PERGAMON Mathematical and Computer Modelling 0 (2003) 1–0
www.elsevier.com/locate/mcm

A Stacked Delta-Nabla Self-Adjoint Problem of Even Order

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(Received December 2002; revised and accepted May 2003)

Abstract—Existence criteria for two positive solutions to a nonlinear, even-order stacked delta-nabla boundary value problem with stacked, vanishing conditions at the two endpoints are found using the method of Green's functions. A few examples are given for standard time scales. The corresponding even-order nabla-delta problem is also discussed in detail.

Keywords—Time scales, Boundary value problem, Fixed points, Cone, Green's function.

1. INTRODUCTION

In this paper, we determine the Green's function for a self-adjoint, even-order boundary-value problem, namely,

$$Bx = h, \qquad \text{where } Bx := (-1)^n \left(x^{\Delta^n}\right)^{\nabla^n},$$

with the boundary conditions

$$\begin{aligned} x^{\Delta^i}(a) &= 0, \qquad 0 \le i \le n-1, \\ \left(x^{\Delta^n}\right)^{\nabla^i}(b) &= 0, \qquad 0 \le i \le n-1, \end{aligned} \tag{1}$$

on a time scale $\mathbb{T}$, where $a \in \mathbb{T}$ and $b \in \mathbb{T}^{\kappa^n}$ with $\sigma^n(a) < \rho^n(b)$, $x : [a, \sigma^n(b)] \to \mathbb{R}$, and $h : [a, b] \to \mathbb{R}$ is a given ld-continuous function. Very little has yet been done on time-scale problems of higher order using the method of Green's functions. Recently, Anderson [1] has considered an n-point right focal problem with delta derivatives, and Hoffacker has looked at delta-based problems such as the focal problem with stacked boundary conditions at the two endpoints [2], and for a $(k, n-k)$ problem [3]. Chyan, Henderson and Pan [4] have worked with

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even-order, delta-derivative, Sturm-Liouville conditions. Even less attention has been focused, however, on mixing delta and nabla derivatives in higher-order derivative operators. Atici and Guseinov [5] broke ground on the problem for order two, but it has been unclear as to how to place the successive derivatives beyond second order. Our motivation for this work is to begin an extension of the self-adjoint problem first suggested in [5, p. 76], by stacking the delta derivatives first, followed by nabla derivatives; as is evident in (1), the boundary conditions are also stacked. Because the delta and nabla derivatives do not commute for the general time scale, one might also consider some alternating scheme in the differential operator. Anderson and Hoffacker [6] have considered one possibility, alternating nabla-delta, nabla-delta or delta-nabla, delta-nabla for an even-order operator; in that case, the boundary conditions at the two endpoints alternated as well, resulting in a different, though somewhat related, problem. As one might expect, the technique of finding the corresponding Green's function for the related homogeneous case has proven to be an effective method in most of these problems. Throughout this work, we assume a working knowledge of time scales and time-scale notation. In the next section, however, we summarize the main points in a quick review.

2. TIME-SCALE ESSENTIALS

Any arbitrary nonempty closed subset of the reals $\mathbb{R}$ can serve as a time scale $\mathbb{T}$, see [7,8].

Definition 1. *For $t \in \mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

It is convenient to have the graininess operators $\mu_\sigma, \mu_\rho : \mathbb{T} \to [0, \infty)$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$.

Definition 2. *A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is denoted by*

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous (ld-continuous) provided it is continuous at all left-dense points of $\mathbb{T}$, and its right-sided limit exists (finite) at right-dense points of $\mathbb{T}$. The set of all left-dense continuous functions is denoted

$$C_{\mathrm{ld}} = C_{\mathrm{ld}}(\mathbb{T}) = C_{\mathrm{ld}}(\mathbb{T}, \mathbb{R}).$$

Define the sets $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa$ by

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} - \{m_2\}, & \text{if } \mathbb{T} \text{ has a right-scattered minimum } m_2, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m_1\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } m_1 \\ \mathbb{T}, & \text{otherwise,} \end{cases}$$

In addition, use the notation $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$, etc.

Definition 3. Delta Derivative. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that*

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \epsilon|\sigma(t) - s|, \qquad \text{for all } s \in U.$$

The function $f^\Delta(t)$ is the delta derivative of f at t.

Definition 4. Delta Integral. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$, then F is a delta antiderivative of f. In this case, the integral is given by the formula*

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a), \qquad \text{for } a, b \in \mathbb{T}.$$

Definition 5. Nabla Derivative. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that*

$$\big|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]\big| \le \epsilon|\rho(t) - s|, \qquad \text{for all } s \in U.$$

The function $f^\nabla(t)$ is the nabla derivative of f at t.

In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

Definition 6. Nabla Integral. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}$, then F is a nabla antiderivative of f. In this case, the integral is given by the formula*

$$\int_a^b f(\tau)\nabla\tau = F(b) - F(a), \qquad \text{for } a, b \in \mathbb{T}.$$

Remark 7. All right-dense continuous functions are delta integrable, and all left-dense continuous functions are nabla integrable.

3. GREEN'S FUNCTION

We now initiate the process of constructing and analyzing the Green's function for $Bx = 0$ with boundary conditions (1). The following two standard lemmas are easily verified.

Lemma 8. *The homogeneous boundary value problem $Bx = 0$, (1), has only the trivial solution.*

Lemma 9. *The nonhomogeneous boundary value problem*

$$\begin{aligned} Bx &= h, \\ x^{\Delta^i}(a) &= \alpha_i, \qquad 0 \le i \le n-1, \\ \left(x^{\Delta^n}\right)^{\nabla^i}(b) &= \beta_i, \qquad 0 \le i \le n-1, \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ for $0 \le i \le n-1$ and h is a given ld-continuous function, has a unique solution.

Define the Cauchy function for this boundary value problem as follows.

DEFINITION 10. *The function $y : [a, \sigma^n(b)] \times [a, b] \to \mathbb{R}$ is the Cauchy function for $Bx = 0$ provided for each fixed $s \in [a, b]$, $y(\cdot, s)$ is the solution to the initial value problem*

$$By(\cdot, s) = 0,$$

$$y^{\Delta^i}(s, s) = 0, \qquad 0 \le i \le n-1, \tag{2}$$

$$\left(y^{\Delta^n}\right)^{\nabla^i}(s, s) = 0, \qquad 0 \le i \le n-2, \tag{3}$$

$$\left(y^{\Delta^n}\right)^{\nabla^{n-1}}(s, s) = (-1)^n. \tag{4}$$

REMARK 11. In order to make calculations with the Cauchy and Green's functions simpler, define the functions $g_{j,k} : [a, \sigma^n(b)] \times [a, b] \to \mathbb{R}$ for $j, k \ge 0$ recursively by

$$\begin{aligned} g_{0,0}(t, s) &\equiv 1, \\ g_{j,0}(t, s) &= \int_s^t g_{j-1,0}(\tau, s)\nabla\tau, \\ g_{j,k}(t, s) &= \int_s^t g_{j,k-1}(\tau, s)\Delta\tau. \end{aligned}$$

Note that $g_{0,1}(t, s) = g_{1,0}(t, s) = t - s$ since the nabla and delta antiderivatives of a constant are equal. If $j, k < 0$, then $g_{j,k}(t, s)$ is taken to be identically zero. The construction of these functions is motivated by similarly defined functions for the delta case [2,3,7,9] and the nabla case [8,10].

EXAMPLE 12. For $\mathbb{T} = \mathbb{R}$, it is easy to see that

$$g_{j,k}(t, s) = \frac{(t-s)^{j+k}}{(j+k)!},$$

for $j, k \ge 0$. For $\mathbb{T} = \mathbb{Z}$, however, we find that

$$\begin{aligned} g_{0,0}(t, s) &= 1, \\ g_{j,0}(t, s) &= \frac{(t-s)^{\overline{j}}}{j!}, && j \ge 0, \\ g_{0,k}(t, s) &= \frac{(t-s)^{\underline{k}}}{k!}, && k \ge 0, \\ g_{j,k}(t, s) &= \frac{(t-s+j-1)^{\underline{j+k}}}{(j+k)!}, && j \ge 1, \quad k \ge 0, \end{aligned}$$

where $t^{\overline{j}} := t(t+1)(t+2)\cdots(t+j-1)$ and $t^{\underline{m}} := t(t-1)(t-2)\cdots(t-m+1)$.

EXAMPLE 13. For $\mathbb{T} = q^{\mathbb{N}_0}$, we find that

$$\begin{aligned} g_{0,0}(t, s) &= 1, \\ g_{j,0}(t, s) &= \prod_{i=0}^{j-1} \frac{tq^i - s}{\sum_{r=0}^{i} q^r}, && j \ge 0, \\ g_{0,k}(t, s) &= \prod_{i=0}^{k-1} \frac{t - q^i s}{\sum_{r=0}^{i} q^r}, && k \ge 0, \\ g_{j,k}(t, s) &= \frac{\left(\prod_{i=0}^{j-1} tq^i - s\right)\left(\prod_{i=1}^{k} t - q^i s\right)}{\prod_{i=0}^{j+k-1} \sum_{r=0}^{i} q^r}, && j, k \ge 1. \end{aligned}$$

Lemma 14. *The Cauchy function for $Bx = 0$ is $(-1)^n g_{n-1,n}(t,s)$.*

Proof. By definition, $g_{n-1,n}(\cdot, s)$ satisfies $Bg_{n-1,n}(\cdot, s) = 0$ for any fixed $s \in [a, b]$. Thus, it only remains to show that $g_{n-1,n}(\cdot, s)$ satisfies the initial values (2)–(4). For $0 \le i \le n-1$,

$$(-1)^n g_{n-1,n}^{\Delta^i}(t,s) = (-1)^n g_{n-1,n-i}(t,s),$$

so $(-1)^n g_{n-1,n}(t,s)$ satisfies the initial conditions (2). In addition, for $0 \le i \le n-2$,

$$\left((-1)^n g_{n-1,n}^{\Delta^n}\right)^{\nabla^i}(t,s) = (-1)^n g_{n-1-i,0}(t,s),$$

so the initial conditions (3) are satisfied. For condition (4), note that

$$\left((-1)^n g_{n-1,n}^{\Delta^n}\right)^{\nabla^{n-1}}(t,s) = (-1)^n g_{0,0}(t,s) = (-1)^n,$$

which completes the proof. ∎

Remark 15. In the proof of the next theorem, it is useful to note that, for $0 \le i \le n-2$,

$$g_{n-i,0}(\rho(t), t) = 0,$$

since

$$\begin{aligned} g_{n-i,0}(\rho(t), t) &= \int_t^{\rho(t)} g_{n-1-i,0}(\tau, t)\nabla\tau \\ &= \mu_\rho(t) g_{n-1-i,0}(t,t) = 0. \end{aligned}$$

Theorem 16. *For each fixed $s \in [a, b]$, let $u(t, s)$ be the unique solution of the boundary value problem*

$$Bu = 0,$$

$$u^{\Delta^i}(a,s) = 0, \qquad 0 \le i \le n-1, \tag{5}$$

$$\left(u^{\Delta^n}\right)^{\nabla^i}(b,s) = -(-1)^n g_{n-1-i,0}(b,s), \qquad 0 \le i \le n-1. \tag{6}$$

Then the Green's function $G : [a, \sigma^n(b)] \times [a, b] \to \mathbb{R}$ for $Bx = 0$, (1), is given by

$$G(t,s) = \begin{cases} u(t,s), & t \le s, \\ u(t,s) + (-1)^n g_{n-1,n}(t,s), & s \le t. \end{cases} \tag{7}$$

Proof. Since for each fixed $s \in [a, b]$, $u(\cdot, s)$ and $(-1)^n g_{n-1,n}(\cdot, s)$ are solutions of $Bx = 0$, we have that, for each fixed $s \in [a, b]$,

$$v(t,s) := u(t,s) + (-1)^n g_{n-1,n}(t,s) \tag{8}$$

is also a solution of $Bx = 0$. It follows from (6) that for each fixed $s \in [a, b]$, $v(\cdot, s)$ satisfies the boundary conditions at b for $0 \le i \le n-1$.

Let h be a given left-dense continuous function on $[a, b]$, and define

$$x(t) := \int_a^b G(t,s)h(s)\nabla s.$$

We wish to show that x is a solution of the nonhomogeneous equation $Bx = h$ satisfying the homogeneous boundary conditions (1). Consider

$$\begin{aligned}
x(t) &= \int_a^b G(t,s)h(s)\nabla s \\
&= \int_a^t G(t,s)h(s)\nabla s + \int_t^b G(t,s)h(s)\nabla s \\
&= \int_a^t (u(t,s) + (-1)^n g_{n-1,n}(t,s))h(s)\nabla s + \int_t^b u(t,s)h(s)\nabla s \\
&= (-1)^n \int_a^t g_{n-1,n}(t,s)h(s)\nabla s + \int_a^b u(t,s)h(s)\nabla s,
\end{aligned}$$

clearly, $x(a) = 0$. Moreover, using Theorem 8.50 (iii) [7],

$$\begin{aligned}
x^{\Delta}(t) &= (-1)^n \int_a^t g^{\Delta}_{n-1,n}(t,s)h(s)\nabla s + (-1)^n g_{n-1,n}(\sigma(t),\sigma(t))h(t) + \int_a^b u^{\Delta}(t,s)h(s)\nabla s \\
&= (-1)^n \int_a^t g_{n-1,n-1}(t,s)h(s)\nabla s + \int_a^b u^{\Delta}(t,s)h(s)\nabla s,
\end{aligned}$$

since $g_{n-1,n}(\sigma(t),\sigma(t)) = 0$. Similarly, for any $0 \le i \le n-1$,

$$x^{\Delta^i}(t) = (-1)^n \int_a^t g_{n-1,n-i}(t,s)h(s)\nabla s + \int_a^b u^{\Delta^i}(t,s)h(s)\nabla s.$$

It follows that, for $0 \le i \le n-1$, $x^{\Delta^i}(a) = 0$, hence, x satisfies the boundary conditions at a. Consider

$$x^{\Delta^n}(t) = (-1)^n \int_a^t g_{n-1,0}(t,s)h(s)\nabla s + \int_a^b u^{\Delta^n}(t,s)h(s)\nabla s.$$

Taking nabla derivatives of the nabla integral for $0 \le i \le n-1$, we have by Remark 15 that

$$\left(x^{\Delta^n}\right)^{\nabla^i}(t) = (-1)^n \int_a^t g_{n-1-i,0}(t,s)h(s)\nabla s + \int_a^b \left(u^{\Delta^n}\right)^{\nabla^i}(t,s)h(s)\nabla s.$$

Using boundary conditions (6), we have for $0 \le i \le n-1$ that

$$\left(x^{\Delta^n}\right)^{\nabla^i}(b) = \int_a^b \left((-1)^n g_{n-1-i,0}(b,s) + \left(u^{\Delta^n}\right)^{\nabla^i}(b,s)\right)h(s)\nabla s = 0.$$

Hence, x satisfies the boundary conditions at b. Now using the fact that

$$(-1)^n g_{0,0}(t,s) \equiv (-1)^n,$$

we have

$$(-1)^n \left(x^{\Delta^n}\right)^{\nabla^n}(t) = (-1)^n \left(\int_a^t (-1)^n h(s)\nabla s\right)^{\nabla} + \int_a^b (-1)^n \left(u^{\Delta^n}\right)^{\nabla^n}(t,s)h(s)\nabla s,$$

since $Bu(t,s) = 0$ for each fixed $s \in [a,b]$,

$$(-1)^n \left(x^{\Delta^n}\right)^{\nabla^n}(t) = h(t). \qquad \blacksquare$$

THEOREM 17. *If*

$$u(t,s) = (-1)^n \begin{vmatrix} 0 & g_{0,n}(t,a) & g_{1,n}(t,a) & \cdots & g_{n-2,n}(t,a) & g_{n-1,n}(t,a) \\ g_{n-1,0}(b,s) & 1 & g_{1,0}(b,a) & \cdots & g_{n-2,0}(b,a) & g_{n-1,0}(b,a) \\ g_{n-2,0}(b,s) & 0 & 1 & \cdots & g_{n-3,0}(b,a) & g_{n-2,0}(b,a) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{1,0}(b,s) & 0 & 0 & \cdots & 1 & g_{1,0}(b,a) \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}, \tag{9}$$

the Green's function for the boundary value problem $Bx = 0$, (1), is given by (7).

PROOF. By Theorem 16, it suffices to show that for each fixed $s \in [a,b]$, $u(\cdot,s)$ satisfies the boundary value problem $Bx = 0$ with boundary conditions at a, and $v(\cdot,s)$ in (8) satisfies the boundary conditions at b. Fix $s \in [a,b]$. By the properties of determinants and the construction of $g_{j,k}(t,s)$, $Bu(\cdot,s) = 0$ and $u^{\Delta^i}(a,s) = 0$ for $0 \le i \le n-1$, for u as in (9). As a result, boundary conditions at a are satisfied for each fixed $s \in [a,b]$ by $u(\cdot,s)$ in (9). It remains to show that $v(\cdot,s)$ satisfies the boundary conditions at b. Now $v(t,s)$ can be written in determinant form as

$$v(t,s) = (-1)^n \begin{vmatrix} g_{n-1,n}(t,s) & g_{0,n}(t,a) & g_{1,n}(t,a) & \cdots & g_{n-2,n}(t,a) & g_{n-1,n}(t,a) \\ g_{n-1,0}(b,s) & 1 & g_{1,0}(b,a) & \cdots & g_{n-2,0}(b,a) & g_{n-1,0}(b,a) \\ g_{n-2,0}(b,s) & 0 & 1 & \cdots & g_{n-3,0}(b,a) & g_{n-2,0}(b,a) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{1,0}(b,s) & 0 & 0 & \cdots & 1 & g_{1,0}(b,a) \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix};$$

it is easy to see that

$$v^{\Delta^n}(t,s) = (-1)^n \begin{vmatrix} g_{n-1,0}(t,s) & g_{0,0}(t,a) & g_{1,0}(t,a) & \cdots & g_{n-2,0}(t,a) & g_{n-1,0}(t,a) \\ g_{n-1,0}(b,s) & 1 & g_{1,0}(b,a) & \cdots & g_{n-2,0}(b,a) & g_{n-1,0}(b,a) \\ g_{n-2,0}(b,s) & 0 & 1 & \cdots & g_{n-3,0}(b,a) & g_{n-2,0}(b,a) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{1,0}(b,s) & 0 & 0 & \cdots & 1 & g_{1,0}(b,a) \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix},$$

and that

$$\left(v^{\Delta^n}\right)^{\nabla^i}(t,s) = (-1)^n \begin{vmatrix} g_{n-1-i,0}(t,s) & g_{0-i,0}(t,a) & g_{1-i,0}(t,a) & \cdots & g_{n-2-i,0}(t,a) & g_{n-1-i,0}(t,a) \\ g_{n-1,0}(b,s) & 1 & g_{1,0}(b,a) & \cdots & g_{n-2,0}(b,a) & g_{n-1,0}(b,a) \\ g_{n-2,0}(b,s) & 0 & 1 & \cdots & g_{n-3,0}(b,a) & g_{n-2,0}(b,a) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{1,0}(b,s) & 0 & 0 & \cdots & 1 & g_{1,0}(b,a) \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix},$$

for $1 \le i \le n-1$ (recalling that if the subscript on the function g is less than 0, the function is taken to be identically 0). For each such i, the first and $(i+2)^{\text{nd}}$ rows will be identical, and hence, $(v^{\Delta^n})^{\nabla^i}(b,s) = 0$ for $0 \le i \le n-1$. Therefore, the boundary conditions at b are likewise satisfied. ∎

EXAMPLE 18. There has been some question as to whether the stacking of n delta derivatives followed by n nabla derivatives, as in the self-adjoint dynamic equation, would lead to a symmetric

Green's function for the corresponding boundary value problem $Bx = 0$, (1). Unfortunately, we cannot expect the Green's function to be symmetric. Take $\mathbb{T} = \mathbb{Z}$. For $n = 2$,

$$\begin{aligned} u(t,s) &= \begin{vmatrix} 0 & g_{0,2}(t,a) & g_{1,2}(t,a) \\ g_{1,0}(b,s) & 1 & g_{1,0}(b,a) \\ 1 & 0 & 1 \end{vmatrix} \\ &= g_{0,2}(t,a)g_{1,0}(b,a) - g_{1,2}(t,a) - g_{0,2}(t,a)g_{1,0}(b,s) \\ &= \frac{1}{6}(t-a)(t-a-1)(3s-t-2a+2) \end{aligned}$$

and

$$\begin{aligned} v(t,s) &= u(t,s) + g_{1,2}(t,s) \\ &= \frac{1}{6}(s-a)(s-a+1)(3t-s-2a-2). \end{aligned}$$

Notice that $G(t,s) \neq G(s,t)$.

EXAMPLE 19. Similarly, one may find the Green's function for the case $\mathbb{T} = q^{\mathbb{N}_0}$ where $n = 2$. Here

$$\begin{aligned} u(t,s) &= g_{0,2}(t,a)g_{1,0}(b,a) - g_{1,2}(t,a) - g_{0,2}(t,a)g_{1,0}(b,s) \\ &= \frac{(t-a)(t-qa)(b-a)}{1+q} - \frac{(t-a)(t-qa)(t-q^2a)}{(1+q)(1+q+q^2)} - \frac{(t-a)(t-qa)(b-s)}{1+q} \\ &= \frac{(t-a)(t-qa)(s-a)}{1+q} - \frac{(t-a)(t-qa)(t-q^2a)}{(1+q)(1+q+q^2)} \end{aligned}$$

and

$$\begin{aligned} v(t,s) &= u(t,s) + g_{1,2}(t,s) \\ &= \frac{(t-a)(t-qa)(s-a)}{1+q} - \frac{(t-a)(t-qa)(t-q^2a)}{(1+q)(1+q+q^2)} + \frac{(t-s)(t-qs)(t-q^2s)}{(1+q)(1+q+q^2)} \end{aligned}$$

and again $G(t,s) \neq G(s,t)$.

LEMMA 20. *Let $G(t,s)$ be the Green's function for the boundary value problem $Bx = 0$, (1). Then, the following hold.*

(i) $(-1)^i (G^{\Delta^n})^{\nabla^i}(t,s) > 0$, $a \le t < s \le b$, $0 \le i \le n-1$.
(ii) $G^{\Delta^i}(t,s) > 0$, $t \in (\sigma^{n-i}(a), b]$, $s \in (a,b]$, $0 \le i \le n-1$.

PROOF. By the previous theorem, the Green's function for this boundary value problem is given by

$$G(t,s) = \begin{cases} u(t,s), & t \le s, \\ v(t,s), & s \le t, \end{cases}$$

where u and v are as given (9) and (8), respectively.

PART (i). For $t \le s$, $G(t,s) = u(t,s)$. Hence, Part (i) is equivalent to showing that

$$(-1)^i \left(u^{\Delta^n}\right)^{\nabla^i}(t,s) > 0, \qquad a \le t < s \le b, \quad 0 \le i \le n-1.$$

In order to determine the sign of $u(t,s)$ and its derivatives, we first consider $v(t,s)$. Fix $s \in [a,b]$. Since $v(\cdot,s)$ satisfies $Bv(\cdot,s) = 0$, $(v^{\Delta^n})^{\nabla^{n-1}}(t,s)$ is a constant. Considering the boundary condition at b, this gives that $(v^{\Delta^n})^{\nabla^{n-1}}(t,s) \equiv 0$, which in turn implies that $(v^{\Delta^n})^{\nabla^{n-2}}(t,s)$ is a constant. Similar work shows that $(v^{\Delta^n})^{\nabla^i}(t,s) \equiv 0$ for $0 \le i \le n-1$ for any fixed $s \in [a,b]$. Therefore, $v^{\Delta^n}(t,s) = 0$ for any fixed s. In particular, using (8), we have that

$$0 \equiv \left(v^{\Delta^n}\right)^{\nabla^i}(t,s) = \left(u^{\Delta^n}\right)^{\nabla^i}(t,s) + (-1)^n g_{n-1-i,0}(t,s).$$

Hence,

$$\begin{aligned}
(-1)^{n-1}\left(u^{\Delta^n}\right)^{\nabla^{n-1}}(t,s) &= g_{0,0}(t,s) \equiv 1 > 0,\\
(-1)^{n-2}\left(u^{\Delta^n}\right)^{\nabla^{n-2}}(t,s) &= -g_{1,0}(t,s) = -(t-s) > 0,\\
(-1)^{n-3}\left(u^{\Delta^n}\right)^{\nabla^{n-3}}(t,s) &= g_{2,0}(t,s) = \int_s^t (\tau - s)\nabla\tau > 0,
\end{aligned}$$

and so on, since $(-1)^j g_{j,0}(t,s) > 0$ for $t < s$. Continuing this process gives the proof of Part (i).

Part (ii). Using Part (i), we have that $u^{\Delta^n}(t,s) = (-1)^n g_{n,0}(t,s) > 0$ for $a \le t < s \le b$. Therefore, $u^{\Delta^{n-1}}(t,s)$ is strictly increasing for $t < s$; using boundary condition (1), $u^{\Delta^{n-1}}(t,s) > 0$ for $a < t < s \le b$. In the same way, we get that $u^{\Delta^i}(t,s) > 0$ for $t \in (\sigma^{n-i}(a), b]$, $s \in [a,b]$ with $s > t$, for $0 \le i \le n-1$. However, it remains to show that $v^{\Delta^i}(t,s) > 0$ where $s \le t$ instead. Recall from above that $v^{\Delta^n}(t,s) = 0$. Therefore, $v^{\Delta^{n-1}}(t,s) = k(s)$ for some function k of s. But

$$k(s) = v^{\Delta^{n-1}}(s,s) = u^{\Delta^{n-1}}(s,s) > 0,$$

for $s \in (a,b]$. It follows that $v^{\Delta^{n-2}}(t,s)$ is increasing in t. Again, we have

$$v^{\Delta^{n-2}}(s,s) = u^{\Delta^{n-2}}(s,s) > 0,$$

so that $v^{\Delta^{n-2}}(t,s) > 0$ for $t \ge s$. Consequently, for $0 \le i \le n-1$ and $t \in [s,b]$, $v^{\Delta^i}(s,s) = u^{\Delta^i}(s,s) > 0$ implies that $v^{\Delta^i}(t,s) > 0$ for $t \ge s$. Hence, $G^{\Delta^i}(t,s) > 0$, where $t \in (\sigma^{n-i}(a), b]$, $s \in (a,b]$, and $0 \le i \le n-1$. ∎

4. EXISTENCE OF AT LEAST TWO POSITIVE SOLUTIONS

Using the Green's function from the previous section, we apply the Avery-Henderson fixed-point theorem [11] to prove the existence of at least two positive solutions to the nonlinear boundary value problem $Bx = f(\cdot, x)$, (1), where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is ld-continuous, f nonnegative for $x \ge 0$. The solutions are the fixed points of the operator $\mathcal{A}$ defined by

$$\mathcal{A}x(t) = \int_a^b G(t,s) f(s, x(s))\nabla s,$$

where $G(t,s)$ is the Green's function as in Theorem 16 for the homogeneous problem $Bx = 0$, (1). Notationally, the cone $\mathcal{P}$ has subsets of the form $\mathcal{P}(\gamma, r) := \{x \in \mathcal{P} : \gamma(x) < r\}$ for a given functional γ.

Theorem 21. *(See [11].) Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$. Let α and γ be increasing, nonnegative continuous functionals on $\mathcal{P}$. Let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\gamma(x) \le \theta(x) \le \alpha(x), \qquad \text{and} \qquad ||x|| \le M\gamma(x)$$

for all $x \in \overline{\mathcal{P}(\gamma, r)}$. Suppose that there exist positive numbers p and q with $p < q < r$ such that

$$\theta(\lambda x) \le \lambda\theta(x), \qquad \text{for all } 0 \le \lambda \le 1 \quad \text{and} \quad x \in \partial\mathcal{P}(\theta, q).$$

Suppose $A : \overline{\mathcal{P}(\gamma, r)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) *$\gamma(Ax) > r$ for all $x \in \partial\mathcal{P}(\gamma, r)$,*
(ii) *$\theta(Ax) < q$ for all $x \in \partial\mathcal{P}(\theta, q)$,*
(iii) *$\mathcal{P}(\alpha, p) \ne \emptyset$ and $\alpha(Ax) > p$ for all $x \in \partial\mathcal{P}(\alpha, p)$.*

Then A has at least two fixed points x_1 and x_2 such that

$$p < \alpha(x_1), \quad \textit{with } \theta(x_1) < q \qquad \textit{and} \qquad q < \theta(x_2), \quad \textit{with } \gamma(x_2) < r.$$

Let $\mathcal{E}$ denote the Banach space $C_{\rm ld}[\rho^n(a), \sigma^n(b)]$ with the norm

$$\|x\| = \sup_{t\in[a,b]} |x(t)|.$$

By Lemma 20, $G(t,s) > 0$ for $t \in (\sigma^n(a), b]$, $s \in (a,b]$, and $G^{\Delta}(t,s) > 0$ for $t \in (\sigma^{n-1}(a), b]$, $s \in (a,b]$. Let $\eta \in (\sigma^n(a), b)$. Then

$$0 < G(\eta, s) \le G(t,s) \le G(b,s),$$

for all $s \in (a,b]$, $t \in [\eta, b]$. Set

$$\ell := \min_{s\in[a,b]} \frac{G(\eta,s)}{G(b,s)}. \tag{10}$$

(If $a \le s < \eta < b$, then $G(\eta,s)/G(b,s) = v(\eta,s)/v(b,s)$, and any $(s-a)$ factors would cancel, leaving ℓ well defined.) Clearly, $0 < \ell < 1$ and

$$\ell G(b,s) \le G(t,s) \le G(b,s),$$

for all $t \in [\eta, b]$, $s \in [a,b]$. Define the cone $\mathcal{P} \subset \mathcal{E}$ by

$$\mathcal{P} = \begin{cases} x & \text{is nonnegative on } [a,b], \\ x & \text{is nondecreasing on } [a,b], \\ x & \text{is positive on } (\sigma^n(a), b], \\ & x(t) \ge \ell\|x\|, \qquad t \in [\eta, b], \end{cases} \qquad x \in \mathcal{E}, \tag{11}$$

where ℓ is given in (10). To accomplish that which follows, we will need the constants

$$m^{-1} := \int_a^b G(b,s)\nabla s \tag{12}$$

and

$$k^{-1} := \ell \int_\eta^b G(b,s)\nabla s. \tag{13}$$

Finally, let the nonnegative, increasing, continuous functionals γ, θ, and α be defined on the cone $\mathcal{P}$ by

$$\gamma(x) := \min_{t\in[\eta,b]} x(t) = x(\eta),$$
$$\theta(x), \alpha(x) := \max_{t\in[\eta,b]} x(t) = x(b).$$

Observe that, for each $x \in \mathcal{P}$,

$$\gamma(x) \le \theta(x) \le \alpha(x) \tag{14}$$

and

$$\|x\| = x(b) \le \frac{1}{\ell} x(\eta) = \frac{1}{\ell}\gamma(x) \le \frac{1}{\ell}\theta(x) = \frac{1}{\ell}\alpha(x). \tag{15}$$

THEOREM 22. *Let ℓ, m, and k be as in (10), (12), and (13), respectively. Suppose there exist positive numbers p, q, and r such that*

$$0 < p < q < r,$$

and suppose an ld-continuous function f satisfies the following conditions:

(i) $f(s,w) \geq 0$ *for* $s \in [a,b]$ *and* $w \in [0, r/\ell]$,
(ii) $f(s,w) > pk$ *for* $s \in [\eta, b]$ *and* $w \in [p, p/\ell]$,
(iii) $f(s,w) < qm$ *for* $s \in [a,b]$ *and* $w \in [q, q/\ell]$,
(iv) $f(s,w) > rk$ *for* $s \in [\eta, b]$ *and* $w \in [r, r/\ell]$.

Then, the even-order boundary value problem $Bx = f(\cdot, x)$, (1), has at least two positive solutions x_1 and x_2 such that

$$p < \max_{t\in[\eta,b]} x_1(t) < q$$

and

$$\max_{t\in[\eta,b]} x_2(t) > q, \qquad \text{with } \min_{t\in[\eta,b]} x_2(t) < r.$$

PROOF. Note that the Green's function $G(\cdot, s)$ is increasing by Lemma 20 on $(\sigma^{n-1}(a), b]$ and positive on $(\sigma^n(a), b]$, for all $s \in (a,b]$. Using the boundary conditions at a, G is nonnegative and nondecreasing for all $(t,s) \in [a,b] \times [a,b]$. For $x \in \mathcal{P}$, $x(t) \geq \ell \|x\|$ for all $t \in [\eta, b]$. Let $t \in [\eta, b]$. Then

$$\begin{aligned} \mathcal{A}x(t) &= \int_a^b G(t,s) f(s, x(s)) \nabla s \\ &\geq \ell \int_a^b G(b,s) f(s, x(s)) \nabla s \\ &= \ell ||Ax||, \end{aligned}$$

so that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. For any $x \in \mathcal{P}$, (14) and (15) imply that

$$\gamma(x) \leq \theta(x) \leq \alpha(x)$$

and

$$\|x\| \leq \frac{1}{\ell}\gamma(x).$$

It is clear that $\theta(0) = 0$, and for all $x \in \mathcal{P}$, $\lambda \in [0,1]$, we have

$$\theta(\lambda x) = \max_{t\in[\eta,b]} (\lambda x)(t) = \lambda \max_{t\in[\eta,b]} x(t) = \lambda\theta(x).$$

Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\alpha, p) \neq \emptyset$. In the following claims, we verify the remaining conditions of Theorem 21.

CLAIM 1. If $x \in \partial\mathcal{P}(\alpha, p)$, then $\alpha(\mathcal{A}x) > p$. Since $x \in \partial\mathcal{P}(\alpha, p)$, $p = \|x\| \leq p/\ell$. Thus,

$$\begin{aligned} \alpha(\mathcal{A}x) &= \max_{t\in[\eta,b]} \int_a^b G(t,s) f(s, x(s)) \nabla s \\ &= \int_a^b G(b,s) f(s, x(s)) \nabla s \\ &\geq \int_\eta^b G(b,s) f(s, x(s)) \nabla s \\ &\geq \ell \int_\eta^b G(b,s) f(s, x(s)) \nabla s \\ &> pk\ell \int_\eta^b G(b,s) \nabla s \\ &= p, \end{aligned}$$

by Hypothesis (ii) and (13).

Claim 2. If $x \in \partial\mathcal{P}(\theta, q)$, then $\theta(\mathcal{A}x) < q$. Note that $x \in \partial\mathcal{P}(\theta, q)$ and (15) imply that $q = \|x\| \le q/\ell$.

$$\begin{aligned}\theta(\mathcal{A}x) &= \max_{t\in[\eta,b]} \int_a^b G(t,s)f(s,x(s))\nabla s \\ &= \int_a^b G(b,s)f(s,x(s))\nabla s \\ &< qm \int_a^b G(b,s)\nabla s \\ &= q,\end{aligned}$$

by Hypothesis (iii) and (12).

Claim 3. If $x \in \partial\mathcal{P}(\gamma, r)$, then $\gamma(\mathcal{A}x) > r$. Since $x \in \partial\mathcal{P}(\gamma, r)$, from (15) we have that $\min_{t\in[\eta,b]} x(t) = r$ and $r \le \|x\| \le \frac{r}{\ell}$.

$$\begin{aligned}\gamma(\mathcal{A}x) &= \min_{t\in[\eta,b]} \int_a^b G(t,s)f(s,x(s))\nabla s \\ &= \int_a^b G(\eta,s)f(s,x(s))\nabla s \\ &\ge \ell \int_\eta^b G(b,s)f(s,x(s))\nabla s \\ &\ge rk\ell \int_\eta^b G(b,s)\nabla s \\ &= r,\end{aligned}$$

by Hypothesis (iv), using arguments as in Claim 1. Therefore, the hypotheses of Theorem 21 are satisfied and there exist at least two positive fixed points x_1 and x_2 of $\mathcal{A}$ in $\overline{\mathcal{P}(\gamma, r)}$. Thus, the even-order boundary value problem $Bx = f(\cdot, x)$, (1), has at least two positive solutions x_1 and x_2 such that

$$p < \alpha(x_1), \qquad \text{with } \theta(x_1) < q,$$

and

$$q < \theta(x_2), \qquad \text{with } \gamma(x_2) < r.$$

∎

5. A STACKED NABLA-DELTA SELF-ADJOINT PROBLEM

One may wonder if stacking the nabla and delta derivatives in the opposite order affects the symmetry of the Green's function or the existence of positive solutions. In this section, we consider the self-adjoint even-order boundary value problem

$$Cx = f, \qquad \text{where } Cx := (-1)^n \left(x^{\nabla^n}\right)^{\Delta^n},$$

with the boundary conditions

$$\begin{aligned} x^{\nabla^i}(a)q &= 0, \qquad 0 \le i \le n-1, \\ \left(x^{\nabla^n}\right)^{\Delta^i}(b) &= 0, \qquad 0 \le i \le n-1, \end{aligned} \tag{16}$$

on a time scale $\mathbb{T}$, where $a \in \mathbb{T}_{\kappa^n}$ and $b \in \mathbb{T}$ with $\sigma^n(a) < \rho^n(b)$, $x : [\rho^n(a), b] \to \mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ is a given rd-continuous function. As in previous sections, we first develop the Green's function and then use it to show the existence of a positive solution. Since the proofs are very similar to those already given, they are omitted here.

Lemma 23. *The homogeneous boundary value problem $Cx = 0$, (16), has only the trivial solution.*

Lemma 24. *The nonhomogeneous boundary value problem*

$$\begin{aligned} Cx &= f, \\ x^{\nabla^i}(a) &= \alpha_i, \qquad 0 \le i \le n-1, \\ (x^{\nabla^n})^{\Delta^i}(b) &= \beta_i, \qquad 0 \le i \le n-1, \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ for $0 \le i \le n-1$ and f is a given rd-continuous function, has a unique solution.

As before, we define the Cauchy function for this boundary value problem as follows.

Definition 25. *The function $y : [\rho^n(a), b] \times [a, b] \to \mathbb{R}$ is the Cauchy function for $Cx = 0$ provided for each fixed $s \in [a, b]$, $y(\cdot, s)$ is the solution to the initial value problem*

$$\begin{aligned} Cy(\cdot, s) &= 0, \\ y^{\nabla^i}(s, s) &= 0, \qquad 0 \le i \le n-1, \\ \left(y^{\nabla^n}\right)^{\Delta^i}(s, s) &= 0, \qquad 0 \le i \le n-2, \\ \left(y^{\nabla^n}\right)^{\Delta^{n-1}}(s, s) &= (-1)^n. \end{aligned}$$

The auxiliary functions similar to $g_{j,k}$ are instead

$$h_{j,k} : [\rho^n(a), b] \times [a, b] \to \mathbb{R},$$

for $j, k \ge 0$, defined recursively by

$$\begin{aligned} h_{0,0}(t, s) &\equiv 1, \\ h_{j,0}(t, s) &= \int_s^t h_{j-1,0}(\tau, s)\Delta\tau, \\ h_{j,k}(t, s) &= \int_s^t h_{j,k-1}(\tau, s)\nabla\tau. \end{aligned}$$

Lemma 26. *The Cauchy function for $Cx = 0$ is $(-1)^n h_{n-1,n}(t, s)$.*

Theorem 27. *For each fixed $s \in [a, b]$, let $u(t, s)$ be the unique solution of the boundary value problem*

$$\begin{aligned} Cu &= 0, \\ u^{\nabla^i}(a, s) &= 0, \qquad 0 \le i \le n-1, \\ \left(u^{\nabla^n}\right)^{\Delta^i}(b, s) &= -(-1)^n g_{n-1-i,0}(b, s), \qquad 0 \le i \le n-1. \end{aligned}$$

Then the Green's function $G : [\rho^n(a), b] \times [a, b] \to \mathbb{R}$ for $Cx = 0$, (16), is given by

$$G(t, s) = \begin{cases} u(t, s), & t \le s, \\ u(t, s) + (-1)^n h_{n-1,n}(t, s), & s \le t. \end{cases} \tag{17}$$

Theorem 28. *If*

$$u(t, s) = (-1)^n \begin{vmatrix} 0 & h_{0,n}(t, a) & h_{1,n}(t, a) & \cdots & h_{n-2,n}(t, a) & h_{n-1,n}(t, a) \\ h_{n-1,0}(b, s) & 1 & h_{1,0}(b, a) & \cdots & h_{n-2,0}(b, a) & h_{n-1,0}(b, a) \\ h_{n-2,0}(b, s) & 0 & 1 & \cdots & h_{n-3,0}(b, a) & h_{n-2,0}(b, a) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{1,0}(b, s) & 0 & 0 & \cdots & 1 & h_{1,0}(b, a) \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix},$$

the Green's function for the boundary value problem $Cx = 0$, (16), is given by (17).

As in Example 18, one may show that the Green's function for the boundary value problem $Cx = 0$, (16) is not symmetric in general. In addition, the existence of two positive solutions of $Cx = f(., x)$, (16) can be shown in a manner similar to that in Theorem 22.

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