Practice problems

1. Define the mathematical terms in your own words and provide examples
    a. Function
    b. Implicit form of a function
    c. Multivalued function
    d. Indeterminate forms
    e. Asymptotic forms and asymptotes
    f. Open and closed curves
    g. L'Hospitals rule: does it always yield useful results? Explain.
2. Show the area under a curve defined by the function $F(x)$ and bounded by $F(a)$ and $F(b)$ can be written as $\int_a^b F(x)dx$.
3. A nonlinear system obeys the equation $\ddot{x} + 0.5\,x^2 = 2 + A\sin t\,; x(0) = 0, \dot{x}(0) = 0.$
    a. Sketch the nonlinear term $0.5\,x^2$ and indicate all possible operating points.
    b. For each operating point you found in part (a), derive the linearized model for the system.
    c. Indicate whether the models derived in part (b) are stable or not.
    d. Sketch the complete system response for each of the models derived in part (b).
4. Given the following differential equation $2\ddot{y} + \alpha\dot{y} + 50y = \sin\omega t$
    a. Determine the damping ratio and natural frequency when $\alpha = 12{,}52$ and $\omega = 5$
    b. Write the form of the free response for each case.
    c. For what values of α does the free response indicate decaying oscillations?
    d. Sketch the complete system response.
5. Given the scalar expression $f(x, y, z) = 2x + 3y^2 - \sin z$
    a. Find its gradient.
    b. Is the gradient a vector or a scalar?
    c. Is it conservative or non-conservative?
    d. If it is conservative, is it also irrotational?
    e. If a vector field is irrotational, is it necessarily conservative? Given the following vector field $\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \rangle$ prove or disprove your answer.
6. Describe the following terms in your own words and illustrate through examples.
    a. Divergence
    b. Curl
    c. Gradient
    d. Laplacian
    e. Mathematically describe Green's theorem and Stoke's theorem and what does each accomplish?
7. Complex analysis
    a. Evaluate the following integral and explicitly determine the real and imaginary parts: $\int_C \frac{\ln^3 z}{z} dz$ where the curve C is the unit circle in the first quadrant of the complex plane.
    b. Solve for all values of the complex variable z that satisfies the equation $e^{-2z} + 1 = 0$.
    c. Sketch the image of the imaginary axis (in the complex z-plane) when mapped to the complex w-plane with the mapping: $w = \frac{z+1}{z-1}$.

8. Find the eigenvalue/eigenvectors for the following matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
9. Given the matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$
   a. Find the eigenvalues/eigenvectors of A.
   b. Find the nonsingular matrix P and diagonal matrix D such that $A = PDP^{-1}$.
   c. Use the diagonalization of part b) to compute A^5.
10. Find all values of h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part, $\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$
11. Use separation of variables to find the solution $u(x,y)$ of the equation $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$
12. Express the Taylor series of the function e^x about $x = 0$. Write out the first four non-zero terms. Use this result and a power series to solve the second order differential equation with initial conditions. Write out the first four non-zero terms $y'' - e^x y = 0; y(0) = 1, y'(0) = 0$.
13. Find the directional derivate of $\nabla \cdot \bar{u}$ where $\bar{u} = x^4\,\hat{\imath} + y^4\hat{\jmath} + z^4\hat{k}$ at the point $P = (4,4,2)$ in the direction of the outward normal from the surface of the sphere given by $x^2 + y^2 + z^2 = 36$
14. Find the Fourier series approximation of the function $f(t) = |t|, -2 < t < 2$. Here $T = 4$ is the period.
15. For the system $y'' + 3y' + 2y = F(t)$, find the response to a unit impulse $F(t) = \delta(t)$ and sketch the response.
16. Find the 3 values of $(-8i)^{1/3}$, that is, the cube roots of $-8i$. Express each complex root in Cartesian form $x + iy$. Also exhibit the roots in the complex plane.
17. Find the general solution of the differential equation $y'' - 4y' + 4y = x\cos x$.
18. Solve the system of equations using Jacobi's method with initial values $(x_1, x_2, x_3) = (0,0,0)$ and 2 iterations. $\begin{aligned} 10x_1 + x_2 - x_3 &= 20 \\ x_1 + 15x_2 + x_3 &= 13 \\ -x_1 + x_2 + 20x_3 &= 17 \end{aligned}$. Repeat using Gauss-Seidel with the same initial values using only 1 iteration. If the same number of iterations were carried out, which method would give a better approximation to the exact solution.
19. Using the Laplace Transform method, solve the differential equation $y' - y = e^{at}$ with initial condition $y(0) = -1$.
20. Find the normal vector to the plane formed by the three points $(1,2,1), (-1,1,3), (-2,-2,-2)$. Then find the equation of the plane of the form $ax + by + cz + d = 0$ containing the three points.
21. The path of a particle moving in the xy-plane is specified by the parametric equations $x = f(t)$ and $y = g(t)$ where t is time. It is required to determine the time at which the particles trajectory will intercept a curve specified by the equation $\phi(x,y) = 0$. If the approximate time is t_0 and the approximate coordinates of the interception are (x_0, y_0), use first order Taylor expansion to linearize the relationships, $\begin{aligned} x_0 + \Delta x - f(t_0 + \Delta t) &= 0 \\ y_0 + \Delta y - g(t_0 + \Delta t) &= 0 \\ \phi(x_0 + \Delta x, y_0 + \Delta y) &= 0 \end{aligned}$ and solve for Δt, and show that this Newton iterative method yields a new estimate $t = t_0 + \Delta t$ where

$$\Delta t = -\frac{\phi_0 + (f_0 - x_0)\partial\phi_0/\partial x + (g_0 - y_0)\partial\phi_0/\partial y}{\frac{\partial f_0}{\partial t}\frac{\partial \phi_0}{\partial x} + \frac{\partial g_0}{\partial t}\frac{\partial \phi_0}{\partial y}}$$