

Stability and Instability for Dynamic Equations on Time Scales

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Abstract—In this paper we examine the stability and instability of the equilibrium solution $\mathbf{x} = \mathbf{0}$ to the first-order system of dynamic equations

$$\mathbf{x}^\Delta = f(t, \mathbf{x}), \quad t \geq t_0, \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n,$$

where t is from a so-called time scale \mathbb{T} with $t_0 \in \mathbb{T}$ and \mathcal{D} is a compact set. Our methods involve the existence of a positive definite Liapunov function V , such that its delta-derivative V^Δ satisfies certain integral, definite or semidefinite sign properties. Finally, we use Liapunov functions to develop an invariance principle regarding solutions to the above dynamic equation. © 2005 Elsevier Science Ltd. All rights reserved.

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1. MOTIVATION

In the study of mechanics, the ideas of stability and instability initially arose as descriptions of the equilibrium of a rigid body. The equilibrium is described as “stable” if the body returns to its original position after every sufficiently small movement. A. M. Liapunov was a pioneer in relating the question of stability or instability of a system of ordinary differential equations to the existence or nonexistence of a Liapunov function V whose derivative possesses certain properties. After Liapunov began work with ordinary differential equations (ODEs), mathematicians realized that stability theory could be applied to concrete problems in the areas of controls, servo-mechanics, and engineering.

Sufficiently motivated by the stability studies involving ODEs, in this paper we examine the stability and instability of the trivial solution $\mathbf{x} = \mathbf{0}$ to the first-order system of dynamic equations

$$\mathbf{x}^\Delta = f(t, \mathbf{x}), \quad t \geq t_0, \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n, \quad (1)$$

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where t is from a so-called time scale \mathbb{T} (an arbitrary nonempty closed subset of \mathbb{R}) with $t_0 \in \mathbb{T}$, and \mathcal{D} is a compact set. Further we assume $f(t, \mathbf{0}) = \mathbf{0} \in \mathcal{D}$, for all $t \in \mathbb{T}$, $t \geq t_0$, so that $\mathbf{x} = \mathbf{0}$ is a solution to equation (1). In addition, we assume that f is continuous and of such a nature that existence and uniqueness of solutions to equation (1) subject to $\mathbf{x}(t_0) = \mathbf{x}_0$ as well as their dependence on initial values, is guaranteed. For clarity we write $\mathbf{x}(t, \mathbf{x}_0, t_0)$ for the solution with initial values $t_0 \in \mathbb{T}$, $\mathbf{x}_0 := \mathbf{x}(t_0) \in \mathcal{D}$.

Our methods involve the existence of a positive definite Liapunov function V , such that its delta-derivative V^Δ satisfies certain integral, definite, or semidefinite sign properties. For more on the stability of dynamic equations via Liapunov functions we refer the reader to [1–5].

We have included the necessary time scale information to make this paper self-contained, however further information on working with dynamic equations on time scales can be found in [6,7].

2. TIME SCALE ESSENTIALS

Any arbitrary nonempty closed subset of the reals \mathbb{R} is a time scale \mathbb{T} .

DEFINITION 1. For $t \in \mathbb{T}$ define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

It is convenient to have the graininess operator $\mu : \mathbb{T} \rightarrow [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$.

By the interval $[t_0, \infty)$ we mean the set $[t_0, \infty) \cap \mathbb{T}$.

DEFINITION 2. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is right dense continuous (rd-continuous) provided it is continuous at all right dense points of \mathbb{T} and its left sided limit exists (finite) at left dense points of \mathbb{T} . The set of all right dense continuous functions on \mathbb{T} is denoted by

$$C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}; \mathbb{R}).$$

DEFINITION 3. DELTA DERIVATIVE. Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t , such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t) [\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

The function $f^\Delta(t)$ is the delta derivative of f at t .

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$.

REMARK 1. It is easy to see that $f(\sigma(t)) = \mu(t)f^\Delta(t) + f(t)$ for any time scale.

DEFINITION 4. DELTA INTEGRAL. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \rightarrow \mathbb{R}$, such that $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}$, then F is a delta antiderivative of f . In this case, the integral is given by the formula

$$\int_a^b f(\tau) \Delta\tau = F(b) - F(a), \quad \text{for } a, b \in \mathbb{T}.$$

REMARK 2. All right dense continuous functions are delta integrable.

3. STABILITY AND LIAPUNOV FUNCTIONS

In this section, we introduce the formal concepts of stability on time scales. We use the conventions $\mathbf{x} := \mathbf{x}(t)$ and $\dot{V}(t, \mathbf{x}) := [V(\mathbf{x}(t))]^\Delta$ throughout.

There are various methods for computing \dot{V} . The most useful are the chain and product rules [6]. Note that $V = V(\mathbf{x})$ and even if the vector field associated with the dynamic equation is autonomous then \dot{V} still depends on t (and \mathbf{x} of course) when the graininess function of \mathbb{T} is nonconstant.

The stability and associated definitions below are motivated by Hahn's monographs [8,9].

DEFINITION 5. A function $\phi : [0, r] \rightarrow [0, \infty)$ is of class \mathcal{K} if it is well-defined, continuous, and strictly increasing on $[0, r]$ with $\phi(0) = 0$.

DEFINITION 6. The equilibrium solution $\mathbf{x} = \mathbf{0}$ of equation (1) is called *stable* if there exists a function $\phi \in \mathcal{K}$, such that

$$|\mathbf{x}(t, \mathbf{x}_0, t_0)| \leq \phi(|\mathbf{x}_0|), \quad \text{for all } t \in \mathbb{T}, \quad t \geq t_0.$$

DEFINITION 7. The equilibrium solution of equation (1) is called *asymptotically stable* if it is stable and if there exists a $\gamma \in \mathbb{R}$, $\gamma > 0$, such that

$$\lim_{t \rightarrow \infty} \mathbf{x}(t, \mathbf{x}_0, t_0) = \mathbf{0}$$

whenever $|\mathbf{x}_0| < \gamma$.

DEFINITION 8. A continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ with $P(\mathbf{0}) = 0$ is called *positive definite* (*negative definite*) on \mathcal{D} if there exists a function $\phi \in \mathcal{K}$, such that $\phi(|\mathbf{x}|) \leq P(\mathbf{x})$ ($\phi(|\mathbf{x}|) \leq -P(\mathbf{x})$) for $\mathbf{x} \in \mathcal{D}$.

DEFINITION 9. A continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ with $P(\mathbf{0}) = 0$ is called *positive semidefinite* (*negative semidefinite*) on \mathcal{D} if $P(\mathbf{x}) \geq 0$ ($P(\mathbf{x}) \leq 0$), for all $\mathbf{x} \in \mathcal{D}$.

DEFINITION 10. A continuous function $Q : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $Q(t, \mathbf{0}) = 0$ is called *positive definite* (*negative definite*) on $[t_0, \infty) \times \mathcal{D}$ if there exists a function $\phi \in \mathcal{K}$, such that $\phi(|\mathbf{x}|) \leq Q(t, \mathbf{x})$ ($\phi(|\mathbf{x}|) \leq -Q(t, \mathbf{x})$), for all $t \in \mathbb{T}$, $t \geq t_0$ and $\mathbf{x} \in \mathcal{D}$.

DEFINITION 11. A continuous function $Q : [t_0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $Q(t, \mathbf{0}) = 0$ is called *positive semidefinite* (*negative semidefinite*) on $[t_0, \infty) \times \mathcal{D}$ if $Q(t, \mathbf{x}) \geq 0$ ($Q(t, \mathbf{x}) \leq 0$), for all $t \in \mathbb{T}$, $t \geq t_0$ and $\mathbf{x} \in \mathcal{D}$.

4. STABILITY AND INSTABILITY THEOREMS

In this section, we present the main stability and instability theorems of the paper.

THEOREM 1. If there exists a continuously differentiable positive-definite function V in a neighborhood of zero with $\dot{V}(t, \mathbf{x})$ negative semidefinite, then the equilibrium solution $\mathbf{x} = \mathbf{0}$ of equation (1) is stable.

PROOF. Since V is continuous, for sufficiently small $r \in \mathbb{R}$ and $|\mathbf{x}| = r$, we have

$$\begin{aligned} V(\mathbf{x}) &\leq \max_{|\mathbf{y}| \leq r} V(\mathbf{y}), \\ V(\mathbf{x}) &\geq \min_{r_1 \leq |\mathbf{y}| \leq r_2} V(\mathbf{y}), \end{aligned}$$

where $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 \leq r \leq r_2$. The monotone function r may be estimated above by functions of class \mathcal{K} . That is, there exists $\phi_1, \phi_2 \in \mathcal{K}$, such that

$$\phi_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \phi_2(|\mathbf{x}|).$$

Since \dot{V} is negative semidefinite, $V(\mathbf{x})$ is a nonincreasing function of t with $V(\mathbf{x}) \leq V(\mathbf{x}_0)$ for $t \in \mathbb{T}$, $t \geq t_0$. Therefore,

$$\phi_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq V(\mathbf{x}_0) \leq \phi_2(|\mathbf{x}_0|),$$

so

$$|\mathbf{x}| \leq \phi_3(|\mathbf{x}_0|) := \phi_1^{-1}(\phi_2(|\mathbf{x}_0|)),$$

where $\phi_3 \in \mathcal{K}$. Thus for $t \geq t_0$,

$$|\mathbf{x}(t, \mathbf{x}_0, t_0)| \leq \phi_3(|\mathbf{x}_0|),$$

yielding stability. ■

THEOREM 2. *If there exists a continuously differentiable, positive definite function V in a neighborhood of zero and there exists a $\xi \in C_{rd}([t_0, \infty); [0, \infty))$ and a $\phi \in \mathcal{K}$, such that*

$$\dot{V}(t, \mathbf{x}) \leq -\xi(t)\phi(|\mathbf{x}|),$$

where

$$\int_{t_0}^t \xi(s)\Delta s \rightarrow \infty, \quad \text{as } t \rightarrow \infty,$$

then the equilibrium solution $\mathbf{x} = \mathbf{0}$ to equation (1) is asymptotically stable.

PROOF. Since the conditions of Theorem 1 are satisfied, the equilibrium solution of equation (1) is at least stable. By way of contradiction assume that there exists a solution \mathbf{x} with $|\mathbf{x}| \leq R$ for some $R \in \mathbb{R}$, $R > 0$, and

$$\lim_{t \rightarrow \infty} |\mathbf{x}(t)| \neq 0.$$

Therefore, there exists an $a \in \mathbb{R}$, $a > 0$, such that \mathbf{x} satisfies

$$a \leq |\mathbf{x}(t)| \leq R, \quad \text{for all } t \in \mathbb{T}, \quad t \geq t_0.$$

Now

$$\int_{t_0}^t \dot{V}(t, \mathbf{x})\Delta s = \int_{t_0}^t [V(\mathbf{x})]^\Delta \Delta s = V(\mathbf{x}) - V(\mathbf{x}_0),$$

by the fundamental theorem of the time scales calculus, so we have

$$\begin{aligned} V(\mathbf{x}) &= V(\mathbf{x}_0) + \int_{t_0}^t [V(\mathbf{x})]^\Delta \Delta s \\ &\leq V(\mathbf{x}_0) - \int_{t_0}^t \xi(s)\phi(|\mathbf{x}|)\Delta s \\ &\leq V(\mathbf{x}_0) - \phi(|\mathbf{x}_0|) \int_{t_0}^t \xi(s)\Delta s. \end{aligned}$$

Since ϕ is a positive, strictly increasing function and $a \leq |\mathbf{x}| \leq R$, this implies that $V(\mathbf{x}) \rightarrow -\infty$ as $t \rightarrow \infty$, which is a contradiction as V is positive definite. \blacksquare

COROLLARY 1. *If there exists a continuously differentiable, positive definite function V in a neighborhood of zero with $\dot{V}(t, \mathbf{x})$ negative definite then the equilibrium solution to equation (1) is asymptotically stable.*

PROOF. This is a special case of Theorem 2 with $\xi = 1$. The proof is virtually identical. \blacksquare

THEOREM 3. *If there exists a positive definite, continuously differentiable function V in a neighborhood of zero, a $\xi \in C_{rd}([t_0, \infty); [0, \infty))$ and a $\phi \in \mathcal{K}$, such that*

$$\dot{V}(t, \mathbf{x}) \leq \xi(t)\phi(|\mathbf{x}|),$$

where

$$\int_{t_0}^t \xi(s)\Delta s \rightarrow \infty, \quad \text{as } t \rightarrow \infty,$$

then the equilibrium solution to equation (1) is unstable.

PROOF. By way of contradiction assume that the equilibrium solution is stable. Therefore, there exists a function $\phi \in \mathcal{K}$, such that

$$|\mathbf{x}(t)| \leq \phi(|\mathbf{x}_0|), \quad \text{for all } t \in \mathbb{T}, \quad t \geq t_0.$$

As in the proof of Theorem 2,

$$\begin{aligned} V(\mathbf{x}) &= V(\mathbf{x}_0) + \int_{t_0}^t [V(\mathbf{x})]^\Delta \Delta s \\ &\leq V(\mathbf{x}_0) + \phi(|\mathbf{x}_0|) \int_{t_0}^t \xi(s)\Delta s, \end{aligned}$$

and we see that V can become arbitrarily large, which is a contradiction. \blacksquare

COROLLARY 2. *If there exists a positive definite, continuously differentiable function V in a neighborhood of zero with $\dot{V}(t, \mathbf{x})$ positive definite then the equilibrium solution to equation (1) is unstable.*

PROOF. This is a special case of Theorem 3 with $\xi = 1$. The proof is virtually identical. \blacksquare

EXAMPLE. We consider the basic problem

$$x^\Delta = -x + y^2, \quad (2)$$

$$y^\Delta = -y - xy, \quad (3)$$

on a general time scale \mathbb{T} . The traditional Liapunov function for this problem is $V(x, y) = x^2 + y^2$ which is strictly positive except at the origin. Calculating \dot{V} by using the product rule we get

$$\begin{aligned} \dot{V}(t, x, y) &= (x^2 + y^2)^\Delta \\ &= x(t)x^\Delta(t) + x^\Delta(t)x(\sigma(t)) + y^\Delta(t)y(\sigma(t)) + y^\Delta(t)y(t). \end{aligned}$$

Using the formula $f(\sigma(t)) = \mu(t)f^\Delta(t) + f(t)$, we have

$$\dot{V}(t, x, y) = x^\Delta(t) (2x(t) + \mu(t)x^\Delta(t)) + y^\Delta(t) (2y(t) + \mu(t)y^\Delta(t)).$$

Substituting in for x^Δ and y^Δ yields

$$\dot{V} = -2x^2 - 2y^2 + \mu(t) (x^2 + y^2 + x^2y^2 + y^4).$$

When $\mathbb{T} = \mathbb{R}$, the system (2),(3) becomes

$$x' = -x + y^2, \quad (4)$$

$$y' = -y - xy, \quad (5)$$

for $t \geq t_0 = 0$, say. For $\mathbb{T} = \mathbb{R}$ the graininess function $\mu = 0$ and so \dot{V} is negative definite. By Corollary 1 the trivial solution to (4), (5) is asymptotically stable.

If $\mathbb{T} = h\mathbb{N}_0 := \{0, h, 2h, \dots\}$, $t_0 = 0$ then (2),(3) becomes

$$\frac{\Delta x(k)}{h} = -x(k) + y^2(k), \quad (6)$$

$$\frac{\Delta y(k)}{h} = -y(k) - x(k)y(k), \quad (7)$$

for $k = 0, 1, \dots$. When $\mathbb{T} = h\mathbb{N}_0$ the graininess function $\mu = h$. Therefore, if $h \geq 2$, then

$$\dot{V} \geq x^2y^2 + y^4 \geq 0,$$

and thus, by Corollary 2 the trivial solution to (6),(7) is unstable.

More generally speaking, if $\mu(t) \geq 2$, for all $t \in \mathbb{T}$, then

$$\dot{V} \geq x^2y^2 + y^4 \geq 0,$$

and by Corollary 2 the trivial solution to (2),(3) is unstable.

If $0 \leq \mu(t) < 2$, for all $t \in \mathbb{T}$, then a function $\phi \in \mathcal{K}$ may be found so that \dot{V} is negative definite, but the region \mathcal{D} on which this occurs shrinks as $\max \mu(t) \rightarrow 2$.

REMARK 3. As seen in the previous example, the stability or instability of equation (1) is inherently dependent on the time scale chosen.

5. INVARIANCE PRINCIPLE

In this section, we use Liapunov functions to develop an invariance principle regarding solutions to (1).

Let $a, b \in \mathbb{T}$, $-\infty \leq a < 0 < b \leq \infty$, and $\phi : (a, b) \rightarrow E$ where E is an open set in \mathbb{R}^n . For convenience we assume that solutions of equation (1) are unique in E .

DEFINITION 12. *The mapping f is called regressive in t if the mapping*

$$I + f(t, \cdot)\mu(t),$$

where I is the identity map is invertible.

One can verify that a mapping f which fulfils the condition

$$|f(t, \mathbf{x}_1) - f(t, \mathbf{x}_2)| \leq L|\mathbf{x}_1 - \mathbf{x}_2|,$$

for $t \in \mathbb{T}$, $L > 0$ is regressive as long as $\mu(t) < 1$, for all $t \in \mathbb{T}$, $t \geq t_0$. In the tradition of LaSalle [10–12], we give the following definitions.

DEFINITION 13. *A point p is said to be a positive (negative) limit point of ϕ if there exists a sequence of points $t_n \in (a, b)$, such that $t_n \rightarrow b$ ($t_n \rightarrow a$) as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \phi(t_n) = p$. The set of all positive (negative) limit points of ϕ , $\Omega(\phi)$ ($A(\phi)$) is called the positive (negative) limit set of ϕ . The interval (a, b) is said to be maximal relative to E if it is empty and if $-\infty < a$ implies that $A(\phi) \cap E$ is empty.*

For each $t \in \mathbb{T}$ define $T(t) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that

$$\begin{aligned} x(t, \mathbf{x}_0) &= T(t)\mathbf{x}_0, \\ x(0, \mathbf{x}_0) &= \mathbf{x}_0 = T(0)\mathbf{x}_0, \quad T(0) = I. \end{aligned}$$

Then from the basic theory of dynamic equations on time scales we have the following properties.

- (1) Each solution $x(t, \mathbf{x}_0)$ of equation (1) satisfying $x(0) = \mathbf{x}_0$ has for each $x_0 \in E$ a maximal interval of definition $I(\mathbf{x}_0) = (a(\mathbf{x}_0), b(\mathbf{x}_0))$.
- (2) Let $s, t \in \mathbb{T}$ be such that $s \in I(\mathbf{x}_0)$ and $t \in I(x(s, \mathbf{x}_0))$, then one can show that $\sigma(t + s) \in I(\mathbf{x}_0)$ and $T(\sigma(s + t))\mathbf{x}_0 = T(\sigma(s))T(t)\mathbf{x}_0$.
- (3) Let x be such that it is continuous at each (t, \mathbf{x}) with right-dense or maximal t , and for $t_n < t$, $t_n \in I(x(t_n, \mathbf{x}_0))$, $(t_n, x(t_n, \mathbf{x}_0)) \rightarrow (t, \mathbf{y}) \in I(\mathbf{y}) \times E$. Then, x is called right-dense continuous (rd-continuous) on $\mathbb{T} \times E$. In this case, one can show that $x(t_n, x(t_n, \mathbf{x}_0)) \rightarrow x(t, \mathbf{y})$ at each left-dense point.

DEFINITION 14. *Relative to equation (1), a set $G \in \mathbb{R}^n$ is said to be positively (negatively) invariant if $\mathbf{x}_0 \in E \cap G$ implies that $T(t)\mathbf{x}_0 \in G$, for all $t \in [0, b(\mathbf{x}_0))$ ($t \in (a(\mathbf{x}_0), 0]$). G is said to be weakly invariant if it is positively and negatively invariant. If in addition $I(\mathbf{x}_0) = \mathbb{T}$ for each $\mathbf{x}_0 \in E \cap G$, G is said to be invariant.*

DEFINITION 15. *A solution $x(t, \mathbf{x}_0) = T(t)\mathbf{x}_0$ is said to be positively (negatively) precompact relative to E if it is bounded, for all $t \in [0, b(\mathbf{x}_0))$ ($t \in (a(\mathbf{x}_0), 0]$) and has no limit points on the boundary of E .*

DEFINITION 16. *A closed invariant set is said to be invariantly connected if it is not the union of two nonempty disjoint closed invariant sets.*

THEOREM 4. *Every positive limit set $\Omega(\mathbf{x}_0)$ is closed and positively invariant.*

PROOF. Each positive limit set $\Omega(\mathbf{x}_0)$ since it contains all of its limit points. To show the positive invariance of $\Omega(\mathbf{x}_0)$, let $\mathbf{y} \in \Omega(\mathbf{x}_0) \cap E$ and $t \in I(\mathbf{y})$. Since $\Omega(\mathbf{x}_0) \cap E$ is nonempty, $b(\mathbf{x}_0) = \infty$. Thus, there is a sequence $t_n \in I(\mathbf{x}_0)$, such that $t_n \rightarrow \infty$ and $x(t_n, \mathbf{x}_0) \rightarrow \mathbf{y}$ as $n \rightarrow \infty$. For sufficiently large n , $I(\mathbf{y}) \cap I(x(t_n, \mathbf{x}_0))$ is nonempty and $x(t, x(t_n, \mathbf{x}_0)) = x(t + t_n, \mathbf{x}_0) \rightarrow x(t, \mathbf{y})$. Hence, $x(t, \mathbf{y}) \in \Omega(\mathbf{x}_0)$, and $\Omega(\mathbf{x}_0)$ is positively invariant. ■

THEOREM 5. *If $x(t, \mathbf{x}_0)$ is positively precompact, then $\Omega(\mathbf{x}_0)$ is in E , and it is nonempty, compact, invariant, invariantly connected, and is the smallest closed set that $x(t, \mathbf{x}_0)$ approaches as $t \rightarrow \infty$.*

PROOF. Since $x(t, \mathbf{x}_0) = T(t)\mathbf{x}_0$ is positively precompact, it is bounded. This in turn implies that $\Omega(\mathbf{x}_0)$ is nonempty and bounded in E , and since $\Omega(\mathbf{x}_0)$ is also closed, it is compact. Since it is positively invariant it is invariant.

We need to show that $x(t, \mathbf{x}_0) = T(t)\mathbf{x}_0 \rightarrow \Omega(\mathbf{x}_0)$ when $T(t)\mathbf{x}_0$ is precompact. Since $T(t)\mathbf{x}_0$ and $\Omega(\mathbf{x}_0)$ are both bounded, the distance between them, denoted $d(T(t)\mathbf{x}_0, \Omega(\mathbf{x}_0))$ is bounded. Thus, if $T(t)\mathbf{x}_0$ does not approach $\Omega(\mathbf{x}_0)$, then there is a sequence $t_n \in I(\mathbf{x}_0)$, such that $t_n \rightarrow \infty$ and $x(t_n, \mathbf{x}_0) = T(t_n)\mathbf{x}_0$ converges but $d(T(t)\mathbf{x}_0, \Omega(\mathbf{x}_0))$ does not approach 0 as $t_n \rightarrow \infty$. This cannot occur since the limit of $x(t_n, \mathbf{x}_0) \in \Omega(\mathbf{x}_0)$. Hence, $x(t_n, \mathbf{x}_0) \rightarrow \Omega(\mathbf{x}_0)$ as $t_n \rightarrow \infty$. If $T(t_n)\mathbf{x}_0 \rightarrow H$ as $t_n \rightarrow \infty$, and H is a closed set, then $\Omega(\mathbf{x}_0) \subset H$. Hence, $\Omega(\mathbf{x}_0)$ is the smallest closed set that $T(t_n)\mathbf{x}_0$ approaches as $t_n \rightarrow \infty$.

To show that $\Omega(\mathbf{x}_0)$ is invariantly connected, we assume by way of contradiction that $\Omega(\mathbf{x}_0)$ is the union of two disjoint closed nonempty invariant sets Ω_1 and Ω_2 . Since $\Omega(\mathbf{x}_0)$ is compact, so are Ω_1 and Ω_2 . Then, there exists disjoint open sets U_1 and U_2 , such that $\Omega_1 \subset U_1$ and $\Omega_2 \subset U_2$. Since for each t , $T(t)$ is continuous on Ω_1 , and thus, uniformly continuous on Ω_1 , there exists an open set V_1 , such that $\Omega_1 \subset V_1$ and $T(t)(V_1) \subset U_1$. Since $\Omega(\mathbf{x}_0)$ is the smallest closed set that $T(t)\mathbf{x}_0$ approaches, there exists a sequence $t_n \in I(\mathbf{x}_0)$, such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and such that $T(t_n)\mathbf{x}_0$ intersects both V_1 and U_2 an infinite number of times. However this implies that there exists a subsequence t_{n_m} , such that $T(t_{n_m})\mathbf{x}_0$ is neither in V_1 or U_2 . This gives rise to a contradiction, thus $\Omega(\mathbf{x}_0)$ is invariantly connected. \blacksquare

Relative to the Liapunov function V relative to equation (1), we introduce the following notation.

- (1) $H = \{\mathbf{x} : \dot{V}(\mathbf{x}) = 0, \mathbf{x} \in \bar{E} \cap E\}$.
- (2) M is the largest invariant set in H .
- (3) M^* is the largest weakly invariant set in H .
- (4) M^+ is the largest positively invariant set in H .
- (5) If M^* is compact, then $M = M^*$. In addition $M \subset M^* \subset M^+$.
- (6) $V^{-1}(c) = \{\mathbf{x} \in \bar{E} \cap E : V(\mathbf{x}) = c\}$.

THEOREM 6. INVARIANCE PRINCIPLE. *Let V be a Liapunov function of equation (1) on E , and let $x(t, \mathbf{x}_0)$ be a solution of equation (1) that remains in E , for all $t \in [0, b(\mathbf{x}_0))$. Then, for some c , $\Omega(\mathbf{x}_0) \cap E \subset M^* \cap V^{-1}(c)$. If $x(t, \mathbf{x}_0)$ is precompact, then $x(t, \mathbf{x}_0) \rightarrow M \cap V^{-1}(c)$.*

PROOF. Assume that $\mathbf{y} \in \Omega(\mathbf{x}_0)$. Then $b(\mathbf{x}_0) = \infty$ and there exists a sequence $t_n \in I(\mathbf{x}_0)$, such that $x(t_n, \mathbf{x}_0) \rightarrow \mathbf{y}$. Since $V(x(t, \mathbf{x}_0))$ is nonincreasing with respect to t , $V(x(t, \mathbf{x}_0)) \geq V(\mathbf{y})$, for all $t \in [0, \infty)$, and $V(x(t, \mathbf{x}_0)) \rightarrow c = V(\mathbf{y})$, for each $\mathbf{y} \in \Omega(\mathbf{x}_0)$. Since $\Omega(\mathbf{x}_0)$ is weakly positively invariant, $\Omega(\mathbf{x}_0) \subset H$, and therefore $\Omega(\mathbf{x}_0) \subset M^*$. Hence, $\Omega(\mathbf{x}_0) \cap E \subset M^* \cap V^{-1}(c)$. If $x(t, \mathbf{x}_0)$ is precompact, then $\Omega(\mathbf{x}_0)$ is invariant and $\Omega(\mathbf{x}_0) \subset M \cap V^{-1}(c)$. Since $x(t, \mathbf{x}_0) \rightarrow \Omega(\mathbf{x}_0)$ as $t \rightarrow \infty$, $x(t, \mathbf{x}_0) \rightarrow M \cap V^{-1}(c)$ as $t \rightarrow \infty$. \blacksquare

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