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Green's Functions and Eigenvalue Comparisons for a Focal Problem on Time Scales

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Abstract—The study of dynamic systems on time scales unifies continuous and discrete processes. In this paper, we shall examine Green's function for an n^{th}-order focal boundary value problem. With this, we can then consider eigenvalue comparisons for higher-order focal boundary value problems on time scales using the theory of operators on a Banach space. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

In this paper, we shall examine Green's function for an n^{th}-order focal boundary value problem. With this, we can then consider eigenvalue comparisons for higher-order focal boundary value problems on time scales.

In Hilger's dissertation [1], the concept of a "time scale" was introduced to help unify the theory of differential and difference equations.

DEFINITION 1. *A time scale (measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, where we assume that $\mathbb{T}$ has the topology that it inherits from $\mathbb{R}$ with the standard topology.*

A more general definition has been given in [2], but for the purposes of this work, the special case given in Definition 1 is sufficient.

DEFINITION 2. *For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\mathbb{T}$ has a maximum t, then we put $\sigma(t) = t$, and $\rho(t) = t$ if $\mathbb{T}$ has a minimum t.

There are four properties a point in the time scale can have:

- if $\sigma(t) = t$, then t is called *right-dense*;

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- if $\sigma(t) > t$, then t is called *right-scattered*;
- if $\rho(t) = t$, then t is called *left-dense*; and
- if $\rho(t) < t$, then t is called *left-scattered*.

It is convenient to have a graininess operator $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$. By the so-called "interval" $[a, b]$, we mean $[a, b] \cap \mathbb{T}$ where $a, b \in \mathbb{T}$. Other intervals are defined similarly.

DEFINITION 3. *We define the interval* $[a, b]^\kappa$ *by*

$$[a,b]^\kappa := \begin{cases} [a,b), & \text{if } b \text{ is left-scattered,} \\ [a,b], & \text{otherwise.} \end{cases}$$

DEFINITION 4. *(See [3].) Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}$. *Then we define* $f^\Delta(t)$ *to be the number (provided it exists) with the property that given any* $\epsilon > 0$, *there is a neighborhood* U *of* t *such that*

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \epsilon|\sigma(t) - s|, \qquad \text{for all } s \in U.$$

We call $f^\Delta(t)$ *the delta derivative of* f *at* t.

THEOREM 1. *(See [3,4].) Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^\kappa$. *Then we have the following.*

(i) *If* f *is differentiable at* t, *then* f *is continuous at* t.

(ii) *If* f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If* t *is right-dense, then* f *is differentiable at* t *if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If* f *is differentiable at* t, *then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

THEOREM 2. *(See [3,4].) Assume* $f, g : \mathbb{T} \to \mathbb{R}$ *are delta differentiable at* $t \in \mathbb{T}^\kappa$. *Then, we have the following.*

(i) $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For any constant* k, $kf : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(kf)^\Delta(t) = kf^\Delta(t).$$

(iii) $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t) = g^\Delta(t) f(t) + g(\sigma(t)) f^\Delta(t).$$

(iv) *If* $f(t) f(\sigma(t)) \neq 0$, *then* $1/f$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^\Delta (t) = -\frac{f^\Delta(t)}{f(t) f(\sigma(t))}.$$

(v) *If* $g(t) g(\sigma(t)) \neq 0$, *then* f/g *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^\Delta (t) = \frac{g(t) f^\Delta(t) - f(t) g^\Delta(t)}{g(t) g(\sigma(t))}.$$

In some cases, it is necessary to take the delta derivative of a function of two variables. In the context of this paper, we make the following definition.

DEFINITION 5. *For a function* $g : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, *we define* $g^{\Delta}(t, s)$ *to be the delta derivative of* g *with respect to* t *for a fixed* s.

In addition, there are two basic integration by parts formulas which are given in the following theorem.

THEOREM 3. *(See [4].) Assume* $c, d \in \mathbb{T}$. *Then,*

(i) $\int_c^d f(\sigma(t))g^{\Delta}(t)\,\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t)g(t)\,\Delta t$,

(ii) $\int_c^d f(t)g^{\Delta}(t)\,\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t)g(\sigma(t))\,\Delta t$.

The following two theorems are well-known formulas which are used frequently in the proofs of this paper.

THEOREM 4. *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is right-dense continuous on* $\mathbb{T}^{\kappa}$. *Then, for* $t \in \mathbb{T}^{\kappa}$,

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \mu(t)f(t).$$

THEOREM 5. *(See [4].) Let* $a \in \mathbb{T}^{\kappa}$, $b \in \mathbb{T}$, *and assume* $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$. *Suppose further that for each fixed* $t \in \mathbb{T}^{\kappa}$, $f(t, \tau)$, *and* $f^{\Delta}(t, \tau)$ *are right-dense continuous and*

$$f^{\Delta}(t, \tau) = \lim_{s \to t} \frac{f(t, \tau) - f(s, \tau)}{t - s}$$

uniformly with respect to τ *on compact subsets of* $\mathbb{T}^{\kappa}$. *Also assume that* $k : \mathbb{T} \to \mathbb{R}$ *is right-dense continuous. Then*

$$g(t) := \int_a^t f(t, \tau)k(\tau)\,\Delta\tau \text{ implies } g^{\Delta}(t) = \int_a^t f^{\Delta}(t, \tau)k(\tau)\,\Delta\tau + f(\sigma(t), t)k(t),$$

$$h(t) := \int_t^b f(t, \tau)k(\tau)\,\Delta\tau \text{ implies } h^{\Delta}(t) = \int_t^b f^{\Delta}(t, \tau)k(\tau)\,\Delta\tau - f(\sigma(t), t)k(t).$$

Some results and definitions for cones on Banach spaces are necessary as well.

DEFINITION 6. *Let* $\mathcal{B}$ *be a Banach space. A closed subset* $\mathcal{P}$ *is said to be a cone provided:*

(i) *if* $u, v \in \mathcal{P}$, *then* $\alpha u + \beta v \in \mathcal{P}$ *for all* $\alpha, \beta \geq 0$;

(ii) *if* $u, -u \in \mathcal{P}$, *then* $u = 0$.

A cone $\mathcal{P}$ *is said to be reproducing provided every* $x \in \mathcal{B}$ *can be written* $x = u - v$ *for some* $u, v \in \mathcal{P}$.

REMARK 1. For the ordering on the cone, we say that for $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ provided $v - u \in \mathcal{P}$. In addition, if M and N are operators on $\mathcal{B}$, then we write $M \leq N$ with respect to $\mathcal{P}$ provided $Mu \leq Nu$ for all $u \in \mathcal{P}$.

DEFINITION 7. *A bounded linear operator* M *on* $\mathcal{B}$ *is* u_0*-positive with respect to the cone* $\mathcal{P}$ *provided* $u_0 \in \mathcal{P}$ *and for every nonzero* $u \in \mathcal{P}$, *there exist positive numbers* k_1, k_2 *such that* $k_1 u_0 \leq Mu \leq k_2 u_0$ *with respect to* $\mathcal{P}$.

Theorems 2.4, 2.10, 2.11, and 2.13 of [5] give the following theorem.

THEOREM 6. *Let* $\mathcal{P}$ *be a reproducing cone. If* L *is a compact* u_0*-positive linear operator then* L *has an essentially unique eigenvector in* $\mathcal{P}$ *and the corresponding eigenvalue is simple, positive, and larger than the modulus of any other eigenvalue.*

The following theorem appears in [6] and is a generalization of Theorem 2.3 of [7].

THEOREM 7. *Let* L *and* M *be bounded linear operators and assume that at least one of the operators is* u_0*-positive. If* $L \leq M$ *and*

$$\begin{aligned} Lu_1 &\geq \lambda_1 u_1, & 0 &\neq u_1 \in \mathcal{P}, & \lambda_1 &> 0, \\ Mu_2 &\leq \lambda_2 u_2, & 0 &\neq u_1 \in \mathcal{P}, & \lambda_1 &> 0, \end{aligned}$$

then $\lambda_1 \leq \lambda_2$, *and if* $\lambda_1 = \lambda_2$, *then* u_1 *is a scalar multiple of* u_2.

2. GREEN'S FUNCTION

In this section, we determine Green's function for an n^{th}-order boundary value problem on a time scale $\mathbb{T}$. Define the operator B_n as follows:

$$B_n x = (-1)^{n-k} \left(px^{\Delta^k}\right)^{\Delta^{n-k}} - \lambda \left\{ \sum_{i=0}^{k-1} p_i x^{\Delta^i} + \sum_{i=0}^{n-k-1} p_{k+i} \left(px^{\Delta^k}\right)^{\Delta^i} \right\}.$$

We are concerned with the focal boundary value problem

$$\begin{aligned} B_n x &= 0, \\ x^{\Delta^i}(a) &= 0, \qquad 0 \le i \le k-1, && (1) \\ \left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) &= 0, \qquad 0 \le i \le n-k-1, && (2) \end{aligned}$$

under the assumptions that $\sigma^k(a) < b$, and $\sigma^n(b) \le \sup \mathbb{T}$, where p and p_i for $0 \le i \le n-1$ are given right-dense continuous functions on $\mathbb{T}$, and $p(t)$ is nonzero on $\mathbb{T}$.

DEFINITION 8. *We say x is a solution of $B_n x = 0$ on $[a, \sigma^n(b)]$ provided x^{Δ^i} is delta differentiable on $[a, \sigma^{n-i}(b)]$, $0 \le i \le k-1$, $(px^{\Delta^k})^{\Delta^i}$ is delta differentiable on $[a, \sigma^{n-k-i}]$, $0 \le i \le n-k-1$, and $(px^{\Delta^k})^{\Delta^{n-k}}$ is right-dense continuous on $[a, b]$.*

The following lemma has a traditional proof, and hence, it will be omitted.

LEMMA 1. *If the homogeneous boundary value problem*

$$\begin{aligned} B_n x &= 0, \\ x^{\Delta^i}(a) &= 0, \qquad 0 \le i \le k-1, \\ \left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) &= 0, \qquad 0 \le i \le n-k-1, \end{aligned}$$

has only the trivial solution, then the nonhomogeneous boundary value problem

$$\begin{aligned} B_n x &= h, \\ x^{\Delta^i}(a) &= \alpha_i, \qquad 0 \le i \le k-1, && (3) \\ \left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) &= \beta_i, \qquad 0 \le i \le n-k-1, && (4) \end{aligned}$$

where $\alpha_0, \dots, \alpha_{k-1}$, $\beta_0, \dots, \beta_{n-k-1}$ are given real constants and h is a given right-dense continuous function on $[a, b]$, has a unique solution.

DEFINITION 9. *The Cauchy function $y(t, s)$ for $B_n x = 0$, defined for $a \le t \le \sigma^n(b)$, $a \le s \le b$, is defined as the function that for each fixed s in $[a, b]$ is the solution of the initial value problem*

$$\begin{aligned} B_n x &= 0, \\ y^{\Delta^i}(\sigma(s), s) &= 0, \qquad 0 \le i \le k, \\ \left(py^{\Delta^k}\right)^{\Delta^i}(\sigma(s), s) &= 0, \qquad 1 \le i \le n-k-2, \\ \left(py^{\Delta^k}\right)^{\Delta^{n-k-1}}(\sigma(s), s) &= (-1)^{n-k}. \end{aligned}$$

THEOREM 8. *Assume the homogeneous boundary value problem* $B_n x = 0$, *(1),(2), has only the trivial solution. For each fixed* $s \in [a,b]$, *let* $u(t,s)$ *be the unique solution of the boundary value problem*

$$\begin{aligned} B_n u &= 0, && \\ u^{\Delta^i}(a,s) &= 0, && 1 \le i \le k, && (5) \\ \left(pu^{\Delta^k}\right)^{\Delta^i}(\sigma(b),s) &= -\left(py^{\Delta^k}\right)^{\Delta^i}(\sigma(b),s), && k+1 \le i \le n, && (6) \end{aligned}$$

where $y(t,s)$ *is the Cauchy function for* $B_n x = 0$. *Then we define Green's function by*

$$G(t,s) = \begin{cases} u(t,s), & t \le s, \\ u(t,s) + y(t,s), & \sigma(s) \le t. \end{cases}$$

For each fixed $s \in [a,b]$, $v(t,s) := u(t,s) + y(t,s)$ *is a solution of* $B_n x = 0$ *and satisfies the boundary conditions (2) for* $k+1 \le i \le n$. *If* $x(t)$ *is defined as*

$$x(t) := \int_a^{\sigma(b)} G(t,s)h(s)\,\Delta s,$$

where h *is assumed to be a right-dense continuous function on* $[a,b]$, *then* $x(t)$ *is a solution of the nonhomogeneous boundary value problem*

$$\begin{aligned} B_n x &= h, && \\ x^{\Delta^i}(a) &= 0, && 1 \le i \le k, \\ \left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) &= 0, && k+1 \le i \le n. \end{aligned}$$

PROOF. The existence of $u(t,s)$ is guaranteed by Lemma 1. Since for each fixed $s \in [a,b]$, $u(t,s)$, and $y(t,s)$ are solutions of $B_n x = 0$, we have that for each fixed $s \in [a,b]$, $v(t,s) := u(t,s) + y(t,s)$ is also a solution of $B_n x = 0$. It follows from (6) that for each fixed $s \in [a,b]$, $v(t,s)$ satisfies the boundary conditions (2) for $k+1 \le i \le n$.

Let $u(t,s)$, $y(t,s)$, and $G(t,s)$ be as in the statement of this theorem, and assume that $h(t)$ is a given right-dense continuous function on $[a,b]$. Then define

$$x(t) := \int_a^{\sigma(b)} G(t,s)h(s)\,\Delta s.$$

We wish to show that $x(t)$ is a solution of the nonhomogeneous equation $B_n x = h$ satisfying the homogeneous boundary conditions (1),(2). Consider

$$\begin{aligned} x(t) &= \int_a^{\sigma(b)} G(t,s)h(s)\,\Delta s \\ &= \int_a^t G(t,s)h(s)\,\Delta s + \int_t^{\sigma(b)} G(t,s)h(s)\,\Delta s \\ &= \int_a^t (u(t,s) + y(t,s))h(s)\,\Delta s + \int_t^{\sigma(b)} u(t,s)h(s)\,\Delta s \\ &= \int_a^t y(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u(t,s)h(s)\,\Delta s. \end{aligned}$$

Theorem 5 can be applied to get

$$\begin{aligned} x^{\Delta}(t) &= \int_a^t y^{\Delta}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u^{\Delta}(t,s)h(s)\,\Delta s + y(\sigma(t),t)h(t) \\ &= \int_a^t y^{\Delta}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u^{\Delta}(t,s)h(s)\,\Delta s \\ &= \int_a^t v^{\Delta}(t,s)h(s)\,\Delta s + \int_t^{\sigma(b)} u^{\Delta}(t,s)h(s)\,\Delta s, \end{aligned}$$

since $y(\sigma(t),t)=0$. Also note that

$$x^{\Delta}(a) = \int_a^{\sigma(b)} u^{\Delta}(a,s)h(s)\,\Delta s = 0.$$

Now, for any $1 < i \le k$, Theorem 5 can again be applied to get the induction step

$$\begin{aligned} x^{\Delta^i}(t) &= \left[\int_a^t y^{\Delta^{i-1}}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u^{\Delta^{i-1}}(t,s)h(s)\,\Delta s\right]^{\Delta} \\ &= \int_a^t y^{\Delta^i}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u^{\Delta^i}(t,s)h(s)\,\Delta s + y^{\Delta^{i-1}}(\sigma(t),t)h(t) \\ &= \int_a^t y^{\Delta^i}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} u^{\Delta^i}(t,s)h(s)\,\Delta s \\ &= \int_a^t v^{\Delta^i}(t,s)h(s)\,\Delta s + \int_t^{\sigma(b)} u^{\Delta^i}(t,s)h(s)\,\Delta s, \end{aligned}$$

since $y^{\Delta^{i-1}}(\sigma(t),t)=0$. Note that for $1 \le i \le k-1$,

$$x^{\Delta^i}(a) = \int_a^{\sigma(b)} u^{\Delta^i}(a,s)h(s)\,\Delta s = 0,$$

and hence, $x(t)$ satisfies the boundary conditions (1). Consider

$$x^{\Delta^k}(t) = \int_a^t v^{\Delta^k}(t,s)h(s)\,\Delta s + \int_t^{\sigma(b)} u^{\Delta^k}(t,s)h(s)\,\Delta s.$$

Then

$$\begin{aligned} \left(px^{\Delta^k}\right)^{\Delta}(t) &= \int_a^t \left(py^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} \left(pu^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s \\ &\quad + \left(py^{\Delta^{k-1}}\right)(\sigma(t),t)h(t) \\ &= \int_a^t \left(py^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s + \int_a^{\sigma(b)} \left(pu^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s \\ &= \int_a^t \left(pv^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s + \int_t^{\sigma(b)} \left(pu^{\Delta^k}\right)^{\Delta}(t,s)h(s)\,\Delta s, \end{aligned}$$

since $y^{\Delta^{k-1}}(\sigma(t),t)=0$. Using boundary conditions (6), we have

$$\left(px^{\Delta^k}\right)^{\Delta}(\sigma(b)) = \int_a^{\sigma(b)} \left(pu^{\Delta^n} + py^{\Delta^k}\right)^{\Delta}(\sigma(b),s)h(s)\,\Delta s = 0.$$

Now, using the fact that $(py^{\Delta^k})^{\Delta^i}(\sigma(t),t)=0$ for $0\le i\le n-k-1$, we have that for $1\le i\le n-k-1$,

$$\begin{aligned}\left(px^{\Delta^k}\right)^{\Delta^i}(t) &= \int_a^t \left(py^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s+\int_a^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s\\ &\quad+\left(py^{\Delta^k}\right)^{\Delta^{i-1}}(\sigma(t),t)h(t)\\ &= \int_a^t \left(py^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s+\int_a^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s\\ &= \int_a^t \left(pv^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s+\int_t^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s.\end{aligned}$$

Note that using the boundary conditions (6), we have that for $1\le i\le n-k-1$,

$$\left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b))=\int_a^{\sigma(b)}\left(pv^{\Delta^k}\right)^{\Delta^i}(\sigma(b),s)h(s)\,\Delta s=0.$$

Hence, $x(t)$ satisfies boundary conditions (2). Now, using the fact that

$$\left(py^{\Delta^k}\right)^{\Delta^{n-k-1}}(\sigma(t),t)=(-1)^{n-k},$$

we have for the $(n-k)^{\text{th}}$ delta derivative

$$\begin{aligned}&(-1)^{n-k}\left(px^{\Delta^k}\right)^{\Delta^{n-k}}(t)\\ &\quad=(-1)^{n-k}\int_a^t\left(py^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+(-1)^{n-k}\int_a^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s\\ &\qquad+(-1)^{n-k}\left(py^{\Delta^k}\right)^{\Delta^{n-k-1}}(\sigma(t),t)h(t)\\ &\quad=(-1)^{n-k}\int_a^t\left(pv^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+(-1)^{n-k}\int_t^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s\\ &\qquad+(-1)^{n-k}\left((-1)^{n-k}h(t)\right)\\ &\quad=(-1)^{n-k}\int_a^t\left(pv^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+(-1)^{n-k}\int_t^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+h(t).\end{aligned}$$

Hence,

$$\begin{aligned}B_nx&=(-1)^{n-k}\left(p(t)x^{\Delta^k}\right)^{\Delta^{n-k}}-\lambda\left\{\sum_{i=0}^{k-1}p_i(t)x^{\Delta^i}(t)+\sum_{i=0}^{n-k-1}p_{k+i}(t)\left(p(t)x^{\Delta^k}(t)\right)^{\Delta^i}\right\}\\ &=(-1)^{n-k}\int_a^t\left(pv^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+(-1)^{n-k}\int_t^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^{n-k}}(t,s)h(s)\,\Delta s+h(t)\\ &\quad-\lambda\left\{\sum_{i=0}^{k-1}p_i(t)\left(\int_a^t v^{\Delta^i}(t,s)h(s)\,\Delta s+\int_t^{\sigma(b)}u^{\Delta^i}(t,s)h(s)\,\Delta s\right)\right.\\ &\quad\left.+\sum_{i=0}^{n-k-1}p_{k+i}(t)\left(\int_a^t\left(pv^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s+\int_t^{\sigma(b)}\left(pu^{\Delta^k}\right)^{\Delta^i}(t,s)h(s)\,\Delta s\right)\right\}\\ &=\int_a^t[B_nv(t,s)]h(s)\,\Delta s+\int_t^{\sigma(b)}[B_nu(t,s)]h(s)\,\Delta s+h(t)\\ &=h(t),\end{aligned}$$

using the fact that $B_nv(t,s)=B_nu(t,s)=0$ for each fixed $s\in[a,b]$. ∎

Consider the special case of $B_n x = 0$, namely,

$$(-1)^{n-k}\left(px^{\Delta^k}\right)^{\Delta^{n-k}} = 0.$$

We now make three recursive definitions which will enable us to get a useful form of Green's function on time scales.

DEFINITION 10. *Let h_0 be identically equal to one on $\mathbb{T}\times\mathbb{T}$, and y_0 be identically equal to $(-1)^{n-k}$ on $\mathbb{T}\times\mathbb{T}$. We define $y_i(t,s)$, $h_i(t,s)$, and $g_{j,i}(t,s)$ for $i \geq 1$ by the following:*

$$\begin{aligned}
y_i(t,s) &= \int_{\sigma(s)}^{t} y_{i-1}(\tau,s)\,\Delta\tau, && 1 \leq i \leq n-k-1, \quad n-k+1 \leq i \leq n-1,\\
y_{n-k}(t,s) &= \int_{\sigma(s)}^{t} \frac{1}{p(\tau)}\, y_{n-k-1}(\tau,s)\,\Delta\tau, &&\\
h_i(t,s) &= \int_{s}^{t} h_{i-1}(\tau,s)\,\Delta\tau, && 1 \leq i \leq n-k-1,\\
g_{i,0}(t,s) &= \frac{1}{p(t)}\, h_i(t,s), && 0 \leq i \leq n-k-1,\\
g_{i,j}(t,s) &= \int_{s}^{t} g_{i,j-1}(\tau,s)\,\Delta\tau, && 1 \leq j \leq k, \quad 0 \leq i \leq n-k-1.
\end{aligned}$$

If the subscript i on the function $h_i(t,s)$ or $y_i(t,s)$ is less than 0, then the functions are taken to be identically zero.

Using the properties of these recursive functions, the following theorem can be easily proved.

THEOREM 9. *The Cauchy function for $(-1)^{n-k}(p(t)x^{\Delta^k})^{\Delta^{n-k}} = 0$, $n \geq 2$, is $y_{n-1}(t,s)$.*

The following lemma has a standard proof which will be omitted.

LEMMA 2. *The boundary value problem $(-1)^{n-k}(p(t)x^{\Delta^k})^{\Delta^{n-k}} = 0$, (1),(2), has only the trivial solution.*

THEOREM 10. *Let*

$$u(t,s) = \begin{vmatrix}
0 & g_{0,k}(t,a) & g_{1,k}(t,a) & \cdots & g_{n-k-1,k}(t,a)\\
y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a)\\
y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a)\\
\vdots & \vdots & \vdots & & \vdots\\
(-1)^{n-k} & 0 & 0 & \cdots & 1
\end{vmatrix}.$$

Then Green's function for the boundary value problem $(-1)^{n-k}(p(t)x^{\Delta^k})^{\Delta^{n-k}} = 0$, (1),(2), is given by

$$G(t,s) = \begin{cases} u(t,s), & t \leq s,\\ u(t,s) + y_{n-1}(t,s), & \sigma(s) \leq t.\end{cases}$$

PROOF. By Theorem 8, it suffices to show that for each fixed $s \in [a,b]$, $u(t,s)$ satisfies the boundary value problem $(-1)^{n-k}(p(t)x^{\Delta^k})^{\Delta^{n-k}} = 0$ with the boundary conditions (1), and

$v(t,s) := y_{n-1}(t,s) + u(t,s)$ satisfies the boundary conditions (2). For $0 \le i \le k-1$,

$$u^{\Delta^i}(t,s) = \left(\begin{vmatrix} 0 & g_{0,k}(t,a) & g_{1,k}(t,a) & \cdots & g_{n-k-1,k}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}\right)^{\Delta^i}$$

$$= \begin{vmatrix} 0 & g_{0,k-i}(t,a) & g_{1,k-i}(t,a) & \cdots & g_{n-k-1,k-i}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

Hence, for $0 \le i \le k-1$, we have

$$u^{\Delta^i}(a,s) = \begin{vmatrix} 0 & g_{0,k-i}(a,a) & g_{1,k-i}(a,a) & \cdots & g_{n-k-1,k-i}(a,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix} = 0,$$

which shows that for each fixed $s \in [a,b]$, $u(t,s)$ satisfies boundary conditions (1). It remains to show that $v(t,s)$ satisfies boundary conditions (2). Consider

$$\begin{aligned} v(t,s) &= u(t,s) + y_{n-1}(t,s) \\ &= \begin{vmatrix} 0 & g_{0,k}(t,a) & g_{1,k}(t,a) & \cdots & g_{n-k-1,k}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix} \\ &\quad + \begin{vmatrix} y_{n-1}(t,s) & 0 & 0 & \cdots & 0 \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix} \\ &= \begin{vmatrix} y_{n-1}(t,s) & g_{0,k}(t,a) & g_{1,k}(t,a) & \cdots & g_{n-k-1,k}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}. \end{aligned}$$

Now, it is easy to see that

$$p(t)v^{\Delta^k}(t,s) = \begin{vmatrix} y_{n-k-1}(t,s) & h_0(t,a) & h_1(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix},$$

and that

$$\left(pv^{\Delta^k}\right)^{\Delta^i}(\sigma(b),s) = \begin{vmatrix} y_{n-k-1-i}(\sigma(b),s) & h_{0-i}(\sigma(b),a) & h_{1-i}(\sigma(b),a) & \cdots & h_{n-k-1-i}(\sigma(b),a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}$$

for $0 \le i \le n-k-1$ (recalling that if the subscript on the function h is less than 0, the function is taken to be identically 0). For each such i, the first and $(i+2)^{\text{nd}}$ rows will be identical, and hence, $(pv^{\Delta^k})^{\Delta^i}(\sigma(b),s) = 0$ for $0 \le i \le n-k-1$. Therefore, boundary conditions (2) are satisfied. Note that, if $i = k$, from above we have

$$p(t)u^{\Delta^k}(t,s) = \begin{vmatrix} 0 & h_0(t,a) & h_1(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

Hence,

$$(-1)^{n-k}\left(pu^{\Delta^k}\right)^{\Delta^{n-k}}(t,s) = \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix} = 0.$$

Therefore, for each fixed $s \in [a,b]$, $(-1)^{n-k}(pu^{\Delta^k})^{\Delta^{n-k}}(t,s) = 0$ and $G(t,s)$ is as stated in this theorem. ∎

The following lemma involves the sign of Green's function for this simplified boundary value problem, which is necessary to do eigenvalue comparisons later. Similar eigenvalue comparisons are done in the Hankerson and Peterson paper [8] for the time scale $\mathbb{T} = \mathbb{Z}$. However, in their case, the sign of Green's function can be found by inspection.

LEMMA 3. *Let $G_p(t,s)$ be Green's function for the boundary value problem*

$$\begin{aligned} (-1)^{n-k}\left(p(t)x^{\Delta^k}\right)^{\Delta^{n-k}} &= 0, \\ x^{\Delta^i}(a) &= 0, \qquad 0 \le i \le k-1, \\ \left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) &= 0, \qquad 0 \le i \le n-k-1. \end{aligned}$$

Then the following hold:

(i) $(-1)^i(p(t)G_p^{\Delta^k}(t,s))^{\Delta^i} > 0$, $a \le t \le s \le b$, $0 \le i \le n-k-1$;

(ii) $G_p^{\Delta^i}(t,s) > 0$, $t \in [\sigma^{k-i}(a), \sigma^{n-i}(b)]$, $s \in [a,b]$, $0 \le i \le k-1$.

PROOF. By Theorem 10, the Green's function for this boundary value problem is given by

$$G(t,s) = \begin{cases} u(t,s), & t \le s, \\ u(t,s) + y_{n-1}(t,s), & \sigma(s) \le t, \end{cases}$$

where

$$u(t,s) = \begin{vmatrix} 0 & g_{0,k}(t,a) & g_{1,k}(t,a) & \cdots & g_{n-k-1,k}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}$$

and $y_{n-1}(t,s)$ is as given in Definition 10. Let $v(t,s) := u(t,s) + y_{n-1}(t,s)$.

PART (i). For $t \leq s$, $G_p(t,s) = u(t,s)$. Hence, Part (i) is equivalent to showing that

$$(-1)^i \left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^i} > 0, \qquad a \leq t \leq s \leq b, \quad 0 \leq i \leq n-k-1.$$

In order to determine the sign of $u(t,s)$ and its delta derivatives, we first consider $v(t,s)$. Fix $s \in [a,b]$ and consider

$$p(t)v^{\Delta^k}(t,s) = \begin{vmatrix} y_{n-k-1}(t,s) & 1 & h_1(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

Therefore,

$$\left(p(t)v^{\Delta^k}(t,s)\right)^{\Delta^{n-k-1}} = \begin{vmatrix} (-1)^{n-k} & 0 & 0 & \cdots & 1 \\ y_{n-k-1}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix} = 0,$$

which implies that $(p(t)v^{\Delta^k}(t,s))^{\Delta^{n-k-2}}$ for any fixed s is a constant.

Evaluating $(p(t)v^{\Delta^k}(t,s))^{\Delta^{n-k-2}}$ at $t = \sigma(b)$ yields that in fact $(p(t)v^{\Delta^k}(t,s))^{\Delta^{n-k-2}} = 0$ for any fixed s as well. Similar work shows that $(p(t)v^{\Delta^k}(t,s))^{\Delta^i}$ is identically 0 for $0 \leq i \leq n-k-1$ for any fixed $s \in [a,b]$. Therefore, $v^{\Delta^k}(t,s) = 0$ for any fixed s. Now, for $1 \leq i \leq n-k-1$, we have

$$\begin{aligned} &(-1)^i \left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^i} \\ &= (-1)^i \begin{vmatrix} 0 & 0 & \cdots & 1 & \cdots & h_{n-k-1-i}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & \cdots & h_i(\sigma(b),a) & \cdots & h_{n-k-1}(\sigma(b),a) \\ y_{n-k-2}(\sigma(b),s) & 0 & \cdots & h_{i-1}(\sigma(b),a) & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 0 & \cdots & 1 \end{vmatrix} \\ &= (-1)^i \begin{vmatrix} 0 & 0 & \cdots & 1 & \cdots & h_{n-k-1-i}(t,a) \\ y_{n-k-2}(\sigma(b),s) & 1 & \cdots & h_{i-1}(\sigma(b),a) & \cdots & h_{n-k-2}(\sigma(b),a) \\ y_{n-k-3}(\sigma(b),s) & 0 & \cdots & h_{i-2}(\sigma(b),a) & \cdots & h_{n-k-3}(\sigma(b),a) \\ \vdots & \vdots & & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 0 & \cdots & 1 \end{vmatrix} \\ &= (-1)^i \begin{vmatrix} 0 & 1 & h_1(t,a) & \cdots & h_{n-k-1-i}(t,a) \\ y_{n-k-1-i}(\sigma(b),s) & 1 & h_1(\sigma(b),a) & \cdots & h_{n-k-1-i}(\sigma(b),a) \\ y_{n-k-i-2}(\sigma(b),s) & 0 & 1 & \cdots & h_{n-k-i-2}(\sigma(b),a) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & 0 & \cdots & 1 \end{vmatrix}. \end{aligned}$$

Letting $i = n-k-1$ yields

$$\begin{aligned}(-1)^{n-k-1}\left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^{n-k-1}} &= (-1)^{n-k-1}\begin{vmatrix} 0 & 1 \\ (-1)^{n-k} & 1 \end{vmatrix} \\ &= (-1)^{2(n-k-1)} \\ &= 1.\end{aligned}$$

Hence, $(-1)^{n-k-1}(p(t)u^{\Delta^k}(t,s))^{\Delta^{n-k-1}} > 0$ for $a \le t \le s \le b$. Therefore,

$$(-1)^{n-k-1}\left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^{n-k-2}}$$

is increasing for $a \le t \le s \le b$. However,

$$(-1)^{n-k-1}\left(pu^{\Delta^k}\right)^{\Delta^{n-k-2}}(\sigma(s),s) = (-1)^{n-k-1}\left(pv^{\Delta^k}\right)^{\Delta^{n-k-2}}(\sigma(s),s) = 0$$

by the above work along with properties of the Cauchy function. Thus,

$$(-1)^{n-k-1}\left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^{n-k-2}} < 0,$$

for $a \le t \le s \le b$, which in turn implies that

$$(-1)^{n-k-2}\left(p(t)u^{\Delta^k}(t,s)\right)^{\Delta^{n-k-2}} > 0,$$

for $a \le t \le s \le b$. Continuing this process and using the fact that, for $1 \le i \le n-k-1$,

$$(-1)^{i-1}\left(pu^{\Delta^k}\right)^{\Delta^i}(\sigma(s),s) = (-1)^{i-1}\left(pv^{\Delta^k}\right)^{\Delta^i}(\sigma(s),s) = 0$$

gives the proof of Part (i) for $0 \le i \le n-k-1$.

PART (ii). Using Part (i), we have that $u^{\Delta^k}(t,s) > 0$ for $a \le t \le s \le b$. Since $u(t,s)$ satisfies the boundary conditions (1), and using techniques as in Part (i), we get that $u^{\Delta^i}(t,s) > 0$ for $t \in [\sigma^{k-i}(a), \sigma^{n-i}(b)]$, $s \in [a,b]$, $t \le s$, and $0 \le i \le k-1$. However, it remains to show that $v^{\Delta^i}(t,s) > 0$ where $\sigma(s) \le t$ instead. This will complete the desired result.

Using Theorem 1(iv), we get that for $0 \le i \le k-1$, and $t \in [\sigma(s), \sigma^{n-i}(b)]$,

$$\begin{aligned}v^{\Delta^i}(\sigma(s),s) &= u^{\Delta^i}(\sigma(s),s) \\ &= u^{\Delta^i}(s,s) + \mu(s)u^{\Delta^{i+1}}(s,s) > 0.\end{aligned}$$

Hence, $G_p^{\Delta^i}(t,s) > 0$, where $t \in [\sigma^{k-i}(a), \sigma^{n-i}(b)]$, $s \in [a,b]$, and $0 \le i \le k-1$. ∎

3. EIGENFUNCTIONS IN THE CONE $\mathcal{P}$

In this section, we are interested in proving the existence of a smallest positive eigenvalue for the focal boundary value problem

$$(-1)^{n-k}\left(p(t)x^{\Delta^k}\right)^{\Delta^{n-k}} = \lambda\left\{\sum_{i=0}^{k-1} p_i(t)x^{\Delta^i} + \sum_{i=0}^{n-k-1} p_{k+i}(t)\left(p(t)x^{\Delta^k}\right)^{\Delta^i}\right\}, \tag{7}$$

$$x^{\Delta^i}(a) = 0, \qquad 0 \le i \le k-1, \tag{8}$$

$$\left(px^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) = 0, \qquad 0 \le i \le n-k-1. \tag{9}$$

Define

$$\mathcal{B} = \left\{ y : [a, \sigma^n(b)] \to \mathbb{R} \mid y^{\Delta^i}(a) = 0,\ 0 \le i \le k-1, \right.$$
$$\left. \left(py^{\Delta^k}\right)^{\Delta^i}(\sigma(b)) = 0,\ 0 \le i \le n-k-1 \right\},$$

and define the norm on $\mathcal{B}$ by $\|y\| = \max\{|y(t)| : a \le t \le \sigma^n(b)\}$. In the Banach space $\mathcal{B}$, define the cone

$$\mathcal{P} = \left\{ y \in \mathcal{B} \mid y^{\Delta^i}(t) \ge 0 \text{ for } t \in [a,b],\ 0 \le i \le k, \right.$$
$$\left. (-1)^i \left(py^{\Delta^k}\right)^{\Delta^i}(t) \ge 0 \text{ for } t \in [a,b],\ 0 \le i \le n-k-1 \right\}.$$

For the following work, it will be required that $\mathcal{P}$ is a reproducing cone. However, if $[a, \sigma^n(b)]$ is a dense set, the cone may not be reproducing. Hence, for the remainder of this chapter, we assume that the set $[a, \sigma^n(b)]$ contains only isolated points. Then $\mathcal{P}$ is a reproducing cone.

PROPOSITION 1. *Assume* $y \in \mathcal{P}$ *and* $y \ne 0$. *If* $1 \le k \le n-2$, *then* $y^{\Delta^i}(t) > 0$ *for* $t \in [\sigma^{k-i}(a), \sigma^{k-i}(b)]$, $0 \le i \le k-1$. *If* $k = n-1$, *then either* $y^{\Delta^{n-2}}(b) > 0$ *or* $y^{\Delta^{n-1}}(b) > 0$.

PROOF. For $1 \le k \le n-2$, assume by way of contradiction that $y \in \mathcal{P}$ and $y \ne 0$ but that there is a fixed i and a fixed $t_0 \in [\sigma^{k-i}(a), \sigma^{k-i}(b)]$ such that $y^{\Delta^i}(t_0) = 0$. Since $y^{\Delta^{i+1}}(t) \ge 0$ for $t \in [a,b]$, $y^{\Delta^i}(t)$ is nondecreasing. Using the boundary conditions at a, it follows that $y^{\Delta^i}(t) = 0$ for $a \le t \le t_0$. Hence, $y^{\Delta^{k-1}}(t) = 0$ for $t \in [a, \rho^{k-i-1}(t_0)]$.

If $y^{\Delta^{k-1}}(t) = 0$ for $t \in [\sigma(a), \sigma(b)]$, then the fact that $y \in \mathcal{P}$ and the boundary conditions at $\sigma(b)$ imply that $y(t) = 0$ for $t \in [a, \sigma^n(b)]$, which is a contradiction. Hence, there is a $t_1 \in [\rho^{k-i-2}(t_0), \sigma(b)]$ such that $y^{\Delta^{k-1}}(t) = 0$ for $t \in [a, \rho(t_1)]$ and $y^{\Delta^{k-1}}(t_1) > 0$. It then follows immediately that

$$\begin{aligned} y^{\Delta^k}(t) &= 0, \qquad a \le t \le \rho^2(t_1), \\ y^{\Delta^k}(\rho(t_1)) &> 0. \end{aligned}$$

But $k \le n-2$, so $y \in \mathcal{P}$ implies that $(py^{\Delta^k})^{\Delta}(t) \le 0$ for $t \in [a,b]$. Using Theorem 1(iv) and the fact that $y^{\Delta^k}(\rho^2(t_1)) = 0$, we have that

$$\begin{aligned} y^{\Delta^k}(\rho(t_1)) &= y^{\Delta^k}\left(\rho^2(t_1)\right) + \mu\left(\rho^2(t_1)\right) y^{\Delta^{k+1}}\left(\rho^2(t_1)\right) \\ &= \mu\left(\rho^2(t_1)\right) y^{\Delta^{k+1}}\left(\rho^2(t_1)\right). \end{aligned}$$

However, it is assumed that all points are isolated, so Theorem 2(iii) can be used along with $y^{\Delta^k}(\rho^2(t_1)) = 0$ to obtain

$$\begin{aligned} \left(py^{\Delta^k}\right)^{\Delta}\left(\rho^2(t_1)\right) &= p\left(\rho(t_1)\right) y^{\Delta^{k+1}}\left(\rho^2(t_1)\right) + p^{\Delta}\left(\rho^2(t_1)\right) y^{\Delta^k}\left(\rho^2(t_1)\right) \\ &= p(\rho(t_1)) y^{\Delta^{k+1}}\left(\rho^2(t_1)\right). \end{aligned}$$

Since $(py^{\Delta^k})^{\Delta}(\rho^2(t_1)) \le 0$ and $p(\rho(t_1)) > 0$, we get that $y^{\Delta^{k+1}}(\rho^2(t_1)) \le 0$, which in turn implies that $y^{\Delta^k}(\rho(t_1)) \le 0$, giving the desired contradiction.

In the case where $k = n-1$, we wish to show that either $y^{\Delta^{n-2}}(b) > 0$ or $y^{\Delta^{n-1}}(b) > 0$. Then $y \in \mathcal{P}$ implies that if this is not the case, then both must be equal to zero. Using Theorem 1(iv), we have that

$$y^{\Delta^{n-2}}(\sigma(b)) = y^{\Delta^{n-2}}(b) + \mu(b) y^{\Delta^{n-1}}(b) = 0.$$

But this contradicts the fact that y is nonzero, completing the proof. ∎

DEFINITION 11. *Define the operator M on $\mathcal{B}$ by*

$$M[y](t) = \int_a^{\sigma(b)} G_p(t,s) \left\{ \sum_{i=0}^{k-1} p_i(s) y^{\Delta^i}(s) + \sum_{i=0}^{n-k-1} p_{k+i}(s) \left(p(s) y^{\Delta^k}(s) \right)^{\Delta^i} \right\} \Delta s,$$

for $t \in [a, \sigma^n(b)]$.

REMARK 2. Let $u(t) = M[y](t)$. Then $u(t)$ is the solution of the dynamic equation

$$(-1)^{n-k} \left(p(t) u^{\Delta^k}(t) \right)^{\Delta^{n-k}} = \sum_{i=0}^{k-1} p_i(t) y^{\Delta^i}(t) + \sum_{i=0}^{n-k-1} p_{k+i}(t) \left(p(t) y^{\Delta^k}(t) \right)^{\Delta^i},$$

satisfying boundary conditions (8),(9).

PROPOSITION 2. *Assume y is an eigenfunction of the operator M. Then y has corresponding eigenvalue $\delta \neq 0$ if and only if $y(t)$ is an eigenfunction of the focal boundary value problem (7)–(9) with corresponding eigenvalue $\lambda = 1/\delta$. In addition, $\lambda = 0$ is not an eigenvalue of (7)–(9).*

LEMMA 4. *Assume*

$$\begin{array}{rll} p_i(t) \geq 0, & \text{for } t \in \left[\sigma^{k-i}(a), b\right], & 0 \leq i \leq k-1, \\ (-1)^i p_{k+i}(t) \geq 0, & \text{for } t \in [a,b], & 0 \leq i \leq n-k-1. \end{array}$$

If $1 \leq k \leq n-2$, then assume $\sum_{i=0}^{k-2} p_i(b) > 0$, whereas, if $k = n-1$, assume $p_{n-2}(b) > 0$ and $p_{n-1}(b) > 0$. Then M is u_0-positive with respect to the cone $\mathcal{P}$.

PROOF. Set $u_0(t) = \int_a^{\sigma(b)} G_p(t,s)\,\Delta s$ for $a \leq t \leq b$. It follows from Lemma 3 and properties of $G_p(t,s)$ that $u_0 \in \mathcal{P}$. Next let $u \in \mathcal{P}$, $u \neq 0$. Then we have

$$(-1)^{n-k} \left(p(t) (M[u])^{\Delta^k}(t) \right)^{\Delta^{n-k}} = \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s) + \sum_{i=0}^{n-k-1} p_{k+i}(s) \left(p(s) u^{\Delta^k}(s) \right)^{\Delta^i}.$$

Integrating both sides from t to $\sigma(b)$ and using the boundary conditions at $\sigma(b)$ yields

$$\begin{aligned}
&(-1)^{n-k-1} \left(p(t) (M[u])^{\Delta^k}(t) \right)^{\Delta^{n-k-1}} \\
&\quad = \int_t^{\sigma(b)} \left\{ \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s) + \sum_{i=0}^{n-k-1} p_{k+i}(s) \left(p(s) u^{\Delta^k}(s) \right)^{\Delta^i} \right\} \Delta s \\
&\quad = \sum_{i=0}^{k-1} \int_t^{\sigma(b)} p_i(s) u^{\Delta^i}(s)\,\Delta s + \sum_{i=0}^{n-k-1} \int_t^{\sigma(b)} p_{k+i}(s) \left(p(s) u^{\Delta^k}(s) \right)^{\Delta^i} \Delta s \\
&\quad \geq \sum_{i=0}^{k-1} \int_b^{\sigma(b)} p_i(s) u^{\Delta^i}(s)\,\Delta s + \sum_{i=0}^{n-k-1} \int_b^{\sigma(b)} p_{k+i}(s) \left(p(s) u^{\Delta^k}(s) \right)^{\Delta^i} \Delta s \\
&\quad = \sum_{i=0}^{k-1} \mu(b) p_i(b) u^{\Delta^i}(b) + \sum_{i=0}^{n-k-1} \mu(b) p_{k+i}(b) \left(p(t) u^{\Delta^k}(t) \right)^{\Delta^i} \Big|_{t=b} \\
&\quad \geq \sum_{i=0}^{k-1} \mu(b) p_i(b) u^{\Delta^i}(b) + \mu(b) p_k(b) p(b) u^{\Delta^k}(b) \\
&\quad = \mu(b) \left(\sum_{i=0}^{k-1} p_i(b) u^{\Delta^i}(b) + p_k(b) p(b) u^{\Delta^k}(b) \right).
\end{aligned}$$

Where the last inequality depends on the fact that $u \in \mathcal{P}$ implies $(-1)^i (py^{\Delta^k})^{\Delta^i}(t) \geq 0$ for $t \in [a,b]$, $0 \leq i \leq n-k-1$, and that we are assuming $(-1)^i p_{k+i}(t) \geq 0$, for $t \in [a,b]$, $0 \leq i \leq n-k-1$. By assumption $\mu(b) \neq 0$, $\sum_{i=0}^{k-1} p_i(b) > 0$ when $1 \leq k \leq n-2$, and $p_{k-1}(b) = p_{n-2}(b) > 0$, $p_k(b) = p_{n-1}(b) > 0$ when $k = n-1$. So using Proposition 1 and the above inequality, we have

$$(-1)^{n-k-1}\left(p(t)(M[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} > 0,$$

for $a \leq t \leq b$. Also

$$(-1)^{n-k-1}\left(p(t)u_0^{\Delta^k}\right)^{\Delta^{n-k-1}} = \int_t^{\sigma(b)} 1\,\Delta s = \sigma(b) - t > 0,$$

for $a \leq t \leq b$. Consequently, there are positive constants k_1 and k_2 such that

$$\begin{aligned} k_1(-1)^{n-k-1}\left(p(t)u_0^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} &\leq (-1)^{n-k-1}\left(p(t)(M[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \\ &\leq k_2(-1)^{n-k-1}\left(p(t)u_0^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \end{aligned}$$

for $a \leq t \leq b$. Integrating from t to $\sigma(b)$ and using the boundary conditions at $\sigma(b)$ yields that for $a \leq t \leq b$, we have

$$\begin{aligned} k_1(-1)^{n-k-2}\left(p(t)u_0^{\Delta^k}(t)\right)^{\Delta^{n-k-2}} &\leq (-1)^{n-k-2}\left(p(t)(M[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-2}} \\ &\leq k_2(-1)^{n-k-2}\left(p(t)u_0^{\Delta^k}(t)\right)^{\Delta^{n-k-2}}. \end{aligned}$$

Continuing this integration process and using the boundary conditions at $\sigma(b)$ yields that for $a \leq t \leq b$, we have

$$k_1 p(t)u_0^{\Delta^k}(t) \leq p(t)(M[u])^{\Delta^k}(t) \leq k_2 p(t)u_0^{\Delta^k}(t).$$

Dividing by $p(t)$ which is positive, integrating from a to t, and using the boundary conditions at a, we obtain

$$k_1 u_0^{\Delta^{k-1}}(t) \leq M^{\Delta^{k-1}}[u](t) \leq k_2 u_0^{\Delta^{k-1}}(t)$$

for $a \leq t \leq b$. Continuing this process and using the boundary conditions at a, we finally obtain

$$k_1 u_0(t) \leq M[u](t) \leq k_2 u_0(t),$$

for $a \leq t \leq b$. Hence, we have shown that $M : \mathcal{P} \to \mathcal{P}$ and $k_1 u_0 \leq M[u] \leq k_2 u_0$ with respect to the cone $\mathcal{P}$. Therefore, M is u_0-positive with respect to the cone $\mathcal{P}$. ∎

THEOREM 11. *Under the hypothesis of Lemma 4, the focal boundary value problem (7)–(9) has a smallest positive eigenvalue* λ_0, *there is an essentially unique eigenfunction* y_0 *corresponding to* λ_0, *and either* y_0 *or* $-y_0$ *satisfies*

$$\begin{aligned} y^{\Delta^i}(t) &> 0, & t &\in \left[\sigma^{k-i}(a), \sigma^{n-i}(b)\right], & 0 &\leq i \leq k-1, \\ (-1)^i\left(p(t)y^{\Delta^k}(t)\right)^{\Delta^i} &> 0, & t &\in [a,b], & 0 &\leq i \leq n-k-1. \end{aligned}$$

Furthermore, λ_0 *is less than the modulus of any other eigenvalue of (7)–(9).*

PROOF. It is a standard argument to show that M is a compact linear operator. By Lemma 4, M is u_0-positive with respect to the cone $\mathcal{P}$. Using Theorem 6 and Proposition 2, we get that the boundary value problem (7)–(9) has a smallest positive eigenvalue λ_0 which is less than the modulus of any other eigenvalue. The corresponding eigenfunction $y_0(t)$ is essentially unique and, without loss of generality, $y_0 \in \mathcal{P}$. Since M is u_0-positive with respect to $\mathcal{P}$, there are constants $k_1 > 0$ and $k_2 > 0$ such that $k_1 u_0 \le M y_0 \le k_2 u_0$ with respect to the cone $\mathcal{P}$. Therefore, $\lambda_0 k_1 u_0 \le \lambda_0 M y_0 = y_0$ with respect to the cone $\mathcal{P}$. It follows from this inequality that $y_0(t)$ satisfies

$$\begin{aligned} y^{\Delta^i}(t) &> 0, \qquad t \in \left[\sigma^{k-i}(a), b\right], \quad 0 \le i \le k-1, \\ (-1)^i \left(p(t) y^{\Delta^k}(t)\right)^{\Delta^i} &> 0, \qquad t \in [a,b], \qquad 0 \le i \le n-k-1. \end{aligned}$$

Note that

$$y^{\Delta^{k-1}}(\sigma(b)) = y^{\Delta^{k-1}}(b) + \mu(b) y^{\Delta^k}(b) > 0$$

by the first inequality. Then advancing and using the boundary condition at $\sigma(b)$, we obtain

$$y^{\Delta^{k-1}}\left(\sigma^j(b)\right) = y^{\Delta^{k-1}}\left(\sigma^{j-1}(b)\right) + \mu(\sigma(b)) y^{\Delta^k}\left(\sigma^{j-1}(b)\right) > 0,$$

for $1 \le j \le n-k+1$. Iterating this process yields that

$$y^{\Delta^i}(t) > 0,$$

$t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$, completing the proof. ∎

We are now interested in the special case of (7) where $p_{k+i}(t) = 0$ on $[a,b]$, $0 \le i \le n-k-1$. In particular, we are interested in the dynamic equation

$$(-1)^{n-k} \left(p(t) u^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \lambda \sum_{i=0}^{k-1} p_i(t) u^{\Delta^i}(t). \tag{10}$$

DEFINITION 12. *Define the operator M_0 on $\mathcal{B}$ by*

$$M_0[y](t) = \int_a^{\sigma(b)} G_p(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\, \Delta s,$$

for $a \le t \le \sigma^n(b)$.

LEMMA 5. *Let $1 \le k \le n-2$. Assume that one of*

(a) *$k > 1$ and $\sum_{i=0}^{k-2} p_i(b) > 0$, or*
(b) *$\int_t^{\sigma(b)} p_{k-1}(s)\, \Delta s > 0$, for $t \in [\sigma(a), b]$,*

holds, and that

(c) *$\int_t^{\sigma(b)} p_i(s)\, \Delta s \ge 0$ for $t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$.*

Then M_0 is u_0-positive with respect to $\mathcal{P}$.

PROOF. Let u_0 be as in the proof of Lemma 4, and let $u \in \mathcal{P}$ be nontrivial. The proof is similar to the proof of Lemma 4 except for the proof of the inequality

$$(-1)^{n-k-1} \left(p(t) (M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} > 0, \tag{11}$$

for $a \le t \le b$. Using properties of the Green's function, we have

$$(-1)^{n-k} \left(p(t) (M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \sum_{i=0}^{k-1} p_i(t) u^{\Delta^i}(t).$$

Integrating both sides from t to $\sigma(b)$ and using the boundary conditions at $\sigma(b)$, we obtain

$$(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} = \int_t^{\sigma(b)} \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s.$$

Then, using the integration by parts formula given in Theorem 3, we get

$$\begin{aligned}
&(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\\
&\quad= \sum_{i=0}^{k-1}\left\{\left[-u^{\Delta^i}(s)\int_s^{\sigma(b)} p_i(\tau)\,\Delta\tau\right]_t^{\sigma(b)} + \int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\}\\
&\quad= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau + \int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\}.
\end{aligned}$$

Let $m_1 = \min\{u^{\Delta^i}(b) : 0 \le i \le k-1\}$. Using Lemma 1, $m_1 > 0$. Taking $t = b$, we have

$$\begin{aligned}
&(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\Big|_{t=b}\\
&\quad= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(b)\int_b^{\sigma(b)} p_i(\tau)\,\Delta\tau + \int_b^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\}\\
&\quad= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(b)\mu(b)p_i(b) + \mu(b)u^{\Delta^{i+1}}(b)\int_{\sigma(b)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\right\}\\
&\quad= \sum_{i=0}^{k-1} u^{\Delta^i}(b)\mu(b)p_i(b) \ge m_1\mu(b)\sum_{i=0}^{k-1} p_i(b)\\
&\quad= m_1\mu(b)\sum_{i=0}^{k-2} p_i(b) + m_1\mu(b)p_{k-1}(b).
\end{aligned}\tag{12}$$

Clearly, Property (c) implies that both terms on the right-hand side of inequality (12) are nonnegative. If (a) holds, then the right-hand side of inequality (12) is strictly positive. If (b) holds, then $\int_t^{\sigma(b)} p_{k-1}(s)\,\Delta s > 0$ for $t \in [\sigma(a), b]$ implies $p_{k-1}(b) > 0$, and again the right-hand side of inequality (12) is strictly positive. Hence, inequality (11) holds at $t = b$.

Now, assume that $a < t < b$. Then

$$\begin{aligned}
&(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\\
&\quad= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau + \int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\}\\
&\quad= \sum_{i=0}^{k-1} u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau + \sum_{i=0}^{k-1}\int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\\
&\quad= \sum_{i=0}^{k-1} u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau + \sum_{i=0}^{k-2}\int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\\
&\qquad+ \int_t^{\sigma(b)} u^{\Delta^k}(s)\int_{\sigma(s)}^{\sigma(b)} p_{k-1}(\tau)\,\Delta\tau\,\Delta s,
\end{aligned}$$

where $u \in \mathcal{P}$ and Property (c) along with Lemma 1 imply that each term on the right-hand side is nonnegative. If Part (a) holds (which implies $k > 1$), then

$$(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \ge \sum_{i=0}^{k-2}\int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s.$$

If $t > \sigma^k(a)$, then by Lemma 1, the right-hand side of the above inequality is strictly positive. If $a < t \le \sigma^k(a)$, let $q_i = \max\{t, \sigma^{k-i}(a)\}$. Then using the boundary conditions at a and Lemma 1, we have

$$\begin{aligned}(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} &\ge \sum_{i=0}^{k-2}\int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &\ge \sum_{i=0}^{k-2}\int_{q_i}^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &> 0.\end{aligned}$$

If Part (b) holds, then

$$(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \ge \sum_{i=0}^{k-1} u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau,$$

which is strictly positive by a similar argument. Hence, inequality (11) holds for $a < t < b$.

Finally, consider the case $t = a$. Using the boundary conditions at a, we get that

$$\begin{aligned}&(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\Big|_{t=a} \\ &= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(a)\int_a^{\sigma(b)} p_i(\tau)\,\Delta\tau + \int_a^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\} \\ &= \sum_{i=0}^{k-1}\int_a^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &= \sum_{i=0}^{k-2}\int_a^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s + \int_a^{\sigma(b)} u^{\Delta^k}(s)\int_{\sigma(s)}^{\sigma(b)} p_{k-1}(\tau)\,\Delta\tau\,\Delta s,\end{aligned}$$

where using Property (c) and Lemma 1, it can be seen that each term on the right-hand side is nonnegative. If Part (a) holds (which implies $k > 1$), then using the boundary conditions at a along with Lemma 1,

$$\begin{aligned}(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\Big|_{t=a} &\ge \sum_{i=0}^{k-2}\int_a^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &= \sum_{i=0}^{k-2}\int_{\sigma^{k-i}(a)}^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &> 0.\end{aligned}$$

Assume (b) holds. Note that, by Theorem 1(ii), we have

$$u^{\Delta^k}(t) = \frac{u^{\Delta^{k-1}}(\sigma(t)) - u^{\Delta^{k-1}}(t)}{\mu(t)}.$$

The boundary conditions at a along with Lemma 1 imply that $u^{\Delta^k}(a) > 0$. Note that $u \in \mathcal{P}$ implies that $u^{\Delta^k}(t) \ge 0$ for $t \in [a, b]$. Thus,

$$(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}}\Big|_{t=a} \ge \int_a^{\sigma(b)} u^{\Delta^k}(s)\int_{\sigma(s)}^{\sigma(b)} p_{k-1}(\tau)\,\Delta\tau\,\Delta s > 0,$$

completing the proof. ∎

THEOREM 12. *Under the hypothesis of Lemma 4, the conclusions of Theorem 11 hold for the boundary value problem (8)–(10).*

We now define a new operator, which along with the operator M_0 will allow us to make eigenvalue comparisons.

DEFINITION 13. *Define* N_0 *on* $\mathcal{B}$ *by*

$$N_0[u](t) = \int_a^{\sigma(b)} G_p(t,s) \sum_{i=0}^{k-1} P_i(s) u^{\Delta^i}(s)\,\Delta s,$$

for $t \in [a, \sigma^n(b)]$.

LEMMA 6. *If* $\int_t^{\sigma(b)} p_i(s)\,\Delta s \le \int_t^{\sigma(b)} P_i(s)\,\Delta s$ *for* $t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$, *then* $M_0 \le N_0$ *with respect to the cone* $\mathcal{P}$.

PROOF. Let $u \in \mathcal{P}$. Using the integration by parts formula in Theorem 3, we get

$$\begin{aligned}
&(-1)^{n-k-1}\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \\
&= \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau + \int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s\right\} \\
&\le \sum_{i=0}^{k-1}\left\{u^{\Delta^i}(t)\int_t^{\sigma(b)} P_i(\tau)\,\Delta\tau + \int_t^{\sigma(b)} u^{\Delta^{i+1}}(s)\int_{\sigma(s)}^{\sigma(b)} P_i(\tau)\,\Delta\tau\,\Delta s\right\} \\
&= \sum_{i=0}^{k-1}\int_t^{\sigma(b)} u^{\Delta^i}(s) P_i(s)\,\Delta s = (-1)^{n-k-1}\left(p(t)(N_0[u])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}},
\end{aligned}$$

for $t \in [a, b]$. The boundary conditions at $\sigma(b)$ imply that

$$(-1)^j\left(p(t)(M_0[u])^{\Delta^k}(t)\right)^{\Delta^j} \le (-1)^j\left(p(t)(N_0[u])^{\Delta^k}(t)\right)^{\Delta^j},$$

for $j = n-k-1, n-k-2, \dots, 0$. In particular,

$$p(t)(M_0[u])^{\Delta^k}(t) \le p(t)(N_0[u])^{\Delta^k}(t).$$

From the boundary conditions at a, we see that

$$(M_0[u])^{\Delta^i}(t) \le (N_0[u])^{\Delta^i}(t),$$

for $t \in [a, b]$, $0 \le i \le k-1$. It then follows that $M_0 u \le N_0 u$ with respect to the cone $\mathcal{P}$. ∎

We now prove a comparison theorem for the focal boundary value problem (10) and

$$(-1)^{n-k}\left(p(t)u^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \Lambda \sum_{i=0}^{k-1} P_i(t) u^{\Delta^i}(t). \tag{13}$$

THEOREM 13. *Let* $1 \le k \le n-2$. *Assume that one of the conditions*

(a) $k > 1$ *and* $\sum_{i=0}^{k-2} p_i(b) > 0$, *or*

(b) $\int_t^{\sigma(b)} p_{k-1}(s)\,\Delta s > 0$, *for* $t \in [\sigma(a), b]$,

holds, and that

(c) $0 \le \int_t^{\sigma(b)} p_i(s)\,\Delta s \le \int_t^{\sigma(b)} P_i(s)\,\Delta s$ *for* $t \in [\sigma^{k-i}(a), b]$, $1 \le i \le k-1$.

Then there exists a smallest positive eigenvalue λ_0 *(which is less than the modulus of any other eigenvalue) of (8)–(10), and a corresponding essentially unique eigenfunction* x_0*, and either* x_0 *or* $-x_0$ *satisfies*

$$\begin{aligned} x^{\Delta^i}(t) &> 0, & t &\in \left[\sigma^{k-i}(a), \sigma^{n-i}(b)\right], & 0 &\le i \le k-1, \\ (-1)^i \left(p(t)x^{\Delta^k}(t)\right)^{\Delta^i} &> 0, & t &\in [a,b], & 0 &\le i \le n-k-1. \end{aligned} \tag{14}$$

Similarly, there exists a smallest positive eigenvalue Λ_0 *of (13), (8), (9), and a corresponding essentially unique eigenfunction* y_0*, and either* y_0 *or* $-y_0$ *satisfies (14). Furthermore,* $\Lambda_0 \le \lambda_0$ *and* $\Lambda_0 = \lambda_0$ *if and only if* $p_i(t) = P_i(t)$ *for* $t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$.

PROOF. By Theorem 12, the boundary value problem (8)–(10) has a smallest positive eigenvalue λ_0. Conditions (a) and (c) imply that $\sum_{i=0}^{k-2} P_i(b) > 0$, and Conditions (b) and (c) imply that $\int_t^{\sigma(b)} P_{k-1}(s)\,\Delta s > 0$ for $t \in [\sigma(a), b]$. Hence, by Theorem 12 applied to the boundary value problem (13), (8), (9), we see that this boundary value problem has a smallest positive eigenvalue Λ_0. By Lemma 6, $M_0 \le N_0$ with respect to the cone $\mathcal{P}$. Then $\Lambda_0 \le \lambda_0$ by Theorem 7 and Proposition 2. The proof that (14) holds for $x_0(t)$ and $y_0(t)$ follows from the fact that M_0 and N_0 are u_0-positive.

We now want to show that if $\Lambda_0 = \lambda_0$, then $p_i(t) = P_i(t)$ for $t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$. To reach this goal, consider the operators L_j defined on $\mathcal{B}$ by

$$L_j[u](t) = \int_a^{\sigma(b)} G_p(t,s) \left\{ \sum_{i=0}^{j-1} P_i(s)u^{\Delta^i}(s) + \sum_{i=j}^{k-1} p_i(s)u^{\Delta^i}(s) \right\} \Delta s,$$

for $t \in [a, \sigma^n(b)]$, $0 \le j \le k$. Note that $L_0 = M_0$ and $L_k = N_0$. The associated dynamic equation is

$$(-1)^{n-k} \left(p(t)u^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \lambda \left\{ \sum_{i=0}^{j-1} P_i(s)u^{\Delta^i}(s) + \sum_{i=j}^{k-1} p_i(s)u^{\Delta^i}(s) \right\}, \tag{15}$$

for $0 \le j \le k$. The first part of this theorem can be applied to get that (15) for the values j and $j+1$ each has a least positive eigenvalue, λ_j and λ_{j+1}, respectively. In addition, $\lambda_{j+1} \le \lambda_j$. Since $\Lambda_0 = \lambda_k \le \cdots \le \lambda_0 = \Lambda_0$, we have that $\lambda_j = \lambda_0$ for $0 \le j \le k$. Let $y_j(t)$ and $y_{j+1}(t)$ be eigenfunctions corresponding to λ_j and λ_{j+1}, respectively. By Theorem 7, $c_j y_j(t) = y_{j+1}(t)$ for some constant c_j. Substituting into (15) yields

$$(-1)^{n-k} \left(p(t)y_{j+1}^{\Delta^k}(t)\right)^{\Delta^{n-k}} = (-1)^{n-k} \left(p(t)cy_j^{\Delta^k}(t)\right)^{\Delta^{n-k}},$$

and hence,

$$\sum_{i=0}^{j} P_i(t)y_j^{\Delta^i}(t) + \sum_{i=j+1}^{k-1} p_i(t)y_j^{\Delta^i}(t) = \sum_{i=0}^{j-1} P_i(t)y_j^{\Delta^i}(t) + \sum_{i=j}^{k-1} p_i(t)y_j^{\Delta^i}(t).$$

Therefore, $p_i(t)y_j^{\Delta^i}(t) = P_i(t)y_j^{\Delta^i}(t)$ for $t \in [a,b]$. Using Proposition 1, we get that $p_i(t) = P_i(t)$ for $t \in [\sigma^{k-j}(a), b]$. ∎

Theorem 6 has been one of the main tools used so far. However, the existence of a maximal positive eigenvalue of a compact linear operator can be obtained under some weaker conditions. The first is Theorem 2.5 of Krasnosel'skii [5].

THEOREM 14. *Let* L *be a compact linear operator which leaves the cone* $\mathcal{P}$ *invariant. Assume there is a* $u \in \mathcal{B}$ *and an* $\epsilon > 0$ *with* $-u \notin \mathcal{P}$ *and* $u = v - w$ *for some* $v, w \in \mathcal{P}$ *such that* $L^p u \ge \epsilon u$ *for some natural number* p*. Then* L *has at least one eigenfunction* x_0 *in* $\mathcal{P}$: $Lx_0 = \lambda_0 x_0$*, where the eigenvalue* λ_0 *satisfies* $\lambda_0 \ge \sqrt[p]{\epsilon}$.

COROLLARY 1. *Assume in addition to the hypothesis of Theorem 14 that $\mathcal{P}$ is reproducing. Then one of the eigenvalues of L corresponding to an eigenfunction in $\mathcal{P}$ is the least upper bound for the modulus of the eigenvalues of L.*

The following theorem is very similar to Theorem 11.

THEOREM 15. *Assume*

$$\begin{aligned} p_i(t) &\geq 0, && \text{for } t \in \left[\sigma^{k-i}(a), b\right], && 0 \leq i \leq k-1, \\ (-1)^i p_{k+i}(t) &\geq 0, && \text{for } t \in [a,b], && 0 \leq i \leq n-k-1, \end{aligned}$$

and that there is an i, $0 \leq i \leq n-1$ such that $p_i(b) \neq 0$. Then M has an eigenfunction $x_0 \in \mathcal{P}$ with corresponding eigenvalue $\gamma_0 > 0$ which is the least upper bound for the modulus of the eigenvalues of M. In particular, the boundary value problem (7)–(9) has a smallest positive eigenvalue λ_0 and a corresponding eigenfunction $y_0(t)$ satisfying

$$\begin{aligned} y^{\Delta^i}(t) &\geq 0, && t \in \left[a, \sigma^{n-i}(b)\right], && 0 \leq i \leq k-1, \\ (-1)^i \left(p(t) y^{\Delta^k}(t)\right)^{\Delta^i} &\geq 0, && t \in [a,b], && 0 \leq i \leq n-k-1. \end{aligned} \tag{16}$$

Furthermore, λ_0 is the greatest lower bound for the modulus of the eigenvalues of (7)–(9).

PROOF. Let $u_0(t) = \int_a^{\sigma(b)} G_p(t,s)\,\Delta s$. Then

$$(-1)^{n-k}\left(p(t)(M[u_0])^{\Delta^k}\right)^{\Delta^{n-k}} = \sum_{i=0}^{k-1} p_i(t) u_0^{\Delta^i}(t) + \sum_{i=0}^{n-k-1} p_{k+i}(t)\left(p(t) u_0^{\Delta^k}(t)\right)^{\Delta^i}.$$

As in Theorem 4,

$$(-1)^{n-k-1}\left(p(t)(M[u_0])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \geq \mu(b)\left(\sum_{i=0}^{k-1} p_i(b) u_0^{\Delta^i}(b) + p_k(b) p(b) u_0^{\Delta^k}(b)\right),$$

which is strictly positive. Hence, there exists an $\epsilon > 0$ such that

$$(-1)^{n-k-1}\left(p(t)(M[u_0])^{\Delta^k}(t)\right)^{\Delta^{n-k-1}} \geq \epsilon(-1)^{n-k-1}\left(p(t) u_0^{\Delta^k}(t)\right)^{\Delta^{n-k-1}},$$

for $t \in [a,b]$. Using arguments as in Lemma 4, it can be shown that $Mu_0 \geq \epsilon u_0$ with respect to the cone $\mathcal{P}$. The result then follows from Corollary 1. ∎

Theorem 14 and Corollary 1 also enable us to obtain the following comparison theorem between the dynamic equations (10) and (13) with the boundary conditions (8),(9) under slightly weaker conditions than those in Theorem 13. The disadvantage of this result in comparison with Theorem 13 is that we do not get the strict inequalities (14) for the solution of (8)–(10).

THEOREM 16. *Let $1 \leq k \leq n-2$. Assume that one of the conditions*

(a) *$k > 1$ and $\sum_{i=0}^{k-2} P_i(b) > 0$, or*
(b) *$\int_t^{\sigma(b)} P_{k-1}(s)\,\Delta s > 0$, for $t \in [\sigma(a), b]$,*

holds, and that

(c) *$p_i(b) > 0$ for some $0 \leq i \leq k-1$,*
(d) *$0 \leq \int_t^{\sigma(b)} p_i(s)\,\Delta s \leq \int_t^{\sigma(b)} P_i(s)\,\Delta s$ for $t \in [\sigma^{k-i}(a), b]$, $1 \leq i \leq k-1$.*

Then there exists a smallest positive eigenvalue λ_0 (which is less than the modulus of any other eigenvalue) of (8)–(10), and a corresponding eigenfunction x_0 satisfying (16). Similarly, there exists a smallest positive eigenvalue Λ_0 of (13), (8), (9), and a corresponding essentially unique eigenfunction y_0, and either y_0 or $-y_0$ satisfies (14). Furthermore, $\Lambda_0 \leq \lambda_0$ and $\Lambda_0 = \lambda_0$ if and only if $p_i(t) = P_i(t)$ for $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$.

4. EIGENFUNCTIONS IN THE CONES $\mathcal{P}_{k-1}$ AND $\mathcal{P}_k$

In this section, we will be concerned with equations of the form

$$(-1)^{n-k}\left(p(t)u^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \lambda \sum_{i=0}^{k-1} p_i(t)u^{\Delta^i}(t), \tag{17}$$

where the p_is satisfy the following "triangular" condition: there is a fixed j, $0 \le j \le k-1$, such that

$$\begin{aligned} &p_i(t) = 0, \qquad \text{for } t \in \left[\rho^{i-j}(b), b\right], \quad j+1 \le i \le k-1, \\ &p_j(b) > 0. \end{aligned} \tag{18}$$

(If $j = k-1$, then condition (18) is $p_{k-1}(b) > 0$.) The primary goal is to compare eigenvalues of (17), (8), (9) with eigenvalues of

$$(-1)^{n-k}\left(P(t)u^{\Delta^k}(t)\right)^{\Delta^{n-k}} = \Lambda \sum_{i=0}^{k-1} P_i(t)u^{\Delta^i}(t), \tag{19}$$

with the boundary conditions (8),(9). In the last section, a comparison theorem was established under the condition $P(t) = p(t)$, $t \in [a,b]$. However, the methods used in that comparison theorem do not apply when $P \ne p$, on $[a,b]$.

We assume throughout this section that both the ps and the Ps satisfy condition (18). Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \left\{u : \left[a, \sigma^j(b)\right] \to \mathbb{R} \mid u^{\Delta^i}(a) = 0,\ 0 \le i \le k-1\right\},$$

where $\|u\| = \max\{|u(t)| : a \le t \le \sigma^j(b)\}$. Let the cones $\mathcal{P}_{k-1}$ and $\mathcal{P}_k$ be defined by

$$\mathcal{P}_{k-1} = \Big\{u \in \mathcal{B} \mid u^{\Delta^i}(t) \ge 0 \text{ for } t \in [a,b],\ 0 \le i \le j,$$
$$u^{\Delta^{i+j}}(t) \ge 0 \text{ for } t \in \left[a, \rho^i(b)\right],\ 1 \le i \le k-1-j\Big\}$$

and

$$\mathcal{P}_k = \Big\{u \in \mathcal{B} \mid u^{\Delta^i}(t) \ge 0 \text{ for } t \in [a, \rho(b)],\ 0 \le i \le j+1,$$
$$u^{\Delta^{i+j}}(t) \ge 0 \text{ for } t \in \left[a, \rho^i(b)\right],\ 2 \le i \le k-j\Big\}.$$

As before, it can be shown that the cones $\mathcal{P}_{k-1}$ and $\mathcal{P}_k$ are reproducing.

Notice that the condition (18) implies that if $u(t)$ is a function defined on $[a, \sigma^n(b)]$, then the expression

$$\sum_{i=0}^{k-1} p_i(t)u^{\Delta^i}(t), \qquad t \in [a,b],$$

does not depend on the values of $u(t)$ at $\sigma^{j+1}(b), \dots, \sigma^n(b)$. Define the operator M on $\mathcal{B}$ by

$$M[u](t) = \int_a^{\sigma(b)} G_p(t,s) \sum_{i=0}^{k-1} p_i(s)u^{\Delta^i}(s)\,\Delta s, \qquad t \in \left[a, \sigma^j(b)\right],$$

where the terms $p_i(t)u^{\Delta^i}(t)$, $t \in [\rho^{i-j-1}(b), b]$, $j+1 \le i \le k-1$, are understood to be zero. Note that since $G_p(t,s)$ is defined for $t \in [a, \sigma^n(b)]$, $M[u](t)$ actually makes sense for $t \in [a, \sigma^n(b)]$. We will make use of this often in the following work.

REMARK 3. Assume u is an eigenvector for M with corresponding eigenvalue $\lambda \ne 0$. Then $M[u](t)$ is defined on $[a, \sigma^n(b)]$ and we know that $M[u](t) = \lambda u(t)$ for $t \in [a, \sigma^j(b)]$. If we use this equation to extend the definition of $u(t)$ to $[a, \sigma^n(b)]$, it follows that $u(t)$ is an eigenfunction for the eigenvalue problem (17), (7), (8) with corresponding eigenvalue $1/\lambda$.

Lemma 7. *Let $u \in \mathcal{P}_{k-1}$, $u \neq 0$. If $0 \leq j \leq k-2$, then $u^{\Delta^j}(b) > 0$; if $1 \leq j = k-1$, then either $u^{\Delta^{k-2}}(b) > 0$ or $u^{\Delta^{k-1}}(b) > 0$. If $u \in \mathcal{P}_k$, $u \neq 0$, then $u^{\Delta^j}(b) > 0$.*

Proof. First, let $u \in \mathcal{P}_{k-1}$, $u \neq 0$, and $0 \leq j \leq k-2$. Since $u^{\Delta^{j+1}}(t) \geq 0$ for $t \in [a, \rho(b)]$, $u^{\Delta^j}(t)$ is nondecreasing on $[a, \rho(b)]$. Note that

$$u^{\Delta^{j+1}}(\rho(b)) = \frac{u^{\Delta^j}(b) - u^{\Delta^j}(\rho(b))}{\mu(\rho(b))} \geq 0,$$

since $u \in \mathcal{P}_{k-1}$. If $u^{\Delta^j}(b) = 0$, then it must be the case that $u^{\Delta^j}(\rho(b)) = 0$ as well. Since $u^{\Delta^j}(t)$ is nondecreasing on $[a, \rho(b)]$, this means that $u^{\Delta^j}(t) = 0$ for $t \in [a, b]$ if $u^{\Delta^j}(b) = 0$, which contradicts the fact that $u \neq 0$.

Now, let $u \in \mathcal{P}_{k-1}$, $u \neq 0$, and $1 \leq j = k-1$. This implies that $u^{\Delta^{k-1}}(t) \geq 0$ for $t \in [a, b]$, which in turn implies that $u^{\Delta^{k-2}}(t)$ is nondecreasing. If $u^{\Delta^{k-2}}(b) = 0$, then $u^{\Delta^{k-2}}(t)$ is identically 0 on $[a, b]$. But

$$u^{\Delta^{k-1}}\left(\rho^i(b)\right) = \frac{u^{\Delta^{k-2}}\left(\rho^{i-1}(b)\right) - u^{\Delta^{k-2}}\left(\rho^i(b)\right)}{\mu\left(\rho^i(b)\right)} = 0,$$

for all i such that $\rho^i(b) \in [a, b]$. Hence, $u^{\Delta^{k-1}}(t) = 0$ for $t \in [a, \rho(b)]$. If in addition $u^{\Delta^{k-1}}(b) = 0$, this will contradict the fact that $u \neq 0$. Therefore, either $u^{\Delta^{k-2}}(b) > 0$ or $u^{\Delta^{k-1}}(b) > 0$.

Lastly, suppose $u \in \mathcal{P}_k$ and $u \neq 0$. Then as before

$$u^{\Delta^j}(b) = \mu(\rho(b)) u^{\Delta^{j+1}}(\rho(b)) + u^{\Delta^j}(\rho(b)) > 0,$$

since $u \in \mathcal{P}_k$, completing the proof. ∎

Lemma 8. *Assume $p_i(t) \geq 0$ for $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$. When $j = k-1$, further assume $p_{j-1}(b) > 0$ if $j \geq 1$, and assume $p_0(t) > 0$ on $[\sigma(a), b]$ if $j = 0$. Then M is u_0-positive with respect to $\mathcal{P}_{k-1}$.*

Proof. Define $u_0(t) = \int_a^{\sigma(b)} G_p(t,s)\,\Delta s$ for $t \in [a, \sigma^n(b)]$. Using properties of Green's function, it can be seen that the restriction of u_0 to $[a, \sigma^j(b)]$ is in $\mathcal{P}_{k-1}$. Let $u \in \mathcal{P}_{k-1}$, $u \neq 0$. Then

$$(M[u])^{\Delta^{k-1}}(t) = \int_a^{\sigma(b)} G_p^{\Delta^{k-1}}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s, \qquad t \in \left[a, \sigma^{n-k+1}(b)\right].$$

We consider three cases: $j \leq k-2$, $1 \leq j = k-1$, and $j = k-1 = 0$. First, assume $j \leq k-2$. Note that

$$\begin{aligned} (M[u])^{\Delta^{k-1}}(t) &\geq \int_b^{\sigma(b)} G_p^{\Delta^{k-1}}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s \\ &= G_p^{\Delta^{k-1}}(t,b) \sum_{i=0}^{k-1} p_i(b) u^{\Delta^i}(b) \mu(b) \\ &\geq G_p^{\Delta^{k-1}}(t,b) p_j(b) u^{\Delta^j}(b) \mu(b), \qquad t \in \left[a, \sigma^{n-k+1}(b)\right]. \end{aligned}$$

Since $G_p^{\Delta^{k-1}}(t,b) > 0$ on $[a, \sigma^{n-k+1}(b)]$, $p_j(b) > 0$, and $u^{\Delta^j}(b) > 0$, we have that $(M[u])^{\Delta^{k-1}}(t) > 0$ for $t \in [a, \sigma^{n-k+1}(b)]$.

Now, assume that $1 \leq j = k-1$. Then

$$\begin{aligned} (M[u])^{\Delta^{k-1}}(t) &\geq G_p^{\Delta^{k-1}}(t,b) \sum_{i=0}^{k-1} p_i(b) u^{\Delta^i}(b) \mu(b) \\ &\geq G_p^{\Delta^{k-1}}(t,b) \left[p_{j-1}(b) u^{\Delta^{j-1}}(b) + p_j(b) u^{\Delta^j}(b)\right] \mu(b) \\ &> 0, \qquad t \in \left[a, \sigma^{n-k+1}(b)\right], \end{aligned}$$

as $p_{j-1}(b) > 0$ and $p_j(b) > 0$ by (18) and at least one of $u^{\Delta^{j-1}}(b)$, $u^{\Delta^j}(b)$ is positive by the preceding remark.

Finally, assume $0 = j = k-1$. Then

$$(M[u])^{\Delta^{k-1}}(t) = M[u](t) = \int_a^{\sigma(b)} G_p(t,s)p_0(s)u(s)\,\Delta s,$$

because $u(a) = 0$. But $u \in \mathcal{P}_{k-1}$, $u \neq 0$ implies that $u(t) \geq 0$ on $[a,b]$, with strict inequality holding for some point in $[\sigma(a), b]$, so $(M[u])^{\Delta^{k-1}}(t) > 0$ for $t \in [a, \sigma^{n-k+1}(b)]$. Then using arguments as in Lemma 4, it can be shown that there exists $k_1, k_2 > 0$ such that $k_1 u_0 \leq M[u] \leq k_2 u_0$ with respect to the cone $\mathcal{P}_{k-1}$, completing the proof. ∎

We now wish to show that M is u_0-positive with respect to the cone $\mathcal{P}_k$. However, we will need to take the delta derivative of $G_p^{\Delta^k}(t,s)$ with respect to s instead of with respect to t as previously done. For notation, we say $\cdot^{\Delta_s}$ is the delta derivative with respect to s.

PROPOSITION 3. *For* $1 \leq i \leq n-k-1$,

$$y_i^{\Delta_s}(t,s) = -\frac{\mu(\sigma(s))}{\mu(s)} y_{i-1}(t,\sigma(s)),$$

where $y_{n-1}(t,s)$ *is the Cauchy function for our focal boundary value problem.*

PROOF. First let $i = 1$. Then using the fact that all the points in $\mathbb{T}$ are isolated, we have

$$\begin{aligned}
y_1^{\Delta_s}(t,s) &= \frac{y_1(t,\sigma(s)) - y_1(t,s)}{\mu(s)} \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^t y_0(\tau,\sigma(s))\,\Delta\tau - \int_{\sigma(s)}^t y_0(\tau,s)\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^t 1\,\Delta\tau - \int_{\sigma(s)}^t 1\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\int_{\sigma^2(s)}^{\sigma(s)} 1\,\Delta\tau = -\frac{\mu(\sigma(s))}{\mu(s)} = -\frac{\mu(\sigma(s))}{\mu(s)} y_0(t,\sigma(s)).
\end{aligned}$$

For the induction step, assume that

$$y_i^{\Delta_s}(t,s) = -\frac{\mu(\sigma(s))}{\mu(s)} y_{i-1}(t,\sigma(s)),$$

for some $1 \leq i < n-k-1$. Since $y_i(\sigma(s), s) = 0$ for $1 \leq i < n-k-1$, we have

$$\begin{aligned}
y_{i+1}^{\Delta_s}(t,s) &= \frac{y_{i+1}(t,\sigma(s)) - y_{i+1}(t,s)}{\mu(s)} \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^t y_i(\tau,\sigma(s))\,\Delta\tau - \int_{\sigma(s)}^t y_i(\tau,s)\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^t y_i(\tau,\sigma(s))\,\Delta\tau - \int_{\sigma^2(s)}^t y_i(\tau,s)\,\Delta\tau - \int_{\sigma(s)}^{\sigma^2(s)} y_i(\tau,s)\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^t (y_i(\tau,\sigma(s)) - y_i(\tau,s))\,\Delta\tau - \mu(\sigma(s))y_i(\sigma(s),s)\right] \\
&= \int_{\sigma^2(s)}^t \frac{y_i(\tau,\sigma(s)) - y_i(\tau,s)}{\mu(s)}\,\Delta\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\sigma^2(s)}^{t} y_i^{\Delta_s}(\tau, s)\,\Delta\tau \\
&= \int_{\sigma^2(s)}^{t} -\frac{\mu(\sigma(s))}{\mu(s)}\, y_{i-1}(\tau, \sigma(s))\,\Delta\tau \\
&= -\frac{\mu(\sigma(s))}{\mu(s)}\, y_i(t, \sigma(s)).
\end{aligned}
$$

∎

We are now ready to find $G_p^{\Delta^k}(t,s)$. Recall that for $a \le t \le s \le b$,

$$
G_p^{\Delta^k}(t,s) = \frac{1}{p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b), s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}.
$$

By Lemma 3, $G_p^{\Delta^k}(t,s) > 0$ for $a \le t \le s \le b$. In addition, from the proof of Lemma 3, we have $G_p^{\Delta^k}(t,s) = 0$ for $\sigma(s) \le t$. We can now use the previous proposition to take the delta derivative of $G_p^{\Delta^k}(t,s)$ with respect to s instead of with respect to t.

$$
\begin{aligned}
\left(G_p^{\Delta^k}\right)^{\Delta_s}(t,s) &= \frac{1}{p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b), s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}^{\Delta_s} \\
&= \frac{1}{p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}^{\Delta_s}(\sigma(b), s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ \left((-1)^{n-k}\right)^{\Delta_s} & 0 & \cdots & 1 \end{vmatrix} \\
&= \frac{1}{p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ -\frac{\mu(\sigma(s))}{\mu(s)}\, y_{n-k-2}(\sigma(b), \sigma(s)) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} \\
&= -\frac{\mu(\sigma(s))}{\mu(s)p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-2}(t,a) \\ y_{n-k-2}(\sigma(b), \sigma(s)) & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}.
\end{aligned}
$$

PROPOSITION 4. *For $1 \le i \le n-k-1$,*

$$
y_i^{\Delta_s}(t,s) = -\frac{\mu(\sigma(s))}{\mu(s)}\, y_{i-1}(t, \sigma(s)),
$$

where $y_{n-1}(t,s)$ is the Cauchy function for our focal boundary value problem.

PROOF. First, let $i = 1$. Then, using the fact that all the points in $\mathbb{T}$ are isolated, we have

$$
\begin{aligned}
y_1^{\Delta_s}(t,s) &= \frac{y_1(t,\sigma(s)) - y_1(t,s)}{\mu(s)} \\
&= \frac{1}{\mu(s)} \left[\int_{\sigma^2(s)}^{t} y_0(\tau, \sigma(s))\,\Delta\tau - \int_{\sigma(s)}^{t} y_0(\tau, s)\,\Delta\tau \right] \\
&= \frac{1}{\mu(s)} \left[\int_{\sigma^2(s)}^{t} 1\,\Delta\tau - \int_{\sigma(s)}^{t} 1\,\Delta\tau \right] \\
&= \frac{1}{\mu(s)} \int_{\sigma^2(s)}^{\sigma(s)} 1\,\Delta\tau = -\frac{\mu(\sigma(s))}{\mu(s)} = -\frac{\mu(\sigma(s))}{\mu(s)}\, y_0(t, \sigma(s)).
\end{aligned}
$$

For the induction step, assume that

$$y_i^{\Delta_s}(t,s) = -\frac{\mu(\sigma(s))}{\mu(s)}\, y_{i-1}(t,\sigma(s)),$$

for some $1 \le i < n-k-1$. Since $y_i(\sigma(s),s) = 0$ for $1 \le i < n-k-1$, we have

$$\begin{aligned}
y_{i+1}^{\Delta_s}(t,s) &= \frac{y_{i+1}(t,\sigma(s)) - y_{i+1}(t,s)}{\mu(s)} \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^{t} y_i(\tau,\sigma(s))\,\Delta\tau - \int_{\sigma(s)}^{t} y_i(\tau,s)\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^{t} y_i(\tau,\sigma(s))\,\Delta\tau - \int_{\sigma^2(s)}^{t} y_i(\tau,s)\,\Delta\tau - \int_{\sigma(s)}^{\sigma^2(s)} y_i(\tau,s)\,\Delta\tau\right] \\
&= \frac{1}{\mu(s)}\left[\int_{\sigma^2(s)}^{t} (y_i(\tau,\sigma(s)) - y_i(\tau,s))\,\Delta\tau - \mu(\sigma(s))y_i(\sigma(s),s)\right] \\
&= \int_{\sigma^2(s)}^{t} \frac{y_i(\tau,\sigma(s)) - y_i(\tau,s)}{\mu(s)}\,\Delta\tau \\
&= \int_{\sigma^2(s)}^{t} y_i^{\Delta_s}(\tau,s)\,\Delta\tau \\
&= \int_{\sigma^2(s)}^{t} -\frac{\mu(\sigma(s))}{\mu(s)}\, y_{i-1}(\tau,\sigma(s))\,\Delta\tau \\
&= -\frac{\mu(\sigma(s))}{\mu(s)}\, y_i(t,\sigma(s)). \qquad \blacksquare
\end{aligned}$$

We are now ready to find $G_p^{\Delta^k}(t,s)$. Recall that for $a \le t \le s \le b$,

$$G_p^{\Delta^k}(t,s) = \frac{1}{p(t)}\begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}.$$

By Lemma 3, $G_p^{\Delta^k}(t,s) > 0$ for $a \le t \le s \le b$. In addition, from the proof of Lemma 3, we have $G_p^{\Delta^k}(t,s) = 0$ for $\sigma(s) \le t$. We can now use the previous proposition to take the delta derivative of $G_p^{\Delta^k}(t,s)$ with respect to s instead of with respect to t.

$$\begin{aligned}
\left(G_p^{\Delta^k}\right)^{\Delta_s}(t,s) &= \frac{1}{p(t)}\begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}(\sigma(b),s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}^{\Delta_s} \\
&= \frac{1}{p(t)}\begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ y_{n-k-1}^{\Delta_s}(\sigma(b),s) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ ((-1)^{n-k})^{\Delta_s} & 0 & \cdots & 1 \end{vmatrix} \\
&= \frac{1}{p(t)}\begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-1}(t,a) \\ -\dfrac{\mu(\sigma(s))}{\mu(s)}\, y_{n-k-2}(\sigma(b),\sigma(s)) & 1 & \cdots & h_{n-k-1}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}
\end{aligned}$$

$$= -\frac{\mu(\sigma(s))}{\mu(s)p(t)} \begin{vmatrix} 0 & h_0(t,a) & \cdots & h_{n-k-2}(t,a) \\ y_{n-k-2}(\sigma(b),\sigma(s)) & 1 & \cdots & h_{n-k-2}(\sigma(b),a) \\ \vdots & \vdots & & \vdots \\ (-1)^{n-k} & 0 & \cdots & 1 \end{vmatrix}.$$

REMARK 4. $G_p^{\Delta^k}(t,s) > 0$ for $a \le t \le s \le b$, and $G_p^{\Delta^k}(t,s) = 0$ for $\sigma(s) \le t$. In addition, from the previous discussion and Lemma 3, we have

$$\frac{\mu(s)}{\mu(\sigma(s))} \left(pG_p^{\Delta^k}\right)^{\Delta_s}(t,s) = -\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s)) > 0,$$

for $a \le t \le s \le b$.

LEMMA 9. *Assume that* $\int_t^{\sigma(b)} p_i(s)\,\Delta s \ge 0$ *for* $t \in [a,b]$, $0 \le i \le k-1$. *Then* M *is* u_0*-positive with respect to* $\mathcal{P}_k$.

PROOF. Let $u \in \mathcal{P}_k$, $u \ne 0$. Then, using the integration by parts formula in Theorem 3, the boundary conditions at a, and Remark 4, we get

$$\begin{aligned}
(M[u])^{\Delta^k}(t) &= \int_a^{\sigma(b)} G_p^{\Delta^k}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s \\
&= \int_a^{t} G_p^{\Delta^k}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s + \int_t^{\sigma(b)} G_p^{\Delta^k}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s \\
&= \int_t^{\sigma(b)} G_p^{\Delta^k}(t,s) \sum_{i=0}^{k-1} p_i(s) u^{\Delta^i}(s)\,\Delta s \\
&= \sum_{i=0}^{k-1} \int_t^{\sigma(b)} G_p^{\Delta^k}(t,s) p_i(s) u^{\Delta^i}(s)\,\Delta s \\
&= \sum_{i=0}^{k-1} -G_p^{\Delta^k}(t,s) u^{\Delta^i}(s) \int_s^{\sigma(b)} p_i(\tau)\,\Delta\tau \Bigg|_{s=t}^{\sigma(b)} \\
&\quad + \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \left(G_p^{\Delta^k}(t,s) u^{\Delta^i}(s)\right)^{\Delta_s} \int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\
&= \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \left(G_p^{\Delta^k}(t,s) u^{\Delta^i}(s)\right)^{\Delta_s} \int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\
&\quad + \sum_{i=0}^{k-1} G_p^{\Delta^k}(t,t) u^{\Delta^i}(t) \int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau.
\end{aligned}$$

First, suppose $t = b$. Then

$$\begin{aligned}
(M[u])^{\Delta^k}(b) &= \sum_{i=0}^{k-1} G_p^{\Delta^k}(b,b) u^{\Delta^i}(b) \int_b^{\sigma(b)} p_i(\tau)\,\Delta\tau \\
&= \sum_{i=0}^{j-1} G_p^{\Delta^k}(b,b) u^{\Delta^i}(b) \int_b^{\sigma(b)} p_i(\tau)\,\Delta\tau + G_p^{\Delta^k}(b,b) u^{\Delta^j}(b) \int_b^{\sigma(b)} p_j(\tau)\,\Delta\tau \\
&\quad + \sum_{i=j+1}^{k-1} G_p^{\Delta^k}(b,b) u^{\Delta^i}(b) \int_b^{\sigma(b)} p_i(\tau)\,\Delta\tau,
\end{aligned}$$

where the first term on the right-hand side is nonnegative, and the last term is zero by the triangular condition on the ps. Hence,

$$(M[u])^{\Delta^k}(b) \ge G_p^{\Delta^k}(b,b)u^{\Delta^j}(b)\int_b^{\sigma(b)} p_j(\tau)\,\Delta\tau > 0$$

using Remark 4 and Lemma 7.

Now, suppose $a \le t < b$. From Theorem 2 and Remark 4, we have

$$\begin{aligned}
\left(G_p^{\Delta^k}(t,s)u^{\Delta^i}(s)\right)^{\Delta_s} &= G_p^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s) + u^{\Delta^i}(s)\left(G_p^{\Delta^k}\right)^{\Delta_s}(t,s) \\
&= G_p^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s) \\
&\quad - u^{\Delta^i}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left[-\frac{\mu(s)}{\mu(\sigma(s))}\left(pG_p^{\Delta^k}\right)^{\Delta_s}(t,s)\right] \\
&= G_p^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s) - u^{\Delta^i}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s)).
\end{aligned}$$

Hence,

$$\begin{aligned}
(M[u])^{\Delta^k}(t) &= \sum_{i=0}^{k-1}\int_t^{\sigma(b)}\left(G_p^{\Delta^k}(t,s)u^{\Delta^i}(s)\right)^{\Delta_s}\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\
&\quad + \sum_{i=0}^{k-1} G_p^{\Delta^k}(t,t)u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau \\
&= \sum_{i=0}^{k-1}\int_t^{\sigma(b)}\left[G_p^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s)\right. \\
&\quad \left. - u^{\Delta^i}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\right]\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\
&\quad + \sum_{i=0}^{k-1} G_p^{\Delta^k}(t,t)u^{\Delta^i}(t)\int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau \\
&\ge \sum_{i=0}^{k-1}\int_t^{\sigma(b)} -u^{\Delta^i}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\
&\ge \sum_{i=0}^{k-1}\int_t^{\sigma(b)} -u^{\Delta^i}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\mu(b)p_i(b)\,\Delta s \\
&\ge \int_t^{\sigma(b)} -u^{\Delta^j}(s)\,\frac{\mu(\sigma(s))}{\mu(s)p(t)}\left(pG_p^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\mu(b)p_j(b)\,\Delta s \\
&> 0,
\end{aligned}$$

using Lemma 7 and Remark 4. We can then proceed as in Lemma 4 to get the desired result. ∎

We require two new operators for comparisons. Define R and N on $\mathcal{B}$ by

$$R[u](t) = \int_a^{\sigma(b)} G_P(t,s)\sum_{i=0}^{k-1} p_i(s)u^{\Delta^i}(s)\,\Delta s, \qquad t \in \left[a,\sigma^j(b)\right],$$

and

$$N[u](t) = \int_a^{\sigma(b)} G_P(t,s)\sum_{i=0}^{k-1} P_i(s)u^{\Delta^i}(s)\,\Delta s, \qquad t \in \left[a,\sigma^j(b)\right],$$

where the coefficients $P_i(t)$ and $p_i(t)$ satisfy the triangular conditions (18).

LEMMA 10. *If $p_i(t) \geq 0$ for $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$, and $0 \leq P(t) \leq p(t)$ for $t \in [a, b]$. Then $M \leq R$ with respect to the cone $\mathcal{P}_{k-1}$.*

PROOF. Using the definition of $M[u](t)$ and previous discussions about the form of $G_p(t,s)$, we have

$$\begin{aligned}(M[u])^{\Delta^k}(t) &= \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \left(G_p^{\Delta^k}(t,s)u^{\Delta^i}(s)\right)^{\Delta_s} \int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &\quad + \sum_{i=0}^{k-1} G_p^{\Delta^k}(t,t)u^{\Delta^i}(t) \int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau \\ &\leq \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \left(G_P^{\Delta^k}(t,s)u^{\Delta^i}(s)\right)^{\Delta_s} \int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &\quad + \sum_{i=0}^{k-1} G_P^{\Delta^k}(t,t)u^{\Delta^i}(t) \int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau \\ &= (R[u])^{\Delta^k}(t).\end{aligned}$$

The remainder of the proof is similar to the end of the proof of Lemma 6. ∎

LEMMA 11. *If*

$$0 \leq \int_t^{\sigma(b)} p_i(s)\,\Delta s \leq \int_t^{\sigma(b)} P_i(s)\,\Delta s,$$

for $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$, then $R \leq N$ with respect to the cone $\mathcal{P}_k$.

PROOF. Let $u \in \mathcal{P}_k$, $u \neq 0$. As in the proof of Lemma 9, we have

$$\begin{aligned}(R[u])^{\Delta^k}(t) &= \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \Big[G_P^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s) \\ &\quad - u^{\Delta^i}(s)\frac{\mu(\sigma(s))}{\mu(s)P(t)}\left(PG_P^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\Big] \int_{\sigma(s)}^{\sigma(b)} p_i(\tau)\,\Delta\tau\,\Delta s \\ &\quad + \sum_{i=0}^{k-1} G_P^{\Delta^k}(t,t)u^{\Delta^i}(t) \int_t^{\sigma(b)} p_i(\tau)\,\Delta\tau,\end{aligned}$$

and

$$\begin{aligned}(N[u])^{\Delta^k}(t) &= \sum_{i=0}^{k-1} \int_t^{\sigma(b)} \Big[G_P^{\Delta^k}(t,\sigma(s))u^{\Delta^{i+1}}(s) \\ &\quad - u^{\Delta^i}(s)\frac{\mu(\sigma(s))}{\mu(s)P(t)}\left(PG_P^{\Delta^k}\right)^{\Delta}(t,\sigma(s))\Big] \int_{\sigma(s)}^{\sigma(b)} P_i(\tau)\,\Delta\tau\,\Delta s \\ &\quad + \sum_{i=0}^{k-1} G_P^{\Delta^k}(t,t)u^{\Delta^i}(t) \int_t^{\sigma(b)} P_i(\tau)\,\Delta\tau.\end{aligned}$$

Since $u \in \mathcal{P}_k$, it follows that $(R[u])^{\Delta^k}(t) \leq (N[u])^{\Delta^k}(t)$ for $t \in [a, b]$. The remainder of the proof is as in the end of Lemma 4. ∎

THEOREM 17. *Assume that the following hold:*

1. *$p_i(t) \geq 0$, $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$, and if $1 \leq j = k-1$, assume $p_{j-1}(b) > 0$, whereas if $j = k-1 = 0$, assume $p_0(t) > 0$ on $[a, b]$;*
2. *$0 \leq \int_t^{\sigma(b)} p_i(s)\,\Delta s \leq \int_t^{\sigma(b)} P_i(s)\,\Delta s$ for $t \in [\sigma^{k-i}(a), b]$, $0 \leq i \leq k-1$;*
3. *$0 \leq P(t) \leq p(t)$ for $t \in [a, b]$.*

Then there exists a smallest positive eigenvalue λ_0 *(which is less than the modulus of any other eigenvalue) of (17), (8), (9), and a corresponding essentially unique eigenfunction* x_0*, and either* x_0 *or* $-x_0$ *satisfies*

$$x^{\Delta^i}(t) > 0, \qquad t \in \left[\sigma^{k-i}(a), \sigma^{n-i}(b)\right], \quad 0 \le i \le k. \tag{20}$$

Similarly, there exists a smallest positive eigenvalue Λ_0 *of (19), (8), (9), and a corresponding essentially unique eigenfunction* y_0*, and either* y_0 *or* $-y_0$ *satisfies (20). Furthermore,* $\Lambda_0 \le \lambda_0$ *and* $\Lambda_0 = \lambda_0$ *if and only if*

$$\begin{aligned} p_i(t) &= P_i(t), \qquad t \in \left[\sigma^{k-i}(a), b\right], \quad 0 \le i \le k-1, \\ p(t) &= P(t), \qquad t \in [a,b]. \end{aligned}$$

PROOF. By Lemma 9, M and N are u_0-positive with respect to $\mathcal{P}_k$. Hence, by Theorem 6 and Remark 3, the eigenvalue problems (17), (8), (9) and (19), (8), (9) have least positive eigenvalues λ_0 and Λ_0, respectively, and the corresponding eigenfunctions can be shown to satisfy (20).

By Lemma 8, M and R are u_0-positive with respect to $\mathcal{P}_{k-1}$. Hence, the eigenvalue problem

$$(-1)^{n-k} \left(P z^{\Delta^k}\right)^{\Delta^{n-k}} (t) = \delta \sum_{i=0}^{k-1} p_i(t) z^{\Delta^i}(t)$$

with boundary conditions (8),(9) has a least positive eigenvalue δ_0. By Lemma 10, $M \le R$ with respect to the cone $\mathcal{P}_{k-1}$, and so we get that $\delta_0 \le \lambda_0$. By Lemma 11, $N \le R$ with respect to the cone $\mathcal{P}_k$, and so $\Lambda_0 \le \delta_0$.

Now, assume that $\lambda_0 = \Lambda_0$. Work as in the proof of Theorem 13 shows that $p_i(t) = P_i(t)$ for $t \in [\sigma^{k-i}(a), b]$, $0 \le i \le k-1$. Now, use the fact that $\delta_0 = \lambda_0$. By Theorem 7, eigenfunctions x_0 and z_0 of M and R in $\mathcal{P}_{k-1}$ corresponding to $1/\lambda_0$ and $1/\delta_0$, respectively, are linearly dependent. It follows that

$$\left(P x_0^{\Delta^k}\right)^{\Delta^{n-k}} (t) = \left(p x_0^{\Delta^k}\right)^{\Delta^{n-k}} (t), \qquad t \in [a,b].$$

Using the boundary conditions (9), we get that

$$P(t) x_0^{\Delta^k}(t) = p(t) x_0^{\Delta^k}(t), \qquad t \in [a,b].$$

Hence, by the boundary conditions (8), $p(t) = P(t)$ for $t \in [a,b]$. ∎

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