function that is linear in the parameters. In [10], a neural network-based adaptive controller was formulated for the class of systems delineated in (1) with the restriction that $G(\cdot)$ be uniformly positive or negative definite. The proposed controller was shown to guarantee the semi-global convergence of the tracking error to a residual set. The drawback of the control

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strategy is that the estimation strategy utilized can lead to loss of controllability in which case the control input tends to zero. In [11], the following subset of MIMO nonlinear systems similar was considered

$$\frac{d^{(n)}x}{dt} = h(x, \dot{x}, \cdots, x^{(n-1)}, \theta_1) + G(x, \dot{x}, \cdots, x^{(n-2)}, \theta_2)u \quad (2)$$

with the $\mathcal{C}^0$ uncertain functions $h(\cdot) \in \mathbb{R}^m$ and $G(\cdot) \in \mathbb{R}^{m\times m}$ being affine in the unknown constant parameter vectors $\theta_i \in \Re^{l_i}$ $\forall$ $i = 1, 2$. The proposed adaptive controller was proven to ensure the global asymptotic convergence of the tracking error to zero.

In this paper, we extend the work in [11] for the broader class of systems given by (1). To satisfy the controllability requirement, it is imperative that the smallest singular value of $G(\cdot)$ be lowerbounded by a positive constant; hence, we make the assumption that $G(\cdot)$ is positive-definite (p.d.). However, we drop the requirement that $G(\cdot)$ be symmetric since many practical nonlinear control systems do not possess a symmetric input gain matrix [12, 13]. Motivated by a matrix decomposition introduced in [14] and subsequently utilized in [5], we decompose $G(\cdot)$ into the product of a symmetric p.d. matrix and a unity upper triangular matrix. The symmetric p.d. matrix is exploited in the Lyapunov based stability analysis while the unity upper triangular matrix allows for an algebraic loop free sequential synthesis of control signals $u_i(t)$ $\forall$ $i = 1, 2, ...m$. The unstructured uncertainty is dealt with by assuming a $\mathcal{C}^2$ smoothness property for $h(\cdot)$ and $G(\cdot)$. The result obtained in this paper is a continuous state feedback control mechanism that achieves semi-global ultimate bounded tracking. In order to broaden the applicability of the approach, we introduce a modular feed-forward scheme which is shown to be achievable via neural network or fuzzy logic compensation.

The rest of this paper is organized as follows. For the sake of clarity, we first present the control design for a first-order, two-input and two-output system of the form (1). We then illustrate how the control method can be applied to the system defined by (1). The theoretical development is complemented with a numerical simulation for a two degrees-of-freedom (DOF) mechanical system. Simulation results prove the efficacy of the control design and clearly illustrate the effectiveness of feed-forward compensation.

2 First-Order MIMO System

To explain the control technique, we first examine a first-order, two-input, two-output, nonlinear system having the following general form

$$\dot{x} = h_p(x) + G_p(x)u \quad (3)$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ is the system state vector, $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ is the control input vector, and $h_p(x) \in \mathbb{R}^2$, $G_p(x) \in \mathbb{R}^{2\times 2}$ are uncertain nonlinear functions. We will assume that $G_p(x)$ is not symmetric but positive definite (*i.e.*, $\nu^T G_p(x)\nu \geq 0$ for $\nu(t) \in \mathbb{R}^2$). To deal with the fact that $G_p(x)$ is not symmetric, we will employ a decomposition tool which is detailed by the following lemma.

Lemma 1 *Any positive-definite, non-symmetric matrix $P \in \mathbb{R}^{m\times m}$ can be decomposed as*

$$P = RQ \quad (4)$$

where $R \in \mathbb{R}^{m\times m}$ is symmetric and positive definite, and $Q \in \mathbb{R}^{m\times m}$ is an unity upper triangular matrix.

Proof: We can use the fact that all leading principal minors of a real, positive definite matrix are positive ([16], Theorem 5.10) along with Lemma 1 given in [5].

Based on Lemma 1, $G_p(x)$ can be decomposed as follows

$$G_p(x) = S(x)T(x) \quad (5)$$

where $S(x) \in \mathbb{R}^{2\times 2}$ is symmetric and positive definite, $T(x) \in \mathbb{R}^{2\times 2}$ is a unity upper triangular matrix explicitly defined as follows

$$T(x) = \begin{bmatrix} 1 & T_{12}(x) \\ 0 & 1 \end{bmatrix}, \quad (6)$$

and $T_{12}(x) \in \mathbb{R}$. We can now utilize (5) to rewrite the dynamic model in (3) as follows

$$M(x)\dot{x} = f(x) + T(x)u \quad (7)$$

where $M(x) \triangleq S^{-1}(x) \in \mathbb{R}^{2\times 2}$ is a symmetric, positive definite matrix, and the auxiliary function $f(x) \in \mathbb{R}^2$ is defined as follows $f(x) \triangleq S^{-1}(x)h_p(x) = \begin{bmatrix} f_1(x) & f_2(x) \end{bmatrix}^T$. To facilitate the control development, we assume that $M(x)$, $f(x)$, and $T(x)$ satisfy the following assumptions:

Assumption F1 The matrix $M(\cdot)$ is assumed to be bounded as follows

$$\underline{m}\left\|\xi\right\|^2 \leqslant \xi^T M(\cdot)\xi \leqslant \bar{m}(\cdot)\left\|\xi\right\|^2 \qquad \forall\xi \in \mathbb{R}^2 \quad (8)$$

where $\underline{m} \in \mathbb{R}$ denotes a positive constant, and $\bar{m}(\cdot)$ denotes a positive, non-decreasing function.

Assumption F2 The functions $M(\cdot)$, $f(\cdot)$ and $T(\cdot)$ are assumed to be second-order differentiable and also assumed to satisfy the following properties

$$\begin{array}{ll} M(\cdot),\ \dot{M}(\cdot),\ \ddot{M}(\cdot) \in \mathcal{L}_\infty & \text{if } x,\ \dot{x},\ \ddot{x} \in \mathcal{L}_\infty \\ f(\cdot),\ \dot{f}(\cdot),\ \ddot{f}(\cdot) \in \mathcal{L}_\infty & \text{if } x,\ \dot{x},\ \ddot{x} \in \mathcal{L}_\infty \\ T(\cdot),\ \dot{T}(\cdot),\ \ddot{T}(\cdot) \in \mathcal{L}_\infty & \text{if } x,\ \dot{x},\ \ddot{x} \in \mathcal{L}_\infty \end{array}. \quad (9)$$

Remark 1 *For simplicity of presentation, we have assumed that $h_p(x)$ and $G_p(x)$ do not depend explicitly on time or on unknown time-varying parameters. However, it should be emphasized that the proposed control approach can compensate for these phenomena provided the time-varying effects satisfy second-order differentiability conditions. That is, the functions $h_p(x)$ and $G_p(x)$ could be easily replaced by $h_p(x, \theta_1(t), t)$ and $G_p(x, \theta_2(t), t)$ where $\theta_i(t)$, $i = 1, 2$ denote unknown time-varying parameter vectors and other time-varying disturbance that may appear nonlinearly in the model. It should be noted that $\theta_i(t)$, $i = 1, 2$ must of course satisfy similar type of assumptions as given in Assumption F2.*

2.1 Control Objective and Control Law

To set up the tracking control problem, we let $x_d(t) = \begin{bmatrix} x_{1d}(t) & x_{2d}(t) \end{bmatrix}^T \in \mathbb{R}^2$ denoted a reference trajectory signal that must be continuous differentiable up to its third derivative such that

$$\frac{d^i x_d(t)}{dt^i} \in \mathcal{L}_\infty \text{ for } i = 0, 1, 2, 3. \tag{10}$$

To quantify the control objective, the tracking error $e(t) = \begin{bmatrix} e_1(t) & e_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ is defined as follows

$$e = x_d - x. \tag{11}$$

Our primary objective requires $x(t)$ to practically track $x_d(t)$ (*i.e.*, UUB tracking) with a continuous control law using full state feedback and norm-based, inequality bounds on the unknown functions $M(x)$, $f(x)$, and $T(x)$. Based on the subsequent stability analysis, we propose the following control law[1] to achieve the stated control objective

$$\begin{aligned} u(t) \quad = \quad & (K_s + I)e(t) - (K_s + I)e(t_0) + \\ & \int_{t_0}^{t} ((K_s + I)e(\tau) + \hat{f}(\tau))d\tau \end{aligned} \tag{12}$$

where $K_s = \begin{bmatrix} k_{s1} & 0 \\ 0 & k_{s2} \end{bmatrix} \in \mathbb{R}^{2\times2}$ is a positive definite, diagonal control gain matrix, $I \in \mathbb{R}^{2\times2}$ denotes the identity matrix, $\hat{f}(t) = \begin{bmatrix} \hat{f}_1(t) & \hat{f}_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ denotes a user designed feed-forward component. It should be noted that it is assumed that the control designer must ensure that $\hat{f}(t) \in \mathcal{L}_\infty$ (The reader is referred to Section 3 for specific details on how this might be done). To simplify the analysis, the scalar control gains k_{s1} and k_{s2} are defined in terms auxiliary control gains as follows

$$\begin{aligned} k_{s1} &= k_{n1} + k_{n2} + k_{n3} \\ k_{s2} &= k_{n1} + k_{n3} \end{aligned} \tag{13}$$

[1] The second term in (12) is used to ensure that $u(t_0) = 0$.

where k_{n1}, k_{n2} and k_{n3} are positive constants. As illustrated in the subsequent sections, the control law of (12) ensures semi-global, uniformly ultimately bounded tracking provided the control gains (*i.e.*, the gains inside the matrix K_s as given by (13)) are selected sufficiently large relative to the norm of the initial tracking error and a reference trajectory-based bound. The proof of this result is presented in the following sections.

2.2 Error System Development

We begin by defining the following filtered tracking error signal [17] $r(t) = \begin{bmatrix} r_1(t) & r_2(t) \end{bmatrix}^T \in \mathbb{R}^2$ as follows

$$r = \dot{e} + e \tag{14}$$

where $e(t)$ was introduced in (11). After differentiating (14) and then multiplying both sides of the resulting equation by $M(x)$, we have

$$M(x)\dot{r} = -\frac{1}{2}\dot{M}(x)r - e - \dot{T}(x)u - T(x)\dot{u} + \tilde{N}_1 + N_{1d} \tag{15}$$

where the auxiliary functions $\tilde{N}_1(\cdot)$, $N_{1d}(\cdot)$, $N_1(\cdot) \in \mathbb{R}^2$ are defined as follows

$$\begin{aligned} \tilde{N}_1 &= N_1 - N_{1d} \\ N_1 &= M(x)\ddot{x}_d + \dot{M}(x)\dot{x} - \dot{f}(x) + M(x)\dot{e} + \frac{1}{2}\dot{M}(x)r + e \\ N_{1d} &= M(x_d)\ddot{x}_d + \dot{M}(x_d)\dot{x}_d - \dot{f}(x_d). \end{aligned} \tag{16}$$

The open loop error system given by (15) can be rewritten as follows

$$\begin{aligned} M(x)\dot{r} \quad = \quad & -\frac{1}{2}\dot{M}(x)r - e - \begin{bmatrix} \dot{T}_{12}(x)u_2 \\ 0 \end{bmatrix} \\ & - \begin{bmatrix} T_{12}(x)\dot{u}_2 \\ 0 \end{bmatrix} - \dot{u} + \tilde{N}_1 + N_{1d} \end{aligned} \tag{17}$$

where (6) and (16) have been utilized. To facilitate further error system development with regard to (17), we utilize (11) and (14) to rewrite $\dot{T}_{12}(x)$ as follows

$$\begin{aligned} \dot{T}_{12}(x) \quad &= \quad \frac{\partial T_{12}(x)}{\partial x}\dot{x} \\ &= \quad \frac{\partial T_{12}(x)}{\partial x}(e - r) + \left(\frac{\partial T_{12}(x)}{\partial x} - \frac{\partial T_{12}(x_d)}{\partial x_d}\right)\dot{x}_d + \\ & \qquad \frac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d \\ &= \quad \tilde{N}_2 + \frac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d \end{aligned} \tag{18}$$

where the auxiliary function $\tilde{N}_2(\cdot) \in \mathbb{R}$ is defined as follows

$$\tilde{N}_2 = \frac{\partial T_{12}(x)}{\partial x}(e - r) + \left(\frac{\partial T_{12}(x)}{\partial x} - \frac{\partial T_{12}(x_d)}{\partial x_d}\right)\dot{x}_d. \tag{19}$$

Likewise, further error system development with regard to (17) is fostered by utilizing (7) to rewrite $u_2(t)$ as follows

$$u_2 = m_{21}(x)\dot{x}_1 + m_{22}(x)\dot{x}_2 - f_2(x) = \tilde{u}_2 + u_{2d} \tag{20}$$

where the auxiliary functions $\tilde{u}_2(\cdot)$, $u_{2d}(\cdot) \in \mathbb{R}$ are defined as follows

$$\begin{aligned} \tilde{u}_2 &= m_{21}(x)\dot{x}_1 + m_{22}(x)\dot{x}_2 - f_2(x) - u_{2d} \\ u_{2d} &= m_{21}(x_d)\dot{x}_{1d} + m_{22}(x_d)\dot{x}_{2d} - f_2(x_d). \end{aligned} \tag{21}$$

We can now use (18) and (20) to rewrite the $\dot{T}_{12}(x)u_2$ term in (17) as follows

$$\dot{T}_{12}(x)u_2 = \tilde{N}_3 + \frac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d u_{2d} \tag{22}$$

where the auxiliary function $\tilde{N}_3(\cdot) \in \mathbb{R}$ is defined as follows

$$\tilde{N}_3 = \tilde{N}_2(\tilde{u}_2 + u_{2d}) + \frac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d \tilde{u}_2. \tag{23}$$

To complete the error system development with regard to (17), we take the time derivative of (12) to obtain

$$\dot{u} = (K_s + I)r + \hat{f} \tag{24}$$

which can be rewritten as

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} (k_{s1}+1)r_1 \\ (k_{s2}+1)r_2 \end{bmatrix} + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} \tag{25}$$

after utilizing (14). The expression in (25) can now be used to rewrite the $T_{12}(x)\dot{u}_2$ term in (17) as follows

$$\begin{aligned} T_{12}(x)\dot{u}_2 &= (\tilde{T}_{12} + T_{12}(x_d))((k_{s2}+1)r_2 + \hat{f}_2) \\ &= \tilde{N}_4 + T_{12}(x_d)\hat{f}_2 \end{aligned} \tag{26}$$

where the auxiliary functions $\tilde{T}_{12}(\cdot)$, $\tilde{N}_4(\cdot) \in \mathbb{R}$ are defined as follows

$$\begin{aligned} \tilde{T}_{12} &= T_{12}(x) - T_{12}(x_d) \\ \tilde{N}_4 &= \tilde{T}_{12}(k_{s2}+1)r_2 + T_{12}(x_d)(k_{s2}+1)r_2 + \tilde{T}_{12}\hat{f}_2. \end{aligned} \tag{27}$$

After substituting (24) into (17) and employing (22) and (26), we can formulate the closed loop error system as follows

$$\begin{aligned} M(x)\dot{r} &= -\frac{1}{2}\dot{M}(x)r - e - (K_s+I)r - \\ &\quad \begin{bmatrix} \tilde{N}_3 \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{N}_4 \\ 0 \end{bmatrix} + \tilde{N}_1 + \bar{N}_d \\ &= -\frac{1}{2}\dot{M}(x)r - e - (K_s+I)r + \tilde{N} + \bar{N}_d \end{aligned} \tag{28}$$

where the auxiliary functions $\tilde{N}(\cdot)$, $\bar{N}_d(\cdot) \in \mathbb{R}$ are defined as follows

$$\begin{aligned} \tilde{N} &= -\begin{bmatrix} \tilde{N}_3 \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{N}_4 \\ 0 \end{bmatrix} + \tilde{N}_1 \\ \bar{N}_d &= N_{1d} - \begin{bmatrix} \dfrac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d u_{2d} \\ 0 \end{bmatrix} - \begin{bmatrix} T_{12}(x_d)\hat{f}_2 \\ 0 \end{bmatrix} - \hat{f}\ . \end{aligned} \tag{29}$$

Remark 2 *To facilitate the stability analysis, we first note that we can use (29) and the fact that $\hat{f}(t) \in \mathcal{L}_\infty$ to show that $\bar{N}_d(t) \in \mathcal{L}_\infty$ It is also not difficult to show by the Mean Value Theorem and Assumption F2 that $\tilde{N}_1(\cdot)$ defined in (16) can be upper bounded as follows*

$$\left\| \tilde{N}_1 \right\| \leqslant \rho_1(\|z\|)\|z\| \tag{30}$$

where $z(t) \in \mathbb{R}^4$ is defined as

$$z = \begin{bmatrix} e^T & r^T \end{bmatrix}^T \tag{31}$$

and $\|\cdot\|$ denotes the standard Euclidean norm, and $\rho_1(\|z\|) \in \mathbb{R}$ denotes a positive, bounding function that is non-decreasing in $\|z\|$ (Note that this bounding function does not contain any control gains). The inequality given by (30) will be utilized in the following stability analysis. Likewise, the Mean Value Theorem and Assumption F2 can be employed to show that $\tilde{N}_3(\cdot)$ defined in (23) and $\tilde{N}_4(\cdot)$ defined in (27) can be upper bounded as follows

$$\begin{aligned} \left\| \tilde{N}_3 \right\| &\leqslant \rho_2(\|z\|)\|z\| \\ \left\| \tilde{N}_4 \right\| &\leqslant \rho_g(\|z\|)\|z\| \end{aligned} \tag{32}$$

where $z(t)$ was defined in (31) and $\rho_2(\|z\|)$, $\rho_g(\|z\|) \in \mathbb{R}$ are positive, bounding functions that are non-decreasing in $\|z\|$. It is easy to show that $\rho_2(\|z\|)$ does not contain any control gains and that $\rho_g(\|z\|)$ does not depend on the control gain k_{n2} introduced in (12).

2.3 Stability Analysis

Before presenting the main result of this section, we state the following lemma which will be invoked later.

Lemma 2 *Let $\gamma_1(\|x(t)\|)$, $\gamma_2(\|x(t)\|) : \mathbb{R}^+ \to \mathbb{R}^+$ denote class $\mathcal{K}_\infty$ functions $\forall (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, $\gamma_3(\|x(t)\|) : \mathbb{R}^+ \to \mathbb{R}^+$ denote a function that assumes positive values, and $V(x(t),t)) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^+$ denote continuously differentiable functions satisfying*

$$\gamma_1(\|x(t)\|) \leqslant V(x(t),t)) \leqslant \gamma_2(\|x(t)\|)\ . \tag{33}$$

If there exists positive constants η_1 and η_2 satisfying $\eta_2 > (\gamma_1^{-1} \circ \gamma_2)(\eta_1)$ and the time derivative of $V(x(t),t))$ along the trajectory of the system satisfies

$$\dot{V}(x(t),t) < -\gamma_3(\|x(t)\|) \quad \textit{for} \quad \eta_1 < \|x(t)\| < \eta_2 \tag{34}$$

then $\|x(t)\|$, with initial state $x(t_0)$, has the following properties.

(i) Locally uniformly bounded. Specifically, $\|x(t_0)\| \leqslant s$ implies $\|x(t)\| \leqslant d(s)$ for all $t \in [t_0, \infty)$, where

$$d(s) = \begin{cases} (\gamma_1^{-1} \circ \gamma_2)(s) & \textit{if } \eta_1 \leqslant s \leqslant (\gamma_2^{-1} \circ \gamma_1)(\eta_2) \\ (\gamma_1^{-1} \circ \gamma_2)(\eta_1) & \textit{if } s \leqslant \eta_1 \end{cases}. \tag{35}$$

(ii) Locally uniformly ultimately bounded. Specifically, given a constant η with $d(\eta_1) < \eta \leqslant \eta_2$, $\|x(t_0)\| \leqslant s$ implies $\|x(t)\| \leqslant \eta$ for all $t \in [t_0 + T(\eta, s), \infty)$, where $T(\eta, s)$ denotes the time in which the ultimate bound is guaranteed.

Proof: See Theorem 2.15 in [18] for proof as well as an explicit formula for $T(\eta, s)$.

Given the above Lemma, the main result of this section can be stated by the following Theorem.

Theorem 1 *The control law of (12) ensures that all system signals are bounded under closed-loop operation and ensures the tracking error variable $z(t)$ exhibits the locally uniformly bounded and the locally uniformly ultimately bounded properties given by Lemma 2.*

Proof: See Appendix A.

Remark 3 *We easily illustrate that the proof of the above theorem and Lemma 2 actually yield a semi-global uniformly bounded result for the tracking error variable $z(t)$ by illustrating that the controller gains can be increased to cover a predetermined value of the initial conditions denoted by $\|z(t_0)\|$. First, by using (14) and (31), we can write an explicit expression for $\|z(t_0)\|$ as follows*

$$\|z(t_0)\| = \sqrt{e^T(t_0)e(t_0) + (\dot{e}(t_0)) + e(t_0))^T(\dot{e}(t_0)) + e(t_0))}. \tag{36}$$

From (7), (11), and the fact that $u(t_0) = 0$ (see (12)), we can develop the following expression for $\dot{e}(t_0)$

$$\dot{e}(t_0) = \dot{x}_d(t_0) - M^{-1}(x(t_0))f(x(t_0)). \tag{37}$$

From the expression given by (36) and (37), it is clear that $\|z(t_0)\|$ does not depend on any of the control gain parameters. Second, we can see from the definition of η_1 given by (66) and the inequality condition given by (61) that η_1 can be made arbitrarily small by increasing the control gain k_{n3}. Lastly, we can see from the definition of η_2 given by (66) and the discussion regarding $\rho_2(\|z\|)$ and $\rho_g(\|z\|)$ given in Remark 2 that η_2 can be made arbitrarily large by increasing the control gains k_{n1} and k_{n2}. We also note that the ultimate bound given in the Lemma 2 can be arbitrarily small by increasing the control gain k_{n3}. That is, as illustrated above, η_1 can be made arbitrarily small by increasing the control gain k_{n3}, and hence, due to the structure of $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ given by (58), the lower bound for the ultimate bound given in Lemma 2 (i.e., $d(\eta_1)$) can be made arbitrarily small.

3 Feed-forward Component Design

In general, one expects that the use of a feed-forward term in control law will reduce the magnitude of the control input. The addition of the feed-forward component $\hat{f}(\cdot)$ in (12) does not alter the type of tracking result presented in Section 2.3 as long as the control designer ensures that $\hat{f}(\cdot) \in \mathcal{L}_\infty$. To illustrate how one might design $\hat{f}(\cdot)$, we now delineate three different methods.

3.1 Best-Guess Estimation

In this case, the functional form of $M(\cdot, \theta_1)$, $f(\cdot, \theta_2)$, and $T(\cdot, \theta_3)$ in (7) is known, but the constant parameter vectors θ_1, θ_2 and θ_3 may not be known precisely. In this case, the following certainty equivalence-type component $\hat{f}(\cdot)$ can be added to the proposed control law $u(t)$ in (12)

$$\begin{aligned} \hat{f}(\cdot) \quad = \quad & N_{1d}(x_d, \dot{x}_d, \ddot{x}_d, \hat{\theta}_1, \hat{\theta}_2, t) - \\ & \begin{bmatrix} \dfrac{\partial T_{12}(x_d, \hat{\theta}_3)}{\partial x_d} \dot{x}_d u_{2d}(x_d, \dot{x}_d, \hat{\theta}_1, \hat{\theta}_2) \\ 0 \end{bmatrix} \end{aligned} \tag{38}$$

where $N_{1d}(\cdot)$ was defined in (16), $u_{2d}(\cdot)$ was defined in (21), $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ denote the best-guess constant estimates of the unknown parameter vectors.

3.2 Neural Network Based Estimation

In this case, the function form of $M(\cdot)$, $f(\cdot)$, and $T(\cdot)$ in (7) is unknown. According to the approximation properties of neural network (See [19] and [20]), the bounded, a continuous uncertain nonlinear function $f(\chi) = N_{1d}(\cdot) - \begin{bmatrix} \dfrac{\partial T_{12}(x_d)}{\partial x_d} \dot{x}_d u_{2d}(\cdot) \\ 0 \end{bmatrix}$ in (29), can be approximate by a two-layer, neural network such that

$$f(\chi) \cong W^T \sigma(V^T \chi) + \varepsilon_1(\chi) \tag{39}$$

for a hidden-layer with p number of neurons where $\chi\left(x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d\right) \in \mathbb{R}^s$ represents the bounded inputs to neural network, $W \in \mathbb{R}^{p\times 2}$ and $V \in \mathbb{R}^{s\times p}$ denote the ideal constant weights, $\sigma(\cdot) \in \mathbb{R}^p$ is the corresponding neuron activation function, $\varepsilon_1(\cdot) \in \mathbb{R}^2$ denotes the bounded reconstruction error[2] (*i.e.*, $\|\varepsilon_1(\chi)\| \leqslant \varepsilon_{d1}$ where ε_{d1} is some positive constant). We can construction a feed-forward neural network to approximate the unknown continuous nonlinear function. It contains a hidden layer with p neurons and a output layer with 2 neurons. The output of neural network is designed as follows

$$\hat{f}(\cdot) = \hat{W}^T \sigma(\hat{V}^T \chi) \tag{40}$$

[2]The reconstruction error is bounded by a constant because it depends on the desired trajectory as opposed to the actual trajectory.

where $\hat{W}(t) \in \mathbb{R}^{p\times 2}$ and $\hat{V}(t) \in \mathbb{R}^{s\times p}$ are the weight estimates of W and V, respectively. To meet the requirement that $\hat{f}(\cdot) \in \mathcal{L}_\infty$, the update law of $\hat{W}(t)$ and $\hat{V}(t)$ should be designed such that $\hat{W}(t)$, $\hat{V}(t) \in \mathcal{L}_\infty$ (In the simulation section, we illustrate how this can be done). There are many choices for the activation function $\sigma(\cdot)$, such as radial basis function, hyperbolic tangent function, or the sigmoid function (See [19] and [20]).

3.3 Fuzzy Logic Based Estimation

In this case, the functional form of $M(\cdot)$, $f(\cdot)$ and $T(\cdot)$ is unknown. Based on the Fuzzy Logic (FL) approximation property (See [19] and [20]), the bounded, continuous nonlinear function $f(\chi) = N_{1d}(\cdot) - \begin{bmatrix} \dfrac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d u_{2d}(\cdot) \\ 0 \end{bmatrix}$ in (29) can be approximated as follows

$$f(\chi) \cong \Phi^T \xi(\chi) + \varepsilon_2(\chi) \tag{41}$$

where $\Phi \in \mathbb{R}^{p\times 2}$ is the ideal FL constant parameters, p is the number of fuzzy logic rules, $\xi(\cdot) \in \mathbb{R}^p$ denotes the output of the fuzzy system, $\chi\left(x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d\right) \in \mathbb{R}^s$, and $\varepsilon_2(\cdot) \in \mathbb{R}^2$ represents the reconstruction error which can be upper bound by some positive constant (i.e., $\|\varepsilon_2(\chi)\| \leqslant \varepsilon_{d2}$ where ε_{d2} is some positive constant). By using the techniques in [20], we can construct a rule-based FL term to approximate the uncertain nonlinear function as follows

$$\hat{f}(\cdot) = \hat{\Phi}(t)^T \xi(\chi) \tag{42}$$

where $\hat{\Phi}(t) \in \mathbb{R}^{p\times 2}$ denote the weight estimates of Φ. To meet the requirement that $\hat{f}(\cdot) \in \mathcal{L}_\infty$, we must ensure that the update law for $\hat{\Phi}(t)$ is designed such that $\hat{\Phi}(t) \in \mathcal{L}_\infty$.

4 Extension to Higher-Order Multi-Input System

In this section, we discuss the extension of the proposed control law for the following system

$$x^{(n)} = f(x, \dot{x}, \cdots, x^{(n-1)}) + G(x, \dot{x}, \cdots, x^{(n-1)})u \tag{43}$$

where $x^{(i)}(t) \in \mathbb{R}^m$, $i = 0, 1, \cdots, n-1$ are the system states, $(\cdot)^{(i)}(t)$ denotes the i^{th} derivative with respect to time, $u(t) \in \mathbb{R}^m$ represents the control input, $f(\cdot) \in \mathbb{R}^m$ and $G(\cdot) \in \mathbb{R}^{m\times m}$ are uncertain nonlinear functions. For presentation purposes, we define $X = [\ x^T \quad \dot{x}^T \quad \cdots \quad (x^{(n-1)})^T\ ]^T \in \mathbb{R}^{mn}$. We assume that $G(\cdot)$ is not symmetric but positive definite. Based on Lemma 1, $G(\cdot)$ can be decomposed as follows

$$G(X) = S(X)T(X) \tag{44}$$

where $S(\cdot) \in \mathbb{R}^{m\times m}$ is symmetric, positive definite, and $T(\cdot) \in \mathbb{R}^{m\times m}$ is an unity upper triangular matrix. We can rewrite the system given (43) as follows

$$M(X)x^{(n)} = F(X) + T(X)u \tag{45}$$

where the auxiliary functions $M(\cdot) \in \mathbb{R}^{m\times m}$ and $F(\cdot) \in \mathbb{R}^m$ are defined as follows

$$M(\cdot) \triangleq S^{-1}(\cdot) \qquad F(\cdot) \triangleq S^{-1}(\cdot)f(\cdot). \tag{46}$$

It is easy to see that $M(\cdot)$ is symmetric and positive definite. We assume that $M(\cdot)$ and $F(\cdot)$ satisfy the following assumptions:

Assumption M1 The matrix $M(\cdot)$ is bounded by

$$\underline{m}\,\|\xi\|^2 \leqslant \xi^T M(\cdot)\xi \leqslant \bar{m}(\cdot)\,\|\xi\|^2 \qquad \forall \xi \in \mathbb{R}^m \tag{47}$$

where $\underline{m} \in \mathbb{R}$ denotes a positive constant, and $\bar{m}(\cdot) \in \mathbb{R}$ denotes a positive, non-decreasing function.

Assumption M2 The functions $M(\cdot)$, $f(\cdot)$, and $T(\cdot)$ are second-order differentiable such that

$$\begin{aligned} &M(\cdot),\, \dot{M}(\cdot),\, \ddot{M}(\cdot), F(\cdot),\, \dot{F}(\cdot),\, \ddot{F}(\cdot), T(\cdot),\, \dot{T}(\cdot), \\ &\ddot{T}(\cdot) \in \mathcal{L}_\infty \text{ if } x,\, \dot{x},\, \ddot{x}, \cdots,\, x^{(n+1)} \in \mathcal{L}_\infty \end{aligned} \tag{48}$$

Let $x_d(t) \in \mathbb{R}^m$ denote the reference trajectory signal that is continuously differentiable such that

$$x_d^{(i)}(t) \in \mathcal{L}_\infty \text{ for } i = 0, 1, ..., n+2. \tag{49}$$

To simplify the analysis, we also define $X_d = [\ x_d^T \quad \dot{x}_d^T \quad \cdots \quad (x_d^{(n-1)})^T\ ]^T \in \mathbb{R}^{mn}$. To quantify the control objective, we define the tracking error signal, denoted by $e_1(t) \in \mathbb{R}^m$, as follows

$$e_1 \triangleq x_d - x. \tag{50}$$

As before, the control objective is to obtain practical tracking with a continuous law with full-state feedback (*i.e.*, $x^{(i)}(t)$, $i = 0, 1, \cdots, n-1$ are assumed measurable) and ensure that all the signals remain bounded during the closed-loop operation. To simplify the subsequent control design and stability analysis, we introduce the following auxiliary error signals $e_i(t) \in \mathbb{R}^m$, $i = 2, 3, \cdots, n$

$$\begin{aligned} e_2 &\triangleq \dot{e}_1 + e_1, \\ e_3 &\triangleq \dot{e}_2 + e_2 + e_1, \\ e_4 &\triangleq \dot{e}_3 + e_3 + e_2, \\ &\vdots \\ e_i &\triangleq \dot{e}_{i-1} + e_{i-1} + e_{i-2}, \\ &\vdots \\ e_n &\triangleq \dot{e}_{n-1} + e_{n-1} + e_{n-2}. \end{aligned} \tag{51}$$

where $e_1(t)$ was defined in (50). An expression can be derived for $e_i(t)$ for $i = 3, 4, \cdots, n$ in terms of $e_1(t)$ and its derivatives as following (see [15] for details)

$$e_i = \sum_{j=0}^{i-1} a_{ij} e_1^{(j)} \tag{52}$$

where the constant coefficients $a_{ij} \in \mathbb{R}$ are defined as follows

$$\begin{array}{l} a_{i0} = \Theta(i) = \dfrac{1}{\sqrt{5}}\left[(\dfrac{1+\sqrt{5}}{2})^i - (\dfrac{1-\sqrt{5}}{2})^i\right],\ i = 2, 3, ..., n \\ a_{ij} = \sum\limits_{k=1}^{i-1} \Theta(i-k-j+1) a_{k+j-1,j-1},\ i = 3, 4, ..., n \\ \text{and } j = 1, 2, ..., i-2 \\ a_{i,i-1} = 1, i = 1, 2, ..., n\ . \end{array} \tag{53}$$

Note that $e_i(t)$ for $i = 1, 2, \cdots, n$ is measurable since it is a function of the system states and the reference trajectory.

4.1 Control Law

Based on subsequent stability analysis, we design the control input $u(t)$ as follows

$$\begin{aligned} u(t) &= (K_s + I_m)e_n(t) - (K_s + I_m)e_n(t_0) + \\ &\int_{t_0}^{t} [(K_s + I_m)e_n(\tau) + \hat{f}(\tau)]d\tau \end{aligned} \tag{54}$$

where $K_s \in \mathbb{R}^{m \times m}$ is positive-definite, diagonal, control gain matrix, $I_m \in \mathbb{R}^{m \times m}$ represents the $m \times m$ identity matrix, $\hat{f}(\cdot) = \begin{bmatrix} \hat{f}_1(\cdot) & \hat{f}_2(\cdot) & \cdots & \hat{f}_m(\cdot) \end{bmatrix}^T \in \mathbb{R}^m$ denotes the user designed feed-forward component (It is assumed that $\hat{f}(\cdot) \in \mathcal{L}_\infty$). It is easy to see that $u(t_0) = 0$. To simplify the analysis, the diagonal components of the control gain matrix K_s are defined as follows

$$\begin{array}{l} k_{s1} = k_m + k_1 + k_n \\ \cdots \\ k_{si} = k_m + k_i + k_n \\ \cdots \\ k_{sm-1} = k_m + k_{m-1} + k_n \\ k_{sm} = k_m + k_n \end{array} \tag{55}$$

where k_{si} denotes the i^{th} diagonal component in K_s $\forall$ $i = 1, 2, \cdots, m$; and k_m, k_n, k_1, $\cdots$, k_{m-1} denotes positive control gains.

4.2 Error System Development

See Appendix B.

4.3 Stability Analysis

The main result of this section can be stated by the following theorem.

Theorem 2 *The control law of (54) ensures that all system signals are bounded under closed-loop operation and ensures the tracking error variable $z(t)$ exhibits locally uniformly bounded and the locally uniformly ultimately bounded properties given by Lemma 2.*

Proof: See Appendix C.

Remark 4 *As done in Remark 3, we easily illustrate that the above the proof of Theorem 2 and Lemma 2 actually yield a semi-global uniformly bounded result for the tracking error variable $z(t)$ by illustrating that the controller gains can be increased to cover a predetermined value of the initial conditions denoted by $\|z(t_0)\|$. In addition, one can develop feed-forward terms for the control law as done in Section 2 without altering the type of tracking result as long as the control designer ensures that $\hat{f}(\cdot) \in \mathcal{L}_\infty$.*

5 Simulation

See Appendix D.

6 Conclusion

In this paper, we considered the tracking control problem for a class of MIMO nonlinear systems for which the input matrix is positive definite but non-symmetric. By utilizing a decomposition property of the input matrix, a continuous control strategy was proposed in order to compensate for uncertain nonlinear functions associated with the system dynamic model and ensure semi-global uniform ultimately bounded tracking under a smoothness restriction on the uncertain system nonlinearities. Best-guess, neural network and fuzzy logic based approximation techniques were employed in the feed-forward control design to broaden the applicability of the approach. Simulation results were presented to demonstrate the efficacy of the control and feed-forward strategies.

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A Proof of Theorem 1

We define the positive function $V(z,t) : \mathbb{R}_+ \times \mathbb{R}^4$ as follows

$$V = \frac{1}{2}e^T e + \frac{1}{2}r^T M r \tag{56}$$

where $z(t)$ was defined in (31). Note that (56) can be bounded as

$$\gamma_1(\|z\|) \leqslant V \leqslant \gamma_2(\|z\|) \tag{57}$$

where the scalar class $\mathcal{K}_\infty$ functions, denoted by $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathbb{R}$, are defined as follows

$$\begin{array}{c} \gamma_1(\|z\|) = \frac{1}{2}\min(1, \underline{m})\|z\|^2 \\ \gamma_2(\|z\|) = \frac{1}{2}\max(1, \bar{m}(x))\|z\|^2 \end{array} \tag{58}$$

where (8) has been utilized.

After taking the time derivative of (56) and substituting from (14) and (28), we have

$$\begin{aligned} \dot{V} &= -e^T e - r^T r - r^T K_s r + r^T \tilde{N} + r^T \bar{N}_d \\ &= -\|z\|^2 + r^T \tilde{N}_1 - r_1 \tilde{N}_3 - r_1 \tilde{N}_4 - k_{n1}\|r\|^2 - \\ &\quad k_{n2}|r_1|^2 - k_{n3}\|r\|^2 + r^T \bar{N}_d \end{aligned} \tag{59}$$

where the definition of K_s in (13) has been utilized. After applying (30), (32) and the fact that $-k_{n3}\|r\|^2 + r^T\bar{N}_d \leqslant \frac{1}{k_{n3}}\|\bar{N}_d\|^2$, we can upper bound the right-hand side of (59) as follows

$$\begin{aligned}\dot{V} \quad &\leqslant \quad -\|z\|^2 + \left[\|r\|\,\bar{\rho}(\|z\|)\,\|z\| - k_{n1}\|r\|^2\right] + \\ &\qquad \left[|r_1|\,\rho_g(\|z\|)\,\|z\| - k_{n2}\,|r_1|^2\right] + \varepsilon \end{aligned} \tag{60}$$

where $\bar{\rho}(\cdot) \in \mathbb{R}$ is defined as $\bar{\rho} = \rho_1 + \rho_2$ and $\varepsilon \in \mathbb{R}$ is some positive constant that satisfies

$$\frac{1}{k_{n3}}\|\bar{N}_d\|^2 \leqslant \varepsilon \tag{61}$$

(See Remark 2 on $\bar{N}_d(\cdot)$). After completing the squares on the bracketed terms in (60), we obtain

$$\dot{V} \leqslant -(1 - \frac{1}{4k_{n1}}\bar{\rho}^2(\|z\|) - \frac{1}{4k_{n2}}\rho_g^2(\|z\|))\,\|z\|^2 + \varepsilon. \tag{62}$$

We can now use (62) to state that

$$\dot{V} \leqslant -\lambda_3\|z\|^2 + \varepsilon \tag{63}$$

provided that the following sufficient conditions are satisfied

$$\begin{aligned} k_{n1} &> \frac{1}{2}\bar{\rho}^2(\|z\|) \text{ (or } \|z\| < \bar{\rho}^{-1}(\sqrt{2k_{n1}})) \\ k_{n2} &> \frac{1}{2}\rho_g^2(\|z\|) \text{ (or } \|z\| < \rho_g^{-1}(\sqrt{2k_{n2}})) \end{aligned} \tag{64}$$

where $\lambda_3 \in \mathbb{R}$ is some positive constant. It is easy to see the gain conditions given in (64) can be satisfied given the facts that $\bar{\rho}(\|z\|)$ does not contain any control gains and that $\rho_g(\|z\|)$ does not depend on the control gain k_{n2} (See Remark 2). From (63), we form the following inequality

$$\dot{V} \leqslant -\gamma_3(\|z\|) \quad \text{for} \quad \eta_1 < \|z(t)\| < \eta_2 \tag{65}$$

where the positive function $\gamma_3(\|z\|) \in \mathbb{R}$ and the positive constants η_1, $\eta_2 \in \mathbb{R}$ are defined as follows

$$\begin{aligned} \gamma_3(\|z\|) = \lambda_4\|z\|^2 \qquad \eta_1 = \sqrt{\varepsilon} \\ \eta_2 = \min(\bar{\rho}^{-1}(\sqrt{2k_{n1}},\ \rho_g^{-1}(\sqrt{2k_{n2}})) \end{aligned} \tag{66}$$

with $\lambda_4 \in \mathbb{R}$ being some positive constant. We can now apply Lemma 2 to (57) and (65) to yield the locally uniformly bounded and the locally uniformly ultimately bounded properties for tracking error variable $z(t)$ given by the theorem statement; hence, if $\|z(t_0)\|$ satisfies the requirements given in the locally uniformly bounded property of Lemma 2, then $z(t) \in \mathcal{L}_\infty$. Standard signal chasing arguments along with Assumption F2 can now be used to show that all the signals remain bounded.

■

B Error System Development

To begin the error system development, we define the filtered error signal $r(t) \in \mathbb{R}^m$ as follows

$$r = \dot{e}_n + e_n \tag{67}$$

where $e_n(t)$ was defined in (51). After taking the time derivative of (67), multiplying both sides of the resulting equation by $M(\cdot)$, and then substituting from the derivative of (52) for $i = n$, we have

$$M\dot{r} = M\sum_{j=0}^{n-1} a_{nj}e_1^{(j+2)} + M\dot{e}_n. \tag{68}$$

After utilizing (50) and the last expression of (53), we can rewrite (68) as follows

$$\begin{aligned} M\dot{r} \quad &= \quad Mx_d^{(n+1)} - Mx^{(n+1)} + M(\sum_{j=0}^{n-2} a_{nj}e_1^{(j+2)} + \dot{e}_n) \\ &= \quad M(x_d^{(n+1)} + \sum_{j=0}^{n-2} a_{nj}e_1^{(j+2)} + \dot{e}_n) + \dot{M}x^{(n)} \\ &\qquad -\dot{F} - \dot{T}u - T\dot{u} \end{aligned} \tag{69}$$

where the time derivative of (45) has been used. We now arrange (69) into the following advantageous form

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{\bar{T}}\,u - \bar{T}\dot{u} - \dot{u} + N_1 \tag{70}$$

where the auxiliary function $N_1(\cdot) \in \mathbb{R}^m$ is defined as follows

$$\begin{aligned} N_1 \quad &\triangleq \quad M(x_d^{(n+1)} + \sum_{j=0}^{n-2} a_{nj}e_1^{(j+2)} + \dot{e}_n) + \dot{M}x^{(n)} + \\ &\qquad \frac{1}{2}\dot{M}r + e_n - \dot{F} \end{aligned} \tag{71}$$

with the strict upper triangular matrix, denoted by $\bar{T}(\cdot) \in \mathbb{R}^{m\times m}$, is defined as follows $\bar{T} = T - I_m$. Finally, after defining the auxiliary expression $N_{1d}(t) \triangleq N(x_d, \dot{x}_d, \cdots, x_d^{(n)}, t) \in \mathbb{R}^m$, (70) can be rewritten as

$$M\dot{r} = -\frac{1}{2}\dot{M}r - e_n - \dot{\bar{T}}\,u - \bar{T}\dot{u} - \dot{u} + N_{1d} + \tilde{N}_1 \tag{72}$$

where the auxiliary function $\tilde{N}_1(\cdot) \in \mathbb{R}^m$ is defined as follows $\tilde{N}_1 = N_1 - N_{1d}$. Note that it is not difficult to show that $N_{1d}(t)$, $\dot{N}_{1d}(t) \in \mathcal{L}_\infty$ after utilizing (48) and (49). In addition, we note that as in Section 2.2, we can show that

$$\begin{aligned} &\left\|\tilde{N}_1\right\| \leqslant \rho_1(\|z\|)\,\|z\| \qquad \text{where} \\ &z = \begin{bmatrix} e_1^T & \cdots & e_n^T & r^T \end{bmatrix}^T \in \mathbb{R}^{m(n+1)} \end{aligned} \tag{73}$$

where $\rho_1(\|z\|) \in \mathbb{R}$ is some positive bounding function that is non-decreasing in $\|z\|$ and does not contain any control gains.

To facilitate the analysis, we note that the matrix $\dot{\bar{T}}\ (X)$ in (72) can be expressed as follows

$$\dot{\bar{T}}= \left[\dot{T}_{ij}(X)\right] \qquad \text{for } i = 1, 2, \cdots, m-1;\ j = i+1, \cdots, m \tag{74}$$

where $T_{ij}(X) \in \mathbb{R}$ is the ij^{th} element of the unit upper triangular matrix $T(X)$. It is obvious that $\dot{\bar{T}}\ (X)$ is a strict upper triangular matrix since all its diagonal elements are zeros. By utilizing (50), we can write $\dot{T}_{ij}(X)$ of (74) as follows

$$\begin{aligned}\dot{T}_{ij}(X) &= \sum_{k=0}^{n-1} \frac{\partial T_{ij}(X)}{\partial x^{(k)}} x^{(k+1)} \\ &= \sum_{k=0}^{n-1} \frac{\partial T_{ij}(X)}{\partial x^{(k)}} (x_d^{(k+1)} - e_1^{(k+1)}) \\ &= \tilde{N}_{2ij} + \sum_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)} \end{aligned} \tag{75}$$

where the auxiliary function $\tilde{N}_{2ij}(\cdot) \in \mathbb{R}$ is defined as follows

$$\begin{aligned}\tilde{N}_{2ij} &= -\sum_{k=0}^{n-1} \frac{\partial T_{ij}(X)}{\partial x^{(k)}} e_1^{(k+1)} + \sum_{k=0}^{n-1} \left(\frac{\partial T_{ij}(X)}{\partial x^{(k)}} x_d^{(k+1)} - \right. \\ &\left. \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)} \right). \end{aligned} \tag{76}$$

It is easy to show that $\tilde{N}_{2ij}(\cdot)$ given in (76) does not contain any control gains.

Similarly, based on (45), we can explicitly express the i^{th} component of the control vector $u(t)$ in (72) as follows

$$u_i = \sum_{j=1}^{m} M_{ij}(X) x_j^{(n)} - F_i(X) - \sum_{j=i+1}^{m} T_{ij}(X) u_j \tag{77}$$

where $i = 1,\ 2,\ \cdots,\ m$, $M_{ij}(\cdot) \in \mathbb{R}$ represents the ij^{th} component in matrix $M(\cdot)$, and $F_i(\cdot) \in \mathbb{R}$ denotes the i^{th} component in vector $F(\cdot)$. We further rewrite (77) as follows

$$u_i = \tilde{u}_i + u_{id} \tag{78}$$

where the auxiliary functions $\tilde{u}_i(t)$, $u_{id}(t) \in \mathbb{R}$ are defined as follows

$$\begin{aligned}\tilde{u}_i &= (\sum_{j=1}^{m} M_{ij}(X) x_j^{(n)} - F_i(X) - \sum_{j=i+1}^{m} T_{ij}(X) u_j) - \\ &(\sum_{j=1}^{m} M_{ij}(X_d) x_{dj}^{(n)} - F_i(X_d) - \\ &\sum_{j=i+1}^{m} T_{ij}(X_d) u_{jd}) \end{aligned} \tag{79}$$

and

$$u_{id} = \sum_{j=1}^{m} M_{ij}(X_d) x_{dj}^{(n)} - F_i(X_d) - \sum_{j=i+1}^{m} T_{ij}(X_d) u_{jd}. \tag{80}$$

It is easy to show that $\tilde{u}_i(t)$ and $u_{id}(t)$ in (78) do not contain any control gains. After utilizing (75) and (79), we now obtain the following expression for $\dot{T}_{ij}(X)u_j$

$$\dot{T}_{ij}(X) u_j = \tilde{N}_{3ij} + (\sum_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) u_{jd} \tag{81}$$

where the auxiliary function $\tilde{N}_{3ij}(\cdot) \in \mathbb{R}$ is defined as follows

$$\tilde{N}_{3ij} = \tilde{N}_{2ij} \tilde{u}_j + u_{jd} \tilde{N}_{2ij} + (\sum_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) \tilde{u}_j. \tag{82}$$

for $i = 1, 2, \cdots, m-1;\ j = i+1, \cdots, m$. The expression given by (81) can now be used to rewrite the vector term $\dot{\bar{T}}\ u \in \mathbb{R}^m$ in (72) as follows

$$\begin{aligned}\dot{\bar{T}}\ u &= \begin{bmatrix} \sum_{j=i+1}^{m} \dot{T}_{ij}(X) u_j \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{N}_{3i} + \sum_{j=i+1}^{m} (\sum_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) u_{jd} \\ 0 \end{bmatrix} \end{aligned} \tag{83}$$

where $i = 1, 2, \cdots, m-1$, and the auxiliary function $\tilde{N}_{3i}(\cdot) \in \mathbb{R}$ is defined as follows

$$\tilde{N}_{3i} = \sum_{j=i+1}^{m} \tilde{N}_{3ij}. \tag{84}$$

From (76), (79), (82), and (84), we can utilize the Mean Value Theorem to upper bound the auxiliary function $\tilde{N}_{3i}(\cdot)$ as follows

$$\left| \tilde{N}_{3i} \right| \leqslant \rho_{2i}(\|z\|) \|z\| \tag{85}$$

where $\rho_{2i}(\|z\|) \in \mathbb{R}$ is a positive bounding function that is non-decreasing in $\|z\|$ and does not contain any control gains. We also note that $\sum_{j=i+1}^{m} (\sum_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) u_{jd}$ in (83) and its time derivative are bounded due to the facts delineated in (48), (49) and (80).

Continuing in the same manner, we can write the vector item $\bar{T}\dot{u} \in \mathbb{R}^m$ in (72) in the following form

$$\bar{T}\dot{u} = \begin{bmatrix} \sum_{j=i+1}^{m} T_{ij}(X) \dot{u}_j \\ 0 \end{bmatrix} \qquad \text{for } i = 1, 2, \cdots, m-1. \tag{86}$$

where the time derivative of (54) can be used to write the term $T_{ij}(X)\dot{u}_j \in \mathbb{R}$ as follows

$$\begin{aligned} T_{ij}(X)\dot{u}_j &= (T_{ij}(X) - T_{ij}(X_d) + T_{ij}(X_d)) \\ &\quad ((k_{s_j}+1)r_j + \hat{f}_j) \\ &= \tilde{N}_{4ij} + T_{ij}(X_d)\hat{f}_j \end{aligned} \tag{87}$$

with the auxiliary functions $\tilde{N}_{4ij}(\cdot) \in \mathbb{R}$ being defined as follows

$$\tilde{N}_{4ij} = \tilde{T}_{ij}(k_{sj}+1)r_j + \tilde{T}_{ij}\hat{f}_j + T_{ij}(X_d)(k_{sj}+1)r_j, \tag{88}$$

and the auxiliary functions $\tilde{T}_{ij}(\cdot) \in \mathbb{R}$ being defined as follows

$$\tilde{T}_{ij} = T_{ij}(X) - T_{ij}(X_d). \tag{89}$$

The expression given by (87) can now be used to can rewrite (86) as follows

$$\bar{T}\dot{u} = \begin{bmatrix} \tilde{N}_{4i} + \sum\limits_{j=i+1}^{m} T_{ij}(X_d)\hat{f}_j \\ 0 \end{bmatrix} \quad \text{for } i = 1, 2, \cdots, m-1 \tag{90}$$

where the auxiliary functions $\tilde{N}_{4i}(\cdot) \in \mathbb{R}$ is defined as $\tilde{N}_{4i}(\cdot) = \sum\limits_{j=i+1}^{m} \tilde{N}_{4ij}(\cdot)$. As similarly done before, we can use the Mean Value Theorem to upper bound $\tilde{N}_{4i}(\cdot)$ as follows

$$\left|\tilde{N}_{4i}\right| \leqslant \rho_{gi}(\|z\|)\,\|z\| \tag{91}$$

where $\rho_{gi}(\|z\|) \in \mathbb{R}$ is a positive bounding function that is non-decreasing in $\|z\|$. It should be noted that $\rho_{gi}(\|z\|)$ does contain the control gain k_{sj} for $j = i+1, \cdots, m$.

Based on (54), (72), (83) and (90), we can now formulate the closed loop system for $r(t)$ as follows

$$\begin{aligned} M\dot{r} &= -\frac{1}{2}\dot{M}r - e_n - \begin{bmatrix} \tilde{N}_{3i} \\ 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} \sum\limits_{j=i+1}^{m} (\sum\limits_{k=1}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) u_{jd} \\ 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} \tilde{N}_{4i} \\ 0 \end{bmatrix} - \bar{T}(X_d)\hat{f} - (K_s + I_m)r \\ &\quad - \hat{f} + \tilde{N}_1 + N_{1d} \end{aligned} \tag{92}$$

where $i = 1, 2, \cdots, m-1$. To facilitate the stability analysis, (92) can be rearranged into the following advantageous form

$$M\dot{r} = --\frac{1}{2}\dot{M}r - e_n - (K_s + I_m)r + \tilde{N} + N_d \tag{93}$$

where the auxiliary functions $\tilde{N}(\cdot)$, $N_d(\cdot) \in \mathbb{R}^m$ are defined as follows

$$\begin{aligned} \tilde{N} &= \tilde{N}_1 - \begin{bmatrix} \tilde{N}_{3i} \\ 0 \end{bmatrix} - \begin{bmatrix} \tilde{N}_{4i} \\ 0 \end{bmatrix} \\ N_d &= N_{1d} - \begin{bmatrix} \sum\limits_{j=i+1}^{m} (\sum\limits_{k=0}^{n-1} \frac{\partial T_{ij}(X_d)}{\partial x_d^{(k)}} x_d^{(k+1)}) u_{jd} \\ 0 \end{bmatrix} - \hat{f} \end{aligned} \tag{94}$$

for $i = 1, 2, \cdots, m-1$. It is not difficult to show that the assumptions given earlier allow us to show that $N_d(\cdot)$, $\bar{T}(\cdot) \in \mathcal{L}_\infty$.

C Proof of Theorem 2

We define the positive function $V(t, z) : \mathbb{R}_+ \times \mathbb{R}^{m(n+1)}$ as follows

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^T e_i + \frac{1}{2} r^T M r. \tag{95}$$

It is easy to see that

$$\gamma_1(\|z\|) \leqslant V \leqslant \gamma_2(\|z\|) \tag{96}$$

where $\gamma_1(\cdot)$, $\gamma_2(\cdot) \in \mathbb{R}$ are the scalar class $\mathcal{K}_\infty$ functions defined as follows

$$\begin{aligned} \gamma_1(\|z\|) &= \tfrac{1}{2}\min(1, \underline{m})\,\|z\|^2 \\ \gamma_2(\|z\|) &= \tfrac{1}{2}\max(1, \bar{m}(x))\,\|z\|^2 \end{aligned} \tag{97}$$

where (8) has been utilized. After taking the time derivative of (95) and substituting from (51), (55), (67) and (93), we obtain

$$\begin{aligned} \dot{V} &= -\sum_{i=1}^{n} e_i^T e_i - e_{n-1}^T e_n - r^T r - r^T K_s r + \\ &\quad r^T \tilde{N} + r^T N_d \\ &= -\sum_{i=1}^{n} e_i^T e_i - e_{n-1}^T e_n - k_m \|r\|^2 - k_n \|r\|^2 - \\ &\quad \sum_{i=1}^{m-1} k_i \|r_i\|^2 + r^T \tilde{N}_1 - \sum_{i=1}^{m-1} r_i^T \tilde{N}_{3i} - \\ &\quad \sum_{i=1}^{m-1} r_i^T \tilde{N}_{4i} + r^T N_d. \end{aligned} \tag{98}$$

By employing the fact that $e_{n-1}^T e_n \leqslant \frac{1}{2}(\|e_{n-1}\|^2 + \|e_n\|^2)$, $-k_n \|r\|^2 + r^T N_d \leqslant \frac{1}{k_n}\|N_d\|^2$, we can use (73), (85) and (91) to obtain an upper bound on (98) as follows

$$\begin{aligned} \dot{V} &\leqslant -\frac{1}{2}\|z\|^2 - k_m \|r\|^2 + r^T \bar{\rho}(\|z\|)\,\|z\| - \\ &\quad \sum_{i=1}^{m-1} k_i \|r\|^2 + \sum_{i=1}^{m-1} \|r_i\|\,\rho_{gi}(\|z\|)\,\|z\| + \frac{1}{k_n}\|N_d\|^2 \\ &\leqslant -(\frac{1}{2} - \frac{\bar{\rho}^2(\|z\|)}{4k_m} - \sum_{i=1}^{m-1} \frac{\rho_{gi}^2(\|z\|)}{4k_i})\,\|z\|^2 + \varepsilon_0 \end{aligned} \tag{99}$$

where the scalar class $\mathcal{K}_\infty$ function $\bar{\rho}(\cdot) \in \mathbb{R}$ is defined as follows

$$\bar{\rho} = \rho_1 + \sum_{i=1}^{m-1} \rho_{2i}. \tag{100}$$

and $\varepsilon_0 \in \mathbb{R}$ denotes some positive constant that satisfies

$$\varepsilon_0 \geqslant \frac{1}{k_n} \|N_d\|^2. \tag{101}$$

Given the fact that $\bar{\rho}(\cdot)$ does not contain control gain k_m, and $\rho_{gi}(\cdot)$ does not contain the control gain k_i, the proof of Theorem 1 can now be followed to prove the result stated in Theorem 2 provided

$$\begin{aligned} k_m &\geqslant \bar{\rho}^2\left(\sqrt{\frac{\lambda_2(\|z(t_0)\|)}{\lambda_1}}\|z(t_0)\|\right), \\ k_i &\geqslant (m-1)\rho_{gi}^2\left(\sqrt{\frac{\|z(t_0)\|}{\lambda_1}}\|z(t_0)\|\right)\ i = 1, \cdots, m-1\ . \end{aligned} \tag{102}$$

We note that (67) and (73) can be used to obtain an explicit expression for $\|z(t_0)\|$ which is independent of all control gains as done in Remark 3. ■

D Simulation

D.1 Simulation Model

In order to demonstrate the performance of the proposed control law, we present the results of a numerical simulation. To this end, the following two degree-of-freedom (DOF) system was considered [17]

$$\begin{aligned} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \\ & \begin{bmatrix} -h\dot{q}_2 & -h\left(\dot{q}_1 + \dot{q}_2\right) \\ h\dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{aligned} \tag{103}$$

where $q_i(t)$ denote the i^{th} DOF position,

$$\begin{aligned} &H_{11} = a_1 + 2a_3 \cos q_2 + 2a_4 \sin q_2 \\ &H_{12} = H_{21} = a_2 + a_3 \cos q_2 + a_4 \sin q_2 \\ &H_{22} = a_2 \\ &h = a_3 \sin q_2 - a_4 \sin q_2 \end{aligned} \tag{104}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \alpha(q_1, q_2) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{105}$$

$a_1 = 4.42$, $a_2 = 0.97$, $a_3 = 1.04$, and $a_4 = 0.6$. In (105), $u_1(t)$, $u_2(t)$ denoted the control inputs, and the function $\alpha(q_1, q_2) = H_{11}H_{22} - H_{12}H_{21} \in \mathbb{R}$. The scalar the function $\alpha(q_1, q_2)$ could also be modified to represent environment related factors as shown in [21, 22] or other effects as shown in [23, 24, 25].

The control objective is to ensure that $q(t) = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}^T$ track the following reference trajectory

$$q_d(t) = \begin{bmatrix} 30\sin(t)\left(1 - \exp(-0.3t^3)\right) \\ 45\sin(t)\left(1 - \exp(-0.3t^3)\right) \end{bmatrix} \text{deg}; \tag{106}$$

hence, the tracking error variable is defined as $e_1(t) = q_d(t) - q(t)$. The initial condition was set to $q_i(0) = 0.06$ degree for $i = 1, 2$. First, we carried on the simulation without feedforward terms in the control input $u(t)$ in (54) (*i.e.*, $\hat{f}(t) = 0$). The feedback control gains were set as follows $K_s = diag\{5, 3\}$. Figure 1 shows the tracking error $e_1(t)$ while Figure 2 shows the control input $u(t)$. We note that one can decrease the magnitude of the tracking error $e_1(t)$ by increasing control gains inside the matrix K_s.

D.2 Neural Network Based Compensation

For simplicity, we employed a radial basis neural network (RBNN) [19] to approximate the unknown nonlinear function $f(\cdot) = N_{1d}(\cdot) - \begin{bmatrix} \dfrac{\partial T_{12}(x_d)}{\partial x_d}\dot{x}_d u_{2d}(\cdot) \\ 0 \end{bmatrix}$ in (29). The RBNN is comprised of a layer of radial basis activation functions with the neurons number $p = 10$ and the output of the neural network system is designed as follows

$$\hat{f}(\cdot) = \hat{W}(t)^T \sigma(\bar{V}^T \chi_1) \tag{107}$$

where $\chi_1\left(1, x_d, \dot{x}_d, \ddot{x}_d, \dddot{x}_d\right) \in \mathbb{R}^9$, the first component in $\chi_1(t)$ is set to 1 to produce a basis, $\bar{V} \in \mathbb{R}^9$ is set to constant values in order to provide a basis [19], $\hat{W}(t) \in \mathbb{R}^{10\times 2}$ denotes the weight estimates of idea weight gain matrix W. A sigmoid function $\sigma(z) = \dfrac{1}{1+\exp(-s)}$ is utilized as the activation function. The following weight tuning law for $\hat{W}(t)$ is utilized

$$\begin{aligned} \dot{\hat{W}} &= -\alpha_1 \hat{W} + \Gamma_1 \sigma(V^T\chi_1)\text{sat}(e_2 + \zeta_1) \\ \zeta_1 &= \frac{1}{\varepsilon_1}(-\eta_1 + e_2) \\ \dot{\eta}_1 &= \frac{1}{\varepsilon_1}(-\eta_1 + e_2) \end{aligned} \tag{108}$$

where $\alpha_1, \varepsilon_1 \in \mathbb{R}$ are some small positive constant, $\Gamma_1 \in \mathbb{R}^{10\times 10}$ is a diagonal, p.d. update gain matrix, $\text{sat}(\cdot) \in \mathbb{R}^2$ is defined as follows

$$\text{sat}(\cdot) = \begin{bmatrix} sat(\xi_1) & sat(\xi_2) \end{bmatrix} \quad \forall \xi = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T \tag{109}$$

with $sat(\cdot)$ as the standard saturation function, while $\zeta_1, \eta_1 \in \mathbb{R}^2$ are auxiliary filter signals. It is not difficult to check that $\hat{W}(t) \in \mathcal{L}_\infty$ as required.

The control gain matrix K_s was selected as in Section D.1 while $\bar{V}$ in (107) was selected as follows

$$\bar{V} = \begin{bmatrix} 1 & 0.1 & 0.2 & 0.3 & 0.11 & 0.25 & 0.6 & 0.27 & 1 \end{bmatrix}^T. \tag{110}$$

The upper and lower value for the saturation function in (108) were set as 100 and -100, respectively. The weight tuning gains in (108) were tuned by trial-and-error until good tracking performance was achieved. The turning procedure resulted in the following gain values

$$\begin{aligned} &\alpha_1 = 0.01 \quad \varepsilon_1 = 0.0001 \\ &\Gamma_1 = diag(100, 220, 200, 120, 150, 150, 240, 200, 160, 280). \end{aligned} \tag{111}$$

The tracking error and control input for the proposed control law are shown in Figure 3 and Figure 4, respectively. While the control gains were set to the same values as in Section D.1, the tracking error $e_1(t)$ was driven to a very small value around zero with the inclusion of neural network based feed-forward component in the control input.

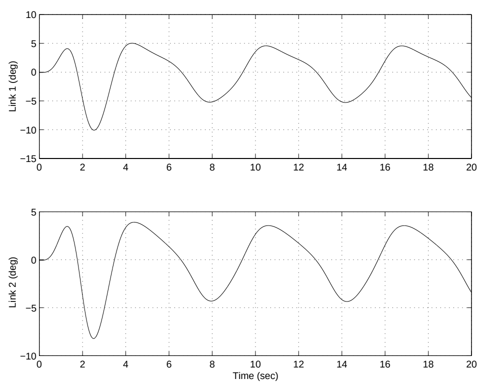

Figure 1: Tracking error for proposed controller without feed forward component $\hat{f}(t)$

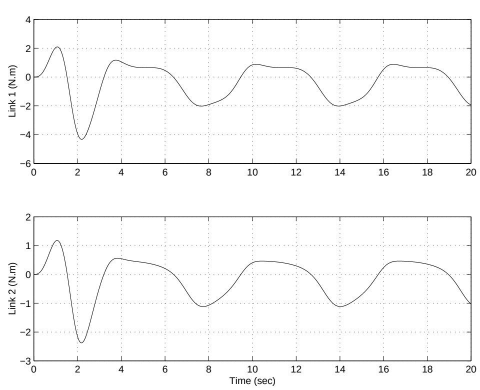

Figure 2: Control input for the proposed controller without feedforward component $\hat{f}(t)$

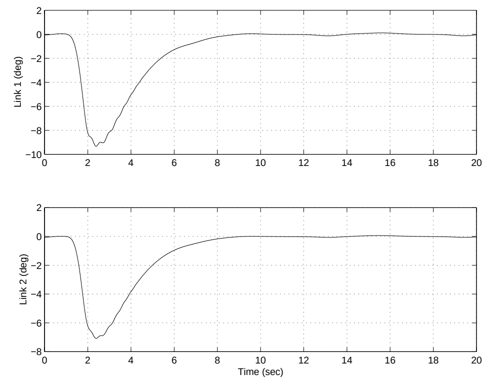

Figure 3: Tracking error for the proposed controller with neural-network based feed forward component $\hat{f}(t)$.

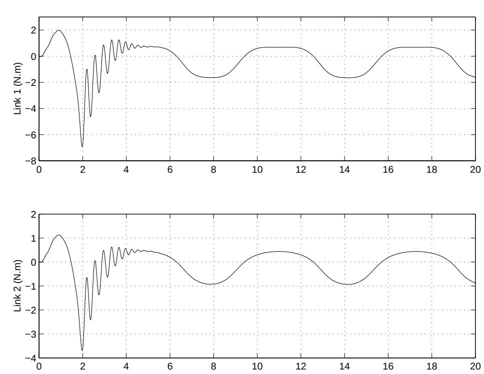

Figure 4: Control input for the proposed controller with neural network based feed forward component $\hat{f}(t)$.