Quality Discrete Representations in Multiple Objective Programming

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Abstract

Within the past ten years, more emphasis has been placed on generating discrete representations of the nondominated set which are truly representative of the nondominated set as a whole. This paper reviews measures for assessing the quality of discrete representations as well as exact solution methods that attempt to produce representations satisfying certain quality criteria. The measures are classified according to the aspect of the representation which they assess: cardinality, coverage, or spacing. The proposed solution methods are categorized according to whether a measure is integrated into the procedure *a priori* (before generation of solution points), *a posteriori* (after the generation of solution points), or not at all. The paper concludes with a comparative discussion of these three approaches and directions for future research.

Keywords: multiple objective programs, discrete representations, quality measures, Pareto set, nondominated set, efficient set

1 Introduction

Multicriteria optimization problems arise frequently and in a wide array of applications. Typically, conflict among the criteria leads to a large, often infinite, number of solutions rather than a single unique optimum. Mathematically, all of the solutions are equivalent, so the selection of the final "best" solution depends upon the preferences and experiences of the decision maker (DM). It is the role of multiple objective programming, then, to present the DM with as much information as possible about the solution set to facilitate the decision making process.

For multiple objective programs (MOPs) with continuous variables, many methods are proposed for generating individual solution points and for approximating the solution set as a whole (see [10], [32] for reviews). It is our opinion that discrete representations of solution sets (collections of solution points) are preferable to approximated solution sets (which use some sort of approximating structure in addition to solution points) for three reasons. First, discrete representations present a

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finite, manageable number of solutions to the DM, whereas approximations do not limit the number of solutions. Second, discrete representations explicitly provide the DM with solution points while solutions are only implicitly available through an approximating structure. Lastly, all the solutions points in a discrete representation are optimal for the MOP; this is not necessarily true for solution points inferred from an approximated solution set.

With respect to discrete representations, several authors (e.g., [1], [2], [14]) suggest that more emphasis should be placed on finding globally representative subsets of the solution sets of MOPs, instead of contenting ourselves with simply finding solutions. That is, solution procedures should be placed into a larger framework where both the relationship among solution points and the relationship between the approximated solution set and the true solution set are considered. Further, Lotov et. al. [23] and Berezkin et. al. [3] discuss the importance of the visualization of solution sets as a tool in the decision making process. Generating representative subsets of the solution sets of MOPs will both provide complete information to the DM as well as aid in her visualization of the solution set (in two and three dimensions).

In the past decade, many researchers have proposed measures for determining the quality of solution sets of MOPs. Others have begun integrating those and other measures into algorithms for generating discrete representations which meet a prespecified quality criterion. In this paper, we review and classify the measures and algorithms that have been published. Note that we include only exact algorithms which produce discrete representations satisfying a stated quality measure or which improve on a certain quality characteristic of a previous method. Further, we do not include any measure that quantifies the "error" of a discrete representation (the distance between the representation and the true solution set) because in the case of exact algorithms, the representation will always be a subset of the true solution set (i.e., there is no error).

We organize the paper as follows. Section 2 gives a brief overview of the terminology and notation that will be used. In Section 3, we present measures that have been proposed independently of an algorithm or that are used in conjunction with a heuristic algorithm. The measures are sorted according to the scheme proposed by Sayin [33] which is discussed at the beginning of that section. Section 4 contains published methods sorted according to whether a measure is integrated before generation of a solution point (*a priori*), after generation of a solution point (*a posteriori*), or not at all. We include the last category because we found several algorithms that were developed to produce quality representations of the solution set but do not integrate any measure per se. Section 5 concludes the paper with closing remarks.

2 Terminology and Notation

The general form of an MOP is given in (1):

$$\begin{array}{ll} \text{minimize} & f(x) = [f_1(x), \ldots, f_p(x)] \\ \text{subject to} & x \in X \end{array} \tag{1}$$

where $f_i : \Re^n \to \Re$ for $i = 1, \ldots, p$ and $X \subseteq \Re^n$. A solution $y^* = f(x^*)$ to (1) is called a *Pareto optimal* point if there does not exist $x \in X$ such that $f(x) \leq f(x^*)$. Further, a solution is called a *weak Pareto optimal* point if there does not exist $x \in X$ such that $f(x) < f(x^*)$. The set of all (weak) Pareto optimal points is called the *(weak) Pareto set* and is denoted by Y_N (Y_{WN}). The pre-image of Y_N (Y_{WN}), in the decision space, is known as the *(weakly) efficient set*, X_E (X_{WE}).

In (1), we assume that our preferences are modeled by the Pareto cone $\Re^p_{\geqq} := \{y \in \Re^p : y_i \geq 0, i = 1, \ldots, p\}$. However, we may generalize the concept of optimality to an aribitrary convex preference cone C. In this case, we say a point $y^* = f(x^*)$ is *nondominated* if $(y^* - C) \cap Y = \{y^*\}$ or is *weakly nondominated* if $(y^* - \text{int } C) \cap Y = \emptyset$, and we call Y_N and Y_{WN} the *nondominated set* and the *weakly nondominated set*, respectively.

Throughout the paper, we denote discrete representations to the true efficient set and nondominated set by $\overline{X}_E$ and $\overline{Y}_N$, respectively. Moreover, for ease of notation, the same symbol may denote slighty different concepts when used in different contexts, but, in such cases, adequate explanation will be given. In particular, unless otherwise stated, $d(\cdot, \cdot)$ denotes a metric for measuring the distance between two points in $\Re^n$; when not specifically defined, the DM should choose the appropriate metric for her situation. Lastly, when reviewing papers, we have maintained the authors' notations as much as possible.

3 Review of Measures

In general, we would like to provide the DM with a "good" representation of the nondominated set. The meaning of "good", here, is ambiguous because no definite consensus has been reached in the mathematical and operations research community on what qualities a good representation of the nondominated set should possess. However, within the past ten years, many authors have suggested quality measures which may be useful to this end. In this section, we present and classify the measures proposed in the literature.

We sort the measures into three main groups as suggested by Sayin [33]: measures of *cardinality*, *coverage*, and *spacing*. Cardinality refers to the number of points in a representation. In general, we desire enough points to fully represent the solution set, but not so many that the DM is overwhelmed with choices. Measures of coverage seek to ensure that all regions of the solution set are represented. That is, we do not want any portion of the solution set to be neglected. Measures of spacing quantify the distance between points in the representation. Typically, we would like a representation to have uniform, or equidistant, spacing, so that all portions of the solution set are represented to an equal degree. It is important to note that, to the best of our knowledge, all of the quality measures in the literature are defined with respect to the Pareto notion of optimality.

3.1 Measures of Cardinality

Measures of cardinality essentially boil down to the same idea: count the number of Pareto points in the representation. Clearly, two straightforward candidates for measures of cardinality are the size of the generated Pareto set (i.e. $|\overline{Y}_N|$) and the size of the generated efficient set (i.e. $|\overline{X}_E|$). Van Veldhuizen [40] proposes the former measure as "overall nondominated vector generation", while Sayin [33] proposes the latter.

On the other hand, in this group we also have the measure proposed by Wu and Azarm [41] called the "number of distinct choices". This measure is similar to the previous two but takes the DM's preferences into account: Pareto solutions within a certain distance of each other (a choice made by the DM) are counted as a single point. Thus, the number of distinct choices of a discrete representation will always be less than or equal to its cardinality. To calculate this measure, let μ $(0 < \mu < 1)$ be chosen so that the DM is indifferent between any two solutions whose difference in each (normalized) criterion is less than or equal to μ. Next, divide the objective space into $\frac{1}{\mu^p}$ hypercubes. These hypercubes are indifference regions. Let $T_\mu(q)$ be the hypercube defined by a point q, and let $NT_\mu(q, \overline{Y}_N)$ be defined as follows:

$$NT_\mu(q, \overline{Y}_N) = \begin{cases} 1, & \exists\, y \in \overline{Y}_N \text{ such that } y \in T_\mu(q) \\ 0, & \text{else} \end{cases} .$$

Then, given the above, the number of distinct choices of a nondominated set, $\overline{Y}_N$, is

$$NDC_\mu(\overline{Y}_N) = \sum_{l_p=0}^{\frac{1}{\mu}-1} \cdots \sum_{l_2=0}^{\frac{1}{\mu}-1} \sum_{l_1=0}^{\frac{1}{\mu}-1} NT_\mu(q, \overline{Y}_N) \tag{2}$$

where $q = (q_1, \ldots, q_p)$ with $q_i = l_i \mu$. Thus, the number of distinct choices of a set $\overline{Y}_N$ is the number of hypercubes containing at least one Pareto point.

Since the cardinality of a discrete representation of the Pareto set is easily controlled, this category of measures is less important than the following two. In general, the cardinality of a representation should be minimized while still maintaining good coverage and spacing.

3.2 Measures of Coverage

Measures of coverage face the challenge of trying to assess something that is unknown because, in general, the true Pareto set is not known *a priori*. However, we must try to ensure that no region of Y_N is neglected. Because of this, we always seek to maximize the coverage of a discrete representation, and unless otherwise noted, the following measures should be maximized as well.

Zitzler and Thiele [43] propose a measure to determine the size of the region dominated by $\overline{Y}_N$. In his dissertation, Ziztler [42] calls this measure the "S-measure". Each point $y \in \overline{Y}_N$ dominates a (hyper)cube with one corner at y and another at $y^{\max}$ where $y^{\max} = (f_1^{\max}, \ldots, f_p^{\max})$ and $f_i^{\max} = \max_{x \in X} f_i(x)$. Thus, the region dominated by $\overline{Y}_N$, which we denote by $D(\overline{Y}_N)$, is found by taking the union of these cubes for all $y \in \overline{Y}_N$. The value of the S-measure is the volume of this union.

$$S(\overline{Y}_N) = \text{volume}(D(\overline{Y}_N)). \tag{3}$$

Zitzler [42] also suggests a measure, called "M_3", to determine the overall range of the representation:

$$M_3(\overline{Y}_N) = \sqrt{\sum_{i=1}^{p} \max\{|\, y_i - \tilde{y}_i| : y, \tilde{y} \in \overline{Y}_N\}}. \tag{4}$$

This measure calculates an average of the ranges of the criteria.

Sayin [33] suggests the measure "coverage" which determines the maximum distance, ϵ, between a point in the true efficient set and its closest neighbor in the representation:

$$\epsilon(X_E, \overline{X}_E) = \max_{x \in X_E} \min_{\tilde{x} \in \overline{X}_E} d(x, \tilde{x}). \tag{5}$$

Because we would like every efficient point in the true efficient set to be represented in our discrete representation, we want this measure to be minimized rather than maximized. Note also that this measure assumes that the true efficient set is known.

Wu and Azarm [41] propose two measures, "overall Pareto spread" and "kth Pareto spread", which calculate the range of the entire representation and of each individual criterion, respectively. These measures are defined as follows:

$$OS(\overline{Y}_N) = \prod_{k} |\max_{y \in \overline{Y}_N} y_k - \min_{y \in \overline{Y}_N} y_k| \tag{6}$$

and

$$OS_k(\overline{Y}_N) = |\max_{y \in \overline{Y}_N} y_k - \min_{y \in \overline{Y}_N} y_k|. \tag{7}$$

Wu and Azarm [41] also propose a measure called "hyperarea difference" which is a slight variation on the S-measure (3) of Ziztler and Thiele. This measure calculates the difference (in terms of volume) between the portions of the objective space which are dominated by the true Pareto set and a given representation of the Pareto set. To overcome the difficulty of not knowing the true Pareto set, they suggest normalizing the objective space so that the volume of the region dominated by the true Pareto set can be estimated as one. Given this, the hyperarea difference is computed as follows:

$$HD(\overline{Y}_N) = 1 - \text{volume}(D(\overline{Y}_N)) \tag{8}$$

where $D(\overline{Y}_N)$ denotes the region of the objective space which is dominated by the set $\overline{Y}_N$, and can be found as discussed in the paragraph preceding (3). Note that if $\overline{Y}_N = Y_N$ then $\text{vol}(D(\overline{Y}_N)) = 1$ and thus, $HD(\overline{Y}_N) = 0$. In general, a small hyperarea difference value is desired.

Meng et. al. [26] propose a measure of coverage called "extension". Let $x^i = \arg\min_{x \in X} f_i(x)$ and let $y^i = f(x^i)$, then we have a set of reference solutions, $\{y^1, \ldots, y^p\}$. Given a discrete representation $\overline{Y}_N$, we calculate the distance between each reference solution and $\overline{Y}_N$ as follows:

$$d(y^i, \overline{Y}_N) = \min\{d(y^i, y) \,|\, y \in \overline{Y}_N\}.$$

Finally, the extension is calculated as follows:

$$\text{EX}(\overline{Y}_N) = \frac{\sqrt{\sum_{i=1}^{p} (d(y^i, \overline{Y}_N))^2}}{p}. \tag{9}$$

Thus, the extension measures an average distance of a representation from its reference solutions. For this measure, a small value is preferred to a larger value because the latter could indicate that the representation is mainly in the center of the true Pareto set with the outskirts being neglected.

3.3 Measures of Spacing

Measures of spacing are abundant in the literature. In general, we desire a discrete representation of the Pareto set with equally-spaced Pareto points, so that each region of the true Pareto set is represented to an equal degree. Note that having equidistant Pareto points, however, does not guarantee that we have good coverage as well, so measures of spacing should always be used in conjunction with a coverage measure.

Schott [36] proposes a measure for bicriteria problems called "spacing" which takes the standard deviation of the distances between nearest-neighbor points:

$$f_{spacing}(\overline{Y}_N) = \sqrt{\frac{1}{|\overline{Y}_N| - 1} \sum_{i=1}^{|\overline{Y}_N|} (\bar{d} - d_i)^2} \tag{10}$$

where each d_i is measured with the l_1−norm and $\bar{d}$ is the average of the d_i. Since we want the spacing of Pareto points to be equidistant, small values for this measure are desired. Note that

Schott's measure can be extended to higher dimensions by changing the definition of d_i.

Ziztler [42] proposes the "M_2" measure which calculates the average cardinality of the set of points which are greater than a fixed distance, σ, from a Pareto point in the representation:

$$M_2(\overline{Y}_N) = \frac{1}{|\overline{Y}_N| - 1} \sum_{y \in \overline{Y}_N} |\{\tilde{y} \in \overline{Y}_N : d(y, \tilde{y}) > \sigma\}|. \tag{11}$$

This measure gives us a sense of the number of redundancies (with respect to the chosen value of σ) which are contained in our representation. Ideally, for a chosen σ, $M_2(\overline{Y}_N) = |\overline{Y}_N|$, indicating no redundancies.

Sayin [33] proposes the measure "uniformity" which is defined as the minimum distance, δ, between any two distinct points in the discrete representation of the efficient set:

$$\delta(\overline{X}_E) = \min_{x,\, \tilde{x} \in \overline{X}_E,\, x \neq \tilde{x}} d(x, \tilde{x}). \tag{12}$$

Wu and Azarm [41] suggest a measure called "cluster" which measures the average size of a redundant cluster of points (with respect to the parameter μ) in the representation. To compute the cluster value, the number of points in the representation is divided by the number of distinct choices, $NDC_\mu(\overline{Y}_N)$ (2): namely,

$$Cluster(\overline{Y}_N) = \frac{|\overline{Y}_N|}{NDC_\mu(\overline{Y}_N)}. \tag{13}$$

We desire no redundancies, so ideally, $|\overline{Y}_N| = NDC_\mu(\overline{Y}_N)$ which gives a cluster value of one. Otherwise, the cluster value will be greater than one.

In [28], Messac and Mattson present a measure of spacing called "evenness". For each point, y_i, in the discrete representation, two (hyper)spheres are constructed: the smallest and the largest spheres that can be formed between y_i and any other point in the set such that no other points are within the spheres. The diameters of the two spheres are denoted by d_l^i and d_u^i, respectively. The evenness, ξ, of a representation is then calculated with the following formula:

$$\xi(\overline{Y}_N) = \frac{\sigma_d}{\bar{d}} \tag{14}$$

where $\bar{d}$ and σ_d are, respectively, the mean and standard deviation of the set of minimum and maximum diameters for each point in the representation. A discrete representation with all points spaced equidistantly will have $\xi = 0$ because the d_l^i and d_u^i will all be equal (i.e., $\sigma_d = 0$).

Meng et. al. [26] propose a measure called "uniformity" which was inspired by wavelet analysis. This measure was developed for comparing two different representations of Y_N. Let $\overline{Y}_N^1$ and $\overline{Y}_N^2$ be two distinct discrete representations of the Pareto set, and suppose that $|\overline{Y}_N^1| = N$ and $|\overline{Y}_N^2| = M$.

Set $l = 1$. For each point in each set we calculate the distance to its nearest neighbor:

$$d_i^1 = d(y^i, \overline{Y}_N^1) = \min_{y^j \in \overline{Y}_N^1,\, y^j \neq y^i} d(y^i, y^j), \qquad i = 1, \ldots, N$$

and

$$d_k^2 = d(y^k, \overline{Y}_N^2) = \min_{y^j \in \overline{Y}_N^2,\, y^j \neq y^k} d(y^k, y^j), \qquad k = 1, \ldots, M.$$

Next we calculate the average distance between nearest neighbor points for both sets:

$$\bar{d}_l^1 = \frac{\sum_{i=1}^N d_i^1}{N} \qquad \text{and} \qquad \bar{d}_l^2 = \frac{\sum_{k=1}^M d_k^1}{M}.$$

Finally, we calculate the spacing measures as follows:

$$SP_l^1(\overline{Y}_N^1) = \sqrt{\frac{\sum_{i=1}^N (1 - F(d_i^1, \bar{d}_l^1))^2}{N-1}} \qquad \text{and} \qquad SP_l^2(\overline{Y}_N^2) = \sqrt{\frac{\sum_{k=1}^M (1 - F(d_k^2, \bar{d}_l^2))^2}{M-1}} \tag{15}$$

where

$$F(a,b) = \begin{cases} \frac{a}{b}, & \text{if } a > b \\ \frac{b}{a}, & \text{else} \end{cases} .$$

If $SP_l^1 < SP_l^2$, then $\overline{Y}_N^1$ has better uniformity, and vice versa. If $SP_l^1 = SP_l^2$ and $l \geq \min(N-1, M-1)$, then $\overline{Y}_N^1$ is the same as $\overline{Y}_N^2$. Else, if $SP_l^1 = SP_l^2$ and $l < \min(N-1, M-1)$, then increment l by one and decrement N and M by one, and recalculate the spacing measure for both sets, ignoring the smallest d_i^1 and d_k^2, respectively. Note that this measure, unlike the others we have presented, is binary because it is used to compare two different discrete representations of the Pareto set; the value of the spacing measure by itself does not have a clear interpretation.

Collette and Siarry [6] propose two different bicriteria spacing measures: "spacing", which is a modification of Schott's measure (10), and the "hole relative size" measure. Both measures require that the generated Pareto points be put in ascending order with respect to the first objective function. Their spacing measure is computed as follows:

$$Spacing(\overline{Y}_N) = \sqrt{\frac{1}{|\overline{Y}_N| - 1} \sum_{i=1}^{|\overline{Y}_N|-1} \left(1 - \frac{d_i}{\bar{d}}\right)^2} \tag{16}$$

where $d_i = \sqrt{(f_1(x_i) - f_1(x_{i+1}))^2 + (f_2(x_i) - f_2(x_{i+1}))^2}$ and $\bar{d}$ is the average of all the d_i. Their hole relative size measure gives the ratio of the largest gap between two adjacent points to the average gap:

$$HRS(\overline{Y}_N) = \frac{\max_i d_i}{\bar{d}} \tag{17}$$

where d_i and $\bar{d}$ are as defined previously. The authors note that the hole relative size measure would not be appropriate for use on a problem with a disconnected Pareto set.

3.4 Hybrid Measures

Several authors propose measures which overlap the above three categories. Deb et al. [8] [9] suggest the "Δ" measure which takes into account both the spacing between generated Pareto points and the coverage of the true Pareto set by the generated representation. This measure calculates the distance between each point and its nearest neighbors (a spacing-type measure) as well as the distance between the individual objective minima and their respective single nearest neighbors (a coverage-type measure). Including the second part of the measure ensures that there will not be a group of equally spaced points in the center of the set, for example, with the outer portions neglected. Deb's measure for bicriteria problems is as follows:

$$\Delta(\overline{Y}_N) = \frac{d_f + d_l + \sum_{i=1}^{|\overline{Y}_N|-1} |d_i - \bar{d}\,|}{d_f + d_l + (|\overline{Y}_N| - 1)\bar{d}} \tag{18}$$

where d_f and d_l are the Euclidean distances between the individual objective minima and the nearest points in the representation, the d_i are the Euclidean distances between each pair of consecutive Pareto points, and $\bar{d}$ is the average of all the d_i. A small value for Δ is desired with the ideal value being $\Delta = 0$: $d_i = \bar{d}$ for all i and $d_f = d_l = 0$, indicating that the individual objective minima are included in the representation. The authors note that this measure can be extended to three or more dimensions, but the formula would change slightly.

Leung and Wang [22] suggest the "U-measure" which measures both coverage and spacing, similar to Deb's measure (18). First, we determine the nearest neighbors of each Pareto point with respect to each axis, as well as the nearest neighbor of each reference point (i.e., the individual objective minima or other points chosen by the DM). Let χ be the set of distances between nearest neighbor solutions and let $\bar{\chi}$ be the set of distances between a reference point and its nearest neighbor solution. For good spacing, we would like the distances in χ to be roughly the same. For good coverage, we would like the distances in $\bar{\chi}$ to be close to zero. For ease of calculation, we combine the sets into one by computing the average of the distances in χ and incrementing each element in $\bar{\chi}$ by this number. We will denote this new set by $\bar{\chi}^{'}$. Now, we need only check that the elements in $\bar{\chi}^{'}$ are close to each other. Given this, we compute the U-measure as follows:

$$U(\overline{Y}_N) = \frac{1}{D} \sum_{i=1}^{D} \left| \frac{d_i^{'}}{d_{\text{ideal}}} - 1 \right| \tag{19}$$

where $d_i^{'} \in \bar{\chi}^{'}$, $d_{\text{ideal}} = \sum_{i=1}^{D} d_i^{'}/D$, and $D = |\bar{\chi}^{'}|$. If the points are equally spaced over the entire set, then $d_i^{'} = d_{\text{ideal}}$ for each i resulting in $U = 0$. This measure calculates the average deviation from the ideal so that a small U-measure indicates a representation that is close to equidistant and covers the entire Pareto set.

Farhang-Mehr and Azarm [13] propose a measure called "entropy" using ideas from the field of information theory. Entropy assesses all three of the quality categories: cardinality, coverage, and spacing. For each Pareto point $\bar{y}^i$ in a discrete representation $\overline{Y}_N$, we define a scalar-valued influence function Ω_i which is decreasing in the distance to $\bar{y}^i$ (e.g., a Gaussian distribution centered on $\bar{y}^i$). Then, the density function, defined for any point in the objective space, is given by the following:

$$D(y) = \sum_{i=1}^{|\overline{Y}_N|} \Omega_i(y), \qquad y \in Y.$$

Next, we create an $a_1 \times a_2 \times \ldots \times a_p$ grid in the objective space so that the DM is indifferent between solutions which share the same (hyper)cube (where a_i is the number of indifference regions with respect to the i^{th} axis). Let $y_{i_1,i_2,\ldots,i_p}$ denote the center point of the cube having grid position $(i_1, i_2, \ldots, i_p)$. Given this, we evaluate $D(y)$ for each center point and then normalize the result as follows:

$$\rho_{i_1,i_2,\ldots,i_p} = \frac{D(y_{i_1,i_2,\ldots,i_p})}{\sum_{k_1=1}^{a_1}\sum_{k_2=1}^{a_2}\cdots\sum_{k_p=1}^{a_p} D(y_{k_1,k_2,\ldots,k_p})}.$$

Finally, the entropy of $\overline{Y}_N$ is given by

$$H(\overline{Y}_N) = -\sum_{i_1=1}^{a_1}\sum_{i_2=1}^{a_2}\cdots\sum_{i_p=1}^{a_p} \rho_{i_1,i_2,\ldots,i_p} \ln(\rho_{i_1,i_2,\ldots,i_p}). \qquad (20)$$

A high entropy value is desired because a set with high entropy maximizes coverage and minimizes redundancies for a given cardinality.

4 Review of Methods

In this section, we review articles which present exact methods for generating a discrete representation of the nondominated set. Recall that we use the term discrete representation to mean a subset of solution points from the nondominated set, while an approximation uses some additional structure. We classify these papers according to whether a measure is incorporated into the method *a priori* (before generation of nondominated points), *a posteriori* (after generation of nondominated points), or not at all. For methods with measures, we also indicate which type of measure is used. Finally, within each category, we present the papers chronologically according to their publication dates.

4.1 Methods With *A Priori* Measures

Despite the prevalence of quality measures in the literature, only a few authors have integrated these measures into algorithms to produce representations of the Pareto set satisfying some prespecified quality criterion. Additionally, a majority of the algorithms in this section are only applicable to specific classes of problems.

Helbig [18] suggests an approach for producing a discrete representation of the Pareto set with good coverage which is applicable to discrete biobjective programs (BOPs) with connected Pareto sets. The convex hull of the individual objective minima is discretized and these points are used as the reference points in the max-ordering method. Helbig presents a method for choosing the discretized points so that the maximum Euclidean distance between a point in the true Pareto set and a point in the representation is at most a prespecified value (chosen by the DM).

In [5], Churkina investigates the Chebyshev method [37] as a method of producing a discrete representation of the Pareto set for convex MOPs. The notion of a delta-grid is used as a measure of coverage. A finite delta-grid, in terms of the Pareto set, is a finite subset in which the maximum distance between a point in the true Pareto set and a point in the representation is at most delta where the distance is measured with the Chebyshev norm. The author proves that for any chosen delta, it is possible to find an epsilon so that a finite epsilon-grid of reference points will produce a delta-grid representation of the Pareto set. However, no method is presented for finding the value of epsilon or the set of reference points.

Sayin [34] proposes a method for multiple criteria linear problems (MOLPs) to produce a representation with a given target coverage value or the maximum coverage possible given a target cardinality when the set of efficient faces is known *a priori*. Here, the coverage of the representation is calculated using the coverage measure (5) from [33]. At each iteration, the point in the true Pareto set which has the maximum Chebyshev distance from the current representation is selected.

Sayin and Kouvelis [35] give a two-stage method for generating representations of the Pareto set for discrete BOPs. The coverage, using the measure in [33], is controlled by continuing to refine an interval between two previously generated Pareto points until its length falls below a prespecified value. Kouvelis and Sayin [21] also improve the coverage of Algorithm Robust, a procedure for generating representations of general BOPs. The same scheme is used as in [35].

Eichfelder [11] [12] is, to the best of our knowledge, the first author to attempt to control the spacing of generated nondominated points. Her method is based on the Pascoletti and Serafini scalarization [30], making it applicable to general MOPs and notions of optimality defined by general cones. She derives sensitivity information in a neighborhood about a nondominated point and uses this information to determine input parameters for the scalarization so that the produced nondominated point will be a prespecified distance from the previous point. Although theoretically sound, issues arise when the method is applied to problems with three or more objective functions.

Ruzika [31] and Hamacher et. al. [17] present two box algorithms for producing representations of the Pareto set for discrete BOPs. The algorithms use the lexicographical epsilon-constraint scalarization to generate Pareto points. Boxes are formed with consecutive Pareto points as the upper left and lower right corner points. The “accuracy” (a coverage measure) of the current representation is calculated as the area of the largest of these rectangles. The representation is refined until the accuracy has met a prespecified value. Alternatively, the authors point out that a specific cardinality may be used as the stopping criterion for the algorithms instead of a desired accuracy, and that the resulting accuracy will be a function of this given cardinality. Filtering is also mentioned as a way to reduce redundancies and, thus, improve the spacing of the representation *a posteriori*.

In [39], Sylva and Crema propose an algorithm for mixed-integer linear MOPs which generates representations of the Pareto set with good coverage. At each iteration, their algorithm finds the Pareto point which maximizes the infinity-norm distance from the set already dominated by previous solutions. The cardinality of the representation can also be used as a stopping criterion.

Most recently, in [24], Masin and Bukchin present the Diversity Maximization Approach to produce representations with good coverage for general MOPs. At each iteration, the most “diverse” solution is added to the representation where the most diverse solution is defined as the one that maximizes the minimum coordinate-wise distance between the new point and all the points already in the representation. The authors note that although this method is applicable to general MOPs, it is recommended predominantly for mixed-integer and combinatorial problems.

4.2 Methods With *A Posteriori* Measures

A posteriori methods are the simplest of the three classes of methods that we present here. In general, these methods consist of generating a discrete representation of the nondominated set and then removing certain points so that the resulting representation satisfies some quality criterion.

As early as 1980, filtering techniques were being proposed to produce discrete representative subsets of the solution sets of MOPs. In [38], Steuer and Harris suggest using a forward and reverse interactive filtering scheme to produce a representative subset of the Pareto extreme points of a MOLP. Forward filtering consists of using a weighted l_p norm to discard Pareto extreme points that are too close together and thus, produce a diverse set of extreme point solutions. Once the DM has chosen her most preferred solution from this set, the reverse filtering is performed. In this process, the closest Pareto extreme points to the preferred solution are reintroduced and Pareto extreme points that are further away are discarded. In this way, the DM is now presented with solutions that are most similar to her preferred solution, allowing her to refine her solution even further.

Also in 1980, Morse [29] suggests a filtering method involving cluster analysis for reducing redundancy in the Pareto sets of MOLPs. The DM sets a minimum redundancy level below which she is indifferent between two Pareto points. The author experiments with several clustering methods among which he found the most useful in this context to be Ward’s Method, the Group Average

Method, and the Centroid Method. Using the desired method and the minimum redundancy level, clusters of Pareto points are formed. The DM is then presented with a representative Pareto point from each cluster.

More recently, in [25], Mattson et. al. suggest a Smart Pareto filter to produce representations with good cardinality and complete coverage which emphasizes areas with high tradeoffs more than areas with low or insignificant tradeoffs. First, an "even" representation of the Pareto set is produced. The authors define an even representation as one where all areas of the Pareto set are represented to an equal degree, and the authors suggest methods proposed in [7], [27], or [28] to produce such a set. Then a Pareto point is selected and the tradeoffs between it and all other Pareto points are calculated. If the tradeoff between the chosen point and another point falls below a prespecified level, the second point is removed. Otherwise, the second point is retained. This process is performed on each point in the representation. The authors denote the resulting representation as the "smart" Pareto set.

4.3 Methods Without Measures

This class includes a variety of methods which aim to produce quality discrete representations but which do not include any measure of quality of the Pareto set. Many authors improve upon the coverage or spacing of existing methods with simple variations. A few authors produce quality representations of other sets (i.e., weights, reference points) and project these sets onto the Pareto set. However, without a quality measure, the degree to which the quality of the method improves can be neither quantified nor guaranteed.

Steuer and Harris [38] propose an intra-set point generation method to go along with their filtering method which is discussed in the previous section. In cases where the Pareto extreme points do not sufficiently describe the Pareto set of a MOLP, Steuer and Harris suggest generating Pareto points within the set (i.e., not extreme points) using intelligently chosen convex combinations of the Pareto extreme points so that the generated points provide good coverage of the entire Pareto set. They present empirical evidence which shows that choosing half of the weights from the uniform distribution and half of the weights from the Weibull distribution results in a well-distributed set.

In [1], Armann develops a method for choosing the epsilon parameters in the hybrid weighted-sum, epsilon-constraint scalarization for general MOPs (proposed by [16], among others). Given the desired number of points in the representation, he solves an integer program to determine the values of epsilon to use in the hybrid scalarization so that in the resulting representation, the distance between neighboring points is maximized. This improves the coverage and the spacing of the hybrid scalarization as compared to using equally spaced values of the epsilon parameter.

Benson and Sayin [2] propose a global shooting procedure to produce a representation of the nondominated set of a general MOP. This method seeks to cover the entire nondominated set without many redundancies. The method begins by constructing a simplex which contains the feasible region in the objective space. Then a subsimplex is chosen and a discrete sample of points from this subsimplex is taken. Finally, the representation is obtained by shooting each of these points in a specific direction toward the nondominated set. The authors stress that it is the method by which the points are sampled from the subsimplex which determines whether the representation will have good coverage.

Das and Dennis [7] introduce the Normal Boundary Intersection method for producing representations of the Pareto set with complete coverage. The convex hull of the individual objective minima is discretized with equally spaced points. Then a series of minimization problems is solved to determine the intersection between the boundary of the feasible region in the objective space and the normal vector emanating from each of these points respectively. The Normal Boundary Intersection method may produce non-Pareto points and not all Pareto points are obtainable using this method. However, if the Pareto set is sufficiently well-behaved the Normal Boundary Intersection method will produce a representation with good coverage.

Buchanan and Gardiner [4] perform a comparative study of two versions of the weighted Chebyshev method [37], one using the ideal point as a reference point and the other using the nadir point. The authors found that when choosing weights from the uniform distribution, discrete representations produced using the nadir solution as the reference point had better coverage than those produced with the ideal point.

In [15], Fu and Diwekar present a variation of the epsilon-constraint method for general MOPs. In their method, the parameter epsilon is chosen in a pseudo-random manner. Empirical evidence is given to show that representations produced using this technique have more complete coverage (measured in terms of the mean and variance of the set) than those produced by the traditional method of using uniformly spaced epsilon values.

Messac and Mattson [28] introduce the Normal Constraint Method for producing representations with complete coverage. The convex hull of the individual objective minima is enlarged slightly so that it covers the entire feasible region. The convex hull is then discretized with equally spaced points. For each discretized point, a single objective optimization problem is solved over a reduced feasible region which is determined using the current point: this produces a Pareto point. Similar to the Normal Boundary Intersection method [7], for well-behaved Pareto sets, the Normal Constraint method produces a representation with good coverage because of the equally spaced points on the enlarged convex hull.

Kim and de Weck [20] propose an algorithm based on the weighted-sum method which seeks to produce complete coverage. The weighted-sum method can only produce nondominated points along convex regions of the Pareto set, resulting in large gaps in the representation if the nondominated set is not entirely convex. However, by refining the feasible region, their adaptive weighted-sum method is able to generate Pareto points in nonconvex regions, thus improving the coverage of the weighted-sum method as well as making it applicable to nonconvex MOPs.

In [19], Karasakal and Köksalan suggest a method for producing discrete representations of the Pareto set with good coverage and spacing as measured by (5) and (12), respectively. First, a weighted l_p-surface is used to approximate the Pareto set. Then, this surface is discretized with equidistant points which are projected onto the Pareto set in the direction of the gradient of the surface at each point. These Pareto points form the discrete representation. Although the method is proposed for general MOPs, the authors emphasize that it is best suited for convex problems.

5 Closing Remarks

In this paper, we reviewed measures proposed in the literature for assessing the quality of a discrete representation of the nondominated set as well as methods that have been proposed for producing a representation of the nondominated set satisfying certain quality criteria. Interestingly though, there is an apparent disconnect between these two areas. The vast majority of quality measures were proposed in the field of engineering. Typically, these measures are used *a posteriori* to evaluate the performances of evolutionary algorithms, but they could just as well be applied to nondominated sets generated by exact algorithms. For the most part, however, these measures are either unknown or unused in the mathematical community which calls for more interdisciplinary research efforts.

Within the proposed methods, several authors suggest filtering algorithms or other schemes to control the quality of a nondominated set after the solution points are generated. This type of method seems inefficient because resources are used to generate a selection of nondominated points, some of which will later be discarded. Why not incorporate a measure into the algorithm so these points are not produced in the first place? Further, methods which do not incorporate measures at all seem to improve the quality of discrete representations but only in a general sense. The quality of a resultant nondominated set cannot be explicitly stated or even guaranteed. Both of these problems are solved by integrating measures into algorithms *a priori*. Each new point is generated so that when combined with the previously produced points, the updated nondominated set will satisfy one or more prespecified quality criteria. Because of this, it is our opinion that the *a priori* class of methods is to be preferred.

However, researchers have only recently begun formulating *a priori* methods, so many of these algorithms are limited to special classes of problems or encounter difficulties in higher dimensions. Hence, further study in this area is needed to develop methods which are applicable to general MOPs and can perhaps be extended to concepts of optimality beyond Pareto optimality (such as in [11], [12]). To this end, more communication between the mathematical and engineering communities would be beneficial. Mathematicians could utilize the wide array of quality measures proposed by engineers and engineers could take advantage of knew theoretical knowledge in the field of mathematics.

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