A robust multiobjective optimization problem with application to Internet routing

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Abstract

Robust optimization addressing decision making under uncertainty has been very well developed for problems with a single objective function and applied to areas of human activity such as portfolio selection, investment decisions, signal processing, and telecommunication-network planning. As these decision problems typically have several decisions or goals, we extend robust single objective optimization to the multiobjective case and examine the column-wise and row-wise uncertainty models in the presence of vector-valued objective functions. For each model, we show that efficient solutions of a robust multiobjective optimization problem can be found as the efficient solutions of a related deterministic problem. Being motivated by the fact that Internet traffic must be maintained in a reliable yet affordable manner, we apply the row-wise model to an intradomain multiobjective routing problem with polyhedral traffic uncertainty. We consider traditional objective functions corresponding to link utilizations and implement the biobjective case using the parametric simplex algorithm to compute Pareto routings. We also present computational results for the Abilene network and analyze their meaning in the context of the application.

1 Introduction

Robust optimization has developed as an alternative to stochastic programming to model decision making problems under conditions of uncertainty and to find decisions that remain optimal over all scenarios by relying on worst-case scenario bounds.

Robust single objective optimization has been well studied. Soyster (1973) studies linear programs with continuous uncertainty sets having specific characteristics. Kouvelis and Yu (1997) present the case of a discrete uncertainty set consisting of realizable scenarios and develop complexity results for many types of uncertain optimization problems. El-Ghaoui and Lebret (1997) examine least squares problems with bounded but uncertain coefficient matrices. Ben-Tal and Nemirovski (1998), Ben-Tal et al. (2009), and Bertsimas et al. (2011) present results for general and convex continuous uncertainty sets. Their approach makes use of a semi-infinite program, which has a finite number of variables but an infinite number of constraints, to model the uncertainty. Bertsimas and Sim (2004, 2006) introduce an approach whose level of conservatism can be adjusted by problem parameters, but provide only a probabilistic guarantee that the solution will remain feasible for every scenario.

Robust multiobjective optimization is a growing area of study. Initially, the concepts and methods of multiobjective optimization were applied to uncertain single objective problems (Iancu and Trichakis 2013, Klamroth et al. 2013, Köbis and Tammer 2012, Ogryczak 2012). In some of these studies, multiple scenarios are modeled with multiple objective functions. However, the interest in robust multiobjective optimization increases both theoretically and in applications, which is reflected in a variety of recent studies. The worst-case scenario approach is carried over from single-objective optimization by Ehrgott et al. (2014), Fliege and Werner (2014), and Kuroiwa and Lee (2012). They follow the classical multiobjective optimization scheme of scalarization and perform robust optimization on the scalarized problem. Another approach to modeling and resolving uncertainty is based on parametric optimization where parameters model the
unknown data. Dellnitz and Witting (2009) combine techniques of multiobjective optimization with path-following algorithms while Witting et al. (2013) consider calculus of variation to solve parameter-dependent multiobjective optimization problems. Kuhn et al. (2013) present a biobjective shortest path problem with one uncertain objective function and propose several concepts of robust solutions.

Of interest to this work is which components of the multiobjective optimization problem are considered to be uncertain. We consider solely the case where uncertainty occurs at the level of the feasibility constraints. However, in other studies the uncertainty is considered solely in the objective functions (Dellnitz and Witting 2009, Ehrgott et al. 2014, Kuhn et al. 2013, Witting et al. 2013) or in both objective and constraint functions (Fliege and Werner 2014, Goberna et al. 2015, Kuroiwa and Lee 2012). Palma and Nelson (2010) and Hu and Mehrrotra (2012) consider uncertain weights in the scalarized objective function.

Applications of robust multiobjective optimization include forestry management (Palma and Nelson 2010), airline scheduling (Kuhn et al. 2013), and portfolio management (Fliege and Werner 2014). Refer to Goberna et al. (2015) for a literature review on robust multiobjective optimization.

Robust Internet routing is a critical direction of research because the Internet today plays a crucial role in everyone’s life. Improving its global performance (e.g., faster, more reliable, higher quality) is one of the main challenges for network operators and principally depends on the management of the underlying routing protocols. Until the 21st century, most of the routing optimization problems in the Internet required some estimate of the traffic demands (e.g., worst-case or average traffic) expected to be carried throughout. Predicting patterns of Internet traffic is a difficult task due to the size and diversity of the Internet (Ben-Ameur 2007, Ben-Ameur et al. 2012). Traffic engineering (TE) involves assigning the number and type of circuits and switching equipment that is necessary to meet these expected traffic demands on a network (TIA 2012). Quality of service (QoS), being a qualitative measure of how well a routing is performing, is a main goal of network operators. Robust routing optimization emerges as a means to achieve trade-offs between the conflicting objectives of TE and QoS. Ben-Ameur et al. (2012) present an overview of robust routing with a focus on the most common uncertainty sets and routing strategies considered in communications networks. Casas (2010) shows that decreases in QoS can be due to overloaded links (utilizations larger than one) allowing the QoS routing problem to be considered a subproblem of the robust routing problem. Hijazi et al. (2013), in particular, focus on one facet of QoS, network response time, and present both theoretical results and numeric experiments regarding the delay constrained routing problem.

Ben-Ameur and Kerivin (2005) introduce the concept of stable robust routing within the context of Virtual Private Network provisioning, which appears to be an application of uncertain linear programs (Ben-Tal and Nemirovski 1999). They represent the traffic uncertainty as a polytope and compute a stable (i.e., fixed) routing scheme which is valid for all the traffic configurations within the polytope. Their model, called the polyhedral traffic model, allows the network operator to exploit, for instance, temporal and geographical correlations and link-traffic measurements.

Many approaches to solving uncertain single objective Internet routing optimization problems differ in the way the semi-infinite program, created by the approach of Ben-Tal and Nemirovski (1999), is managed. Ben-Tal and Nemirovski (1999), Ben-Ameur and Kerivin (2005), and Casas (2010) deal with the semi-infinite programs by reducing the uncertainty set to its convex hull. This technique eliminates the infinite number of constraints because a convex hull can be described by a finite number of extreme points. Altin et al. (2010), Belotti and Pinar (2008), and Tabatabaei et al. (2007) deal with the semi-infinite programs by taking the dual of these programs. Kodialam et al. (2006) create a separation oracle linear program related to the semi-infinite program, and apply a two-phase routing technique to take the dual of the separation oracle linear program. Gunnar and Johansson (2011) eliminate the use of the semi-infinite program by adopting techniques combining column and constraint generation to find a robust routing.

Another distinguishing characteristic of many applications is the network representation. Altin et al. (2010), Belotti and Pinar (2008), Ben-Ameur and Kerivin (2005), Casas (2010), Gunnar and Johansson (2011), Minoux (2009), and Tabatabaei et al. (2007) deal with the path decomposition formulation of the robust routing problem. One the other hand, Kodialam et al. (2006) describe a problem that has both arc and path aspects due to the two-phase nature of the routing.

A drawback to considering a single objective is that too much focus is set on a specific TE or QoS criterion.
while other criteria are ignored. However, only a few telecommunications studies deal with a multiobjective
problem, none with more than two objectives. Because optimization under uncertainty is often seen as
more complex than deterministic optimization, researchers who have attempted to consider two objectives
in those problems tend to use linear combinations of both objective functions with a priori chosen scalars
as in Casas (2010). Robust routing under traffic uncertainty with multiple objectives has not been studied
in a multiobjective context. However, robust routing under conditions of equipment failure with multiple
objectives has been previously considered. In particular, the works of Nucci et al. (2003) and Yuan (2003)
consider the latter.

The first objective of this paper is to extend robust single objective optimization to the multiobjective
case. We examine two prominent approaches, each with its own level of robustness. They include column-
wise uncertainty (Soyster 1973) and row-wise uncertainty (Ben-Tal and Nemirovski 1999). We extend their
validity in the presence of a vector-valued objective and without scalarization. For each uncertainty model,
we show that efficient solutions of a robust multiobjective optimization problem can be found as the efficient
solutions of a related deterministic problem. The second objective of this paper is to develop an approach to
robust multiobjective optimization that is applicable to Internet routing. We make use of the multiobjective
extension of row-wise uncertainty (Ben-Tal and Nemirovski 1999), which is well-suited to routing, and the
polyhedral traffic model of Ben-Ameur and Kerivin (2005), which is commonly used in telecommunications
networks, to develop a new model for Internet routing. We formulate a multicommodity flow problem to
model traffic routing on the network, described in terms of its arc representation. We compute robust efficient
solutions to an uncertain biobjective routing problem and indicate its relevance to decision making.

The paper is organized in five sections. In Section 2, we present theoretical results on the extension of
robust optimization to the multiobjective case, while a robust biobjective routing problem with the necessary
notation is developed in Section 3. The computational results are reported and discussed in Section 4. We
conclude with Section 5.

2 Robust multiobjective optimization

We consider an optimization problem of the form

\[
\begin{align*}
\min_{x} & \quad f(x) \\
\text{s.t.} & \quad g(x, u) \in \mathcal{K}, \\
& \quad x \in \mathcal{X},
\end{align*}
\]

(1)

with an uncertainty vector \( u \) that belongs to a given compact uncertainty set \( \mathcal{U} \subseteq \mathbb{R}^{r} \). Here \( x \in \mathbb{R}^{n} \) represents
the vector of decision variables, \( g : \mathbb{R}^{n} \times \mathbb{R}^{r} \to \mathbb{R}^{m}, \mathcal{K} \subseteq \mathbb{R}^{m} \), and \( \mathcal{X} \subseteq \mathbb{R}^{n} \) are the structural elements of the
constraints, and \( f : \mathbb{R}^{n} \to \mathbb{R}^{p} \) is the vector-valued objective function with component functions \( f_{k} : \mathbb{R}^{n} \to \mathbb{R} \)
for all \( k \in \{1, \ldots, p\} \). Hereafter, problem (1) is referred to as an uncertain multiobjective optimization
problem.

2.1 Terminology and reformulation

With every uncertain vector \( u \in \mathcal{U} \), one associates a deterministic multiobjective optimization problem

\[
\min_{x} \{ f(x) : g(x, u) \in \mathcal{K}, x \in \mathcal{X} \},
\]

(2)

called an instance of the uncertain multiobjective optimization problem. Let \( \mathcal{X}_{u} \) denote the set of feasible
solutions to deterministic multiobjective optimization problem (2) associated with \( u \in \mathcal{U} \), that is,

\[
\mathcal{X}_{u} = \{ x \in \mathcal{X} : g(x, u) \in \mathcal{K} \}.
\]

An essential attribute of problem (1) lies in the uncertain constraints

\[
g(x, u) \in \mathcal{K},
\]

(3)
which must be satisfied no matter what the actual realization of uncertainty vector $u$ is, provided the latter belongs to $\mathcal{U}$. Because uncertain vector $u$ is unknown at the time of solving problem (1), but uncertainty set $\mathcal{U}$ is given, the solution vector(s) to problem (1) can be obtained by solving the following problem, called the robust counterpart of uncertain multiobjective optimization problem (1),

$$\begin{align*}
\min_x & \quad f(x) \\
\text{s.t.} & \quad g(x, u) \in K \quad \text{for all} \ u \in \mathcal{U}, \\
& \quad x \in \mathcal{X}.
\end{align*} \tag{4}$$

Robust counterpart (4) deals with solutions which remain feasible to uncertain multiobjective optimization problem (1) regardless of the future realization of vector $u$ in $\mathcal{U}$. Robust counterpart methodology, initially developed by Soyster (1973) and later by Ben-Tal and Nemirovski (1998, 1999) and El-Ghaoui and Lebret (1997), then corresponds to a worst-case-oriented approach.

Let $\mathcal{X}_{\text{RC}}$ denote the set of feasible solutions to robust counterpart (4), that is,

$$\mathcal{X}_{\text{RC}} = \bigcap_{u \in \mathcal{U}} \mathcal{X}_u. \tag{5}$$

The feasible set $\mathcal{X}_{\text{RC}}$ is a subset of the decision space $\mathbb{R}^n$, whereas its image $\mathcal{Y}_{\text{RC}} = f(\mathcal{X}_{\text{RC}})$ is a subset of the objective space $\mathbb{R}^p$. A vector in $\mathcal{X}_{\text{RC}}$ (in $\mathcal{Y}_{\text{RC}}$, respectively) then is said to be a robust feasible solution (a robust outcome of, respectively) to uncertain multiobjective optimization problem (1). A direct consequence of definition (5) is the following proposition.

**Proposition 1.** If the robust counterpart (4) is feasible, then so are all the instances (2).

The converse of Proposition 1 is not necessarily true, even in the case that constraint (3) involves linear inequalities with uncertain coefficients, the set $\mathcal{X} = \mathbb{R}^+_n$, and the uncertainty set $\mathcal{U}$ is a polytope (Ben-Tal and Nemirovski 1999).

When a single objective function (i.e., $p = 1$) is considered, an optimal solution $x^*$ to robust counterpart (4) is called robust optimal to uncertain optimization problem (1) and the objective-function value at $x^*$ is the robust optimal value of problem (1) (Ben-Tal et al. 2009). In multiobjective optimization, the existence of feasible solutions simultaneously minimizing all the objective functions (i.e., the so-called ideal point (Ehrgott 2005)) is extremely rare due to conflict among objectives (e.g., cost versus quality of service for Internet routing problems). To cope with the necessity of trade-offs between objectives, decision makers generally consider some preferences between the outcomes and then seek preferred solutions and outcomes. One of the most common concepts to model these preferences is the well-known Pareto optimality (Pareto 1896) which is associated with the classical partial orderings on $\mathbb{R}^p$

$$\begin{align*}
y' \leq y & \iff y'_k \leq y_k \text{ for all } k \in \{1, \ldots, p\}, \\
y' \leq y & \iff y'_k \leq y_k \text{ for all } k \in \{1, \ldots, p\} \text{ and } y' \neq y, \\
y' < y & \iff y'_k < y_k \text{ for all } k \in \{1, \ldots, p\}.
\end{align*}$$

A vector $y \in \mathbb{R}^p$ is said to dominate a vector $y' \in \mathbb{R}^p$ if and only if $y \leq y'$.

**Definition 1.** Outcome $y \in \mathcal{Y}_{\text{RC}} \subset \mathbb{R}^p$ is called nondominated (or Pareto optimal) to robust counterpart (4) if there does not exist $y' \in \mathcal{Y}_{\text{RC}}$ which dominates $y$, that is, such that $y' \leq y$.

The set of all nondominated outcomes to robust counterpart (4) is denoted by $N(\mathcal{Y}_{\text{RC}})$. The elements composing the pre-images of the nondominated outcomes in $N(\mathcal{Y}_{\text{RC}})$ are called efficient to robust counterpart (4) and they form the efficient set to robust counterpart (4)

$$E(\mathcal{X}_{\text{RC}}) = \{x \in \mathcal{X}_{\text{RC}} : f(x) \in N(\mathcal{Y}_{\text{RC}})\},$$

where $\mathcal{X}_{\text{RC}} = f^{-1}(\mathcal{Y}_{\text{RC}}) \subset \mathbb{R}^n$ is the pre-image of $\mathcal{Y}_{\text{RC}}$. If the pre-image $f^{-1}(y) = \{x \in \mathcal{X} : f(x) = y\}$ of nondominated outcome $y$ in $N(\mathcal{Y}_{\text{RC}})$ is a singleton, the unique element of $E(\mathcal{X}_{\text{RC}})$ belonging to $f^{-1}(y)$
then is called a strictly efficient solution to robust counterpart (4). Let \( s-E(\mathcal{X}_{RC}) \) denote the set composed of all the strictly efficient solutions to robust counterpart (4).

Besides the nondominated outcomes, other notions of preferred solutions are widely used, including the weakly nondominated outcomes (Ehrgott 2005).

The latter form a larger set than \( N(\mathcal{Y}_{RC}) \), and despite being less useful in practice, tend to be easier to generate than nondominated outcomes. A vector \( y \in \mathbb{R}^p \) is said to strictly dominate a vector \( y' \in \mathbb{R}^p \) if and only if \( y < y' \).

**Definition 2.** Outcome \( y \in \mathcal{Y}_{RC} \) is called weakly nondominated to robust counterpart (4) if there does not exist \( y' \in \mathcal{Y}_{RC} \) which strictly dominates \( y \), that is, such that \( y' < y \).

The set of all weakly nondominated outcomes to robust counterpart (4) is denoted by \( w-N(\mathcal{Y}_{RC}) \). The weakly efficient set to robust counterpart (4) then is composed of the union of the pre-images of the weakly nondominated outcomes in \( w-N(\mathcal{Y}_{RC}) \), that is,

\[
    w-E(\mathcal{X}_{RC}) = \{ x \in \mathcal{X}_{RC} : f(x) \in w-N(\mathcal{Y}_{RC}) \},
\]

and its elements are called weakly efficient to robust counterpart (4).

Robust multiobjective optimization thus consists of finding the (strictly/weakly) efficient set of robust counterpart (4). A (strictly/weakly) efficient solution to robust counterpart (4) is called a robust (strictly/weakly) efficient solution to uncertain multiobjective optimization problem (1). Similarly one talks about robust (weakly) nondominated outcomes of uncertain multiobjective optimization problem (1).

As a single-objective optimization problem may have no optimal solutions, the efficient set of a multiobjective optimization problem does not necessarily exist. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called lower semicontinuous throughout \( \mathbb{R}^n \) if for every \( \alpha \in \mathbb{R} \), the level set \( S_\alpha = \{ x : f(x) \leq \alpha \} \) is closed (Rockafellar 1997). Combining the lower semicontinuity property on the component functions \( f_1, \ldots, f_p \) of the vector-valued objective function \( f \) with additional properties on the feasible set \( X \), sufficient conditions for the existence of the efficient set have been obtained as stated in the following theorem (see, e.g., Theorem 2.19 in Ehrgott (2005)).

**Theorem 1.** The efficient set to a multiobjective optimization problem \( \min\{ (f_1(x), \ldots, f_p(x)) : x \in \mathcal{X} \subseteq \mathbb{R}^n \} \) is nonempty, if the feasible set \( \mathcal{X} \) is nonempty and compact, and the scalar-valued functions \( f_k \), for all \( k \in \{1, \ldots, p\} \), are lower semicontinuous.

To ensure the existence of robust efficient solutions to uncertain multiobjective optimization problem (1), we introduce the following assumptions on the structural elements of its constraints, that is, \( g \), \( \mathcal{X} \), and \( K \) and on the component functions \( f_k \) of the vector-valued objective function \( f \).

**Assumption 1.** From now on, we assume that

(i) \( \mathcal{X} \subseteq \mathbb{R}^n \) is a nonempty compact set;

(ii) functions \( g(\cdot, u) \) are continuous on \( \mathcal{X} \) for all \( u \in \mathcal{U} \);

(iii) \( K \subseteq \mathbb{R}^m \) is closed;

(iv) functions \( f_k \) are lower semicontinuous on \( \mathcal{X} \) for all \( k \in \{1, \ldots, p\} \).

The assumptions on the structural elements of the constraints ensure that the feasible set to robust counterpart (4) is a compact set in \( \mathbb{R}^n \), as stated in the next proposition.

**Proposition 2.** Set \( \mathcal{X}_{RC} \) is compact.

**Proof.** The case \( \mathcal{X}_{RC} = \emptyset \) is trivial, so assume that \( \mathcal{X}_{RC} \) is nonempty. Let \( u \in \mathcal{U} \). Because compactness is preserved by continuous functions, by Assumption 1(i)-(ii), the set \( g(\mathcal{X}, u) \), that is, the set consisting of the images of all \( x \in \mathcal{X} \) under \( g(\cdot, u) \), is a compact set in \( \mathbb{R}^m \). The intersection of a compact set with a closed set being compact, Assumption 1(iii) implies that the set \( g(\mathcal{X}, u) \cap K \) is compact. Consequently, the set \( \bigcap_{u \in \mathcal{U}} (g(\mathcal{X}, u) \cap K) \) is compact in \( \mathbb{R}^m \). By the continuity of \( g(\cdot, u) \) for every \( u \in \mathcal{U} \), the inverse image of \( \bigcap_{u \in \mathcal{U}} (g(\mathcal{X}, u) \cap K) \), which is \( \mathcal{X}_{RC} \), is a closed set in \( \mathbb{R}^n \). As \( \mathcal{X}_{RC} \) is a closed subset of the compact set \( \mathcal{X} \), \( \mathcal{X}_{RC} \) is a compact set in \( \mathbb{R}^n \). \( \square \)
Proposition 2, combined with Assumption 1(iv) on the component functions of the objective function to robust counterpart (4), immediately implies the following corollary of Theorem 1, showing that a robust efficient solution to uncertain multiobjective optimization problem (1) exists as long as a robust feasible solution exists to robust counterpart (4).

**Corollary 1.** If feasible set $X_{RC}$ is nonempty, then so is the efficient set $E(X_{RC})$.

Because an efficient solution also is a weakly efficient one, Assumption 1 ensures the existence of the weakly efficient set $w-E(X_{RC})$ as well.

We now wish to establish some connections between the efficient solutions to the robust counterpart (4) and those to the instances (2). As pointed out by Ben-Tal and Nemirovski (1998) in the single-objective case, there might exist a gap between the optimal value to the robust counterpart and the optimal values to all the instances; the former being greater than the latter. In robust optimization, uncertainty can be represented by the triplet $(g, K, U)$, that is, by the two structural elements of the uncertain constraints, $g$ and $K$, of problem (1) and the uncertainty set $U$. In the case of uncertain convex single-objective optimization problems, Ben-Tal and Nemirovski (1998) considered some additional properties on the uncertainty $(g, K, U)$, that we will mention in Subsection 2.2, that make this gap vanish. To obtain a similar result in the multiobjective case, we introduce the following type of uncertainty that encompasses the one introduced by Ben-Tal and Nemirovski (1998).

**Definition 3.** Uncertainty $(g, K, U)$ is called robust-counterpart suitable if for every compact set $X \subseteq \mathbb{R}^n$, robust counterpart (4) is feasible whenever all the instances (2) are feasible.

Proving that uncertainty is robust-counterpart suitable may be difficult in practice, so we also consider a more restrictive definition where convexity is enforced on the deterministic part of the constraints (i.e., set $X$). This new definition of robust-counterpart suitability will appear useful later in this section.

**Definition 4.** Uncertainty $(g, K, U)$ is called convex robust-counterpart suitable if for every convex compact set $X \subseteq \mathbb{R}^n$, robust counterpart (4) is feasible whenever all the instances (2) are feasible.

Under robust-counterpart-suitable uncertainty, Proposition 1 implies that robust counterpart (4) is feasible if and only if all the instances (2) are feasible. This feasibility equivalence allows us to prove the following proposition connecting the (strictly/weakly) efficient solutions to robust counterpart (4) with those to all the instances (2).

**Proposition 3.** If the uncertainty is robust-counterpart suitable, then every (strictly/weakly) efficient point to robust counterpart (4) is (strictly/weakly) efficient to at least one instance (2).

**Proof.** We only prove for efficient points, the cases of strictly or weakly efficient points are similar and left to the reader.

Let $\bar{x}$ be an efficient solution to robust counterpart (4), that is, $\bar{x} \in E(X_{RC})$. Because $\bar{x} \in X_{RC}$, definition (5) implies that $\bar{x}$ is feasible to all the instances (2).

Suppose that $\bar{x}$ is efficient to no instance (2), that is, for every $u \in U$ there exists $x^u \in X_u$ such that $f(x^u) \leq f(\bar{x})$. Let $X'(u)$ denote, for a given $u \in U$ the set of points feasible to that instance (2) whose image under $f$ dominates $f(\bar{x})$, that is, there exists a non-empty set $X'(u)$ defined as

$$X'(u) = \{x^u \in X_u : f(x^u) \leq f(\bar{x})\}.$$  

Further, define $X' = \bigcup_{u \in U} X'(u)$. We then have

$$\sum_{k=1}^{p} f_k(x^u) < \sum_{k=1}^{p} f_k(\bar{x}) \quad \text{for all } x^u \in X'.$$  \hspace{1cm} (6)

Let $\epsilon$ be a scalar chosen so that

$$\max \left\{ \sum_{k=1}^{p} f_k(x^u) : x^u \in X' \right\} \leq \epsilon < \sum_{i=k}^{p} f_k(\bar{x}).$$  \hspace{1cm} (7)
By inequality (6), such ε exists. Consider the subset \( \mathcal{P}_\epsilon(\mathbf{x}) \) of \( \mathbb{R}^n \) defined by the following inequalities

\[
f_k(\mathbf{x}) \leq f_k(\mathbf{\bar{x}}) \quad \text{for } k \in \{1, \ldots, p\},
\]
\[
\sum_{k=1}^p f_k(\mathbf{x}) \leq \epsilon.
\]

Because the functions \( f_k \) are lower semicontinuous and \( f_k(\mathbf{x}^u) \) is finite on \( \mathcal{X}' \) for any \( k = \{1, \ldots, p\} \), then \( \sum_{k=1}^p f_k(\mathbf{x}^u) \) is also lower semicontinuous on \( \mathcal{X}' \). Thus, the level sets defining \( \mathcal{P}_\epsilon(\mathbf{x}) \) are closed and so is \( \mathcal{P}_\epsilon(\mathbf{x}) \). Note that for each \( u \in \mathcal{U} \), the vector \( \mathbf{x}^u \) belongs to \( \mathcal{P}_\epsilon(\mathbf{x}) \), yet the vector \( \mathbf{x} \) does not belong to \( \mathcal{P}_\epsilon(\mathbf{x}) \).

Consider now the uncertain optimization problem

\[
\min \{ f(\mathbf{x}) : g(\mathbf{x}, u) \in K, \mathbf{x} \in \mathcal{X}, \mathbf{x} \in \mathcal{P}_\epsilon(\mathbf{\bar{x}}) \} \quad \text{for all } u \in \mathcal{U}.
\]

Note that problem (8) has the same uncertainty \((g, K, \mathcal{U})\) which is robust-counterpart suitable. For every \( u \in \mathcal{U} \), the associated instance of problem (8) is feasible, because \( \mathbf{x}^u \) belongs to both \( \mathcal{X} \) and \( \mathcal{P}_\epsilon(\mathbf{x}) \). The robust counterpart of uncertain optimization problem (8) is

\[
\min \{ f(\mathbf{x}) : g(\mathbf{x}, u) \in K \text{ for all } u \in \mathcal{U}, \mathbf{x} \in \mathcal{X}, \mathbf{x} \in \mathcal{P}_\epsilon(\mathbf{\bar{x}}) \}.
\]

Note that \( \mathcal{X} \cap \mathcal{P}_\epsilon(\mathbf{x}) \) is compact (being the intersection of a compact set and a closed set) and all instances of problem (8) are feasible. Then, because the uncertainty is robust-counterpart suitable, robust counterpart (9) is feasible. Because \( \mathcal{X} \cap \mathcal{P}_\epsilon(\mathbf{x}) \subseteq \mathcal{X} \), every feasible solution to robust counterpart (9) belongs to \( \mathcal{X}_{RC} \). Moreover, set \( \mathcal{X} \cap \mathcal{P}_\epsilon(\mathbf{x}) \) is only composed of points whose images under \( f \) dominate \( f(\mathbf{x}) \). Therefore, there exists at least one point \( \mathbf{x}' \) in \( \mathcal{X}_{RC} \) such that \( f(\mathbf{x}') \leq f(\mathbf{x}) \), a contradiction with \( \mathbf{x} \) being efficient to robust counterpart (4).

For convex robust-counterpart suitable uncertainty, obtaining a similar relationship between the efficient solutions to the robust counterpart and those to the related instances follows by considering some additional convexity-preserving property on the objective component function. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be quasiconvex if for all \( \alpha \in \mathbb{R} \), the level sets \( S_\alpha = \{ x \in \mathbb{R}^n : f(x) \leq \alpha \} \) are convex. Adding the quasiconvex property on the component functions of the vector-valued objective function of robust counterpart (4) enables us to prove a similar result of Proposition 3.

**Proposition 4.** If the uncertainty is convex robust-counterpart suitable, the functions \( f_i \) are quasiconvex for \( i \in \{1, \ldots, p\} \), and their sum also is quasiconvex, then every (strictly/weakly) efficient point to robust counterpart (4) is (strictly/weakly) efficient to at least one instance (2).

**Proof.** We only prove for efficient points, the cases of strictly or weakly efficient points are similar and left to the reader.

As in the proof of Proposition 3, let \( \mathbf{x} \) be an efficient solution to robust counterpart (4), that is, \( \mathbf{x} \in E(\mathcal{X}_{RC}) \). Because \( \mathbf{x} \in \mathcal{X}_{RC} \), definition (5) implies that \( \mathbf{x} \) is feasible to all the instances (2).

Further, suppose that \( \mathbf{x} \) is efficient to no instance (2), that is, for every \( u \in \mathcal{U} \) there exists \( \mathbf{x}^u \in \mathcal{X}_u \) such that \( f(\mathbf{x}^u) \leq f(\mathbf{x}) \). Proceeding as in the proof of Proposition 3, by inequality (6), an \( \epsilon \) exists which satisfies inequality (7). Consider the subset \( \mathcal{P}_\epsilon(\mathbf{x}) \) of \( \mathbb{R}^n \) defined by the following inequalities

\[
f_k(\mathbf{x}) \leq f_k(\mathbf{\bar{x}}) \quad \text{for } k \in \{1, \ldots, p\},
\]
\[
\sum_{k=1}^p f_k(\mathbf{x}) \leq \epsilon.
\]

Because the functions \( f_k \) are quasiconvex, the level sets defined by inequalities (10) are convex. The convexity of the level set associated with inequality (11) comes directly from the quasiconvexity of the sum of the functions \( f_k \). The set \( \mathcal{P}_\epsilon(\mathbf{x}) \) is the intersection of convex sets and therefore is convex. Further, because the functions \( f_k \) are lower semicontinuous and \( f_k(\mathbf{x}^u) \) is finite on \( \mathcal{X}' \) for any \( k = \{1, \ldots, p\} \), then \( \sum_{k=1}^p f_k(\mathbf{x}^u) \)
is also lower semicontinuous on \( \mathcal{X}' \). Thus, the level sets defining \( \mathcal{P}_\epsilon(\mathbf{x}) \) are closed and so is \( \mathcal{P}_\epsilon(\mathbf{x}) \). Note that for each \( u \) of \( U \), the vector \( \mathbf{x}^u \) belongs to \( \mathcal{P}_\epsilon(\mathbf{x}) \), yet the vector \( \mathbf{x} \) does not belong to \( \mathcal{P}_\epsilon(\mathbf{x}) \). Consider now the uncertain optimization problem

\[
\min \{ f(x) : g(x, u) \in K, x \in X, x \in \mathcal{P}_\epsilon(x) \} u \in U.
\]  

(12)

Note that problem (12) has the same uncertainty \((g, K, U)\) that is convex robust-counterpart suitable. Using a similar argument as in the end of the proof of Proposition 3, a contradiction to \( x \) being efficient can be reached.

So far, uncertainty has only been considered in constraints (3), and each component function of the objective function \( f \) of uncertain multiobjective optimization problem (1) is assumed to be deterministic. If, however, some component functions of \( f \) are subject to uncertainty and the uncertain multiobjective optimization problem is formulated as

\[
\begin{align*}
\min_{x} & \quad f(x, u) \\
\text{s.t.} & \quad g(x, u) \in K, \\
& \quad x \in \mathcal{X}, \\
& \quad u \in U,
\end{align*}
\]

(13)

then the worst-case-oriented approach, commonly used in robust optimization, leads us to consider

\[
\min_{x} F(x),
\]

where, for \( x \in \mathbb{R}^n \), \( F(x) \) is the \( p \)-dimensional vector whose components are

\[
F_k(x) = \max \{ f_k(x, u) : u \in U \} \text{ for all } k \in \{1, \ldots, p\}.
\]

Similarly to what was done for problem (1), we define the concepts of robust (strictly/weakly) efficient solutions and (weakly) nondominated outcomes to problem (13) through the (strictly/weakly) efficient solutions and (weakly) nondominated outcomes to its robust counterpart

\[
\begin{align*}
\min_{x} & \quad F(x) \\
\text{s.t.} & \quad g(x, u) \in K \text{ for all } u \in U, \\
& \quad x \in \mathcal{X}.
\end{align*}
\]

(14)

Using \( p \) additional variables \( \beta_1, \ldots, \beta_p \) and \( p \) additional constraints

\[
F_k(x) \leq \beta_k \text{ for all } k \in \{1, \ldots, p\},
\]

the next proposition introduces an uncertain multiobjective optimization problem equivalent to problem (13), which has a deterministic objective function and uncertainty only in the constraints as in problem (1).

**Proposition 5.** Let \( \beta \in \mathbb{R}^p \) and associate with problem (13) the following uncertain multiobjective optimization problem

\[
\begin{align*}
\min_{(x, \beta)} & \quad \beta \\
\text{s.t.} & \quad f(x, u) - \beta \in \mathbb{R}^p, \\
& \quad g(x, u) \in K, \\
& \quad x \in \mathcal{X} \\
& \quad u \in U.
\end{align*}
\]

(15)

The following statements hold.

(i) If \((x, \beta)\) is robust (strictly/weakly) efficient to problem (15), then \( x \) is robust (strictly/weakly) efficient to problem (13).
(ii) If \( \bar{x} \) is robust (strictly/weakly) efficient to problem (13), then \( (\bar{x}, F(\bar{x})) \) is robust (strictly/weakly) efficient to problem (15).

Proof. We first notice that if \( x \) is feasible to robust counterpart (14) (and recalling that this is the robust counterpart to problem (13)), then \( (x, F(x)) \) is feasible to the robust counterpart of uncertain multiobjective optimization problem (15)

\[
\begin{align*}
\min_{(x, \beta)} & \quad \beta \\
\text{s.t.} & \quad f(x, u) - \beta \in \mathbb{R}^p_\geq \quad \text{for all } u \in \mathcal{U}, \\
& \quad g(x, u) \in \mathcal{K} \quad \text{for all } u \in \mathcal{U}, \\
& \quad x \in \mathcal{X}.
\end{align*}
\]

(16)

Conversely, any feasible solution \((x, \beta)\) to robust counterpart (16) has its \( x \)-component feasible to robust counterpart (14). We prove the result with respect to efficiency to both robust counterparts, a similar argument works for strict efficiency and weak efficiency.

Let \((\bar{x}, \bar{\beta})\) be an efficient solution to robust counterpart (16), that is, \( \bar{x} \) is feasible to robust counterpart (14) and \( F(\bar{x}) \leq \bar{\beta} \). Suppose that \( \bar{x} \) is not efficient to robust counterpart (14). There then exists \( x^* \), feasible to robust counterpart (14), such that

\[ F(x^*) \leq F(\bar{x}) \leq \bar{\beta}. \]

Because \((x^*, F(x^*))\) is feasible to robust counterpart (16) and outcome \( F(x^*) \) dominates outcome \( \bar{\beta} \), the solution \((\bar{x}, \bar{\beta})\) then is not efficient to robust counterpart (16).

Consider now an efficient solution \( \tilde{x} \) to robust counterpart (14). Suppose that \((\tilde{x}, F(\tilde{x}))\) is not efficient to robust counterpart (16). There then exists \((x', \beta')\) such that \( x' \) is feasible to robust counterpart (14) and \( \beta' \leq F(\tilde{x}) \). Because \( F(x') \leq \beta' \) by the feasibility of \((\tilde{x}, F(\tilde{x}))\) for problem (16), we deduce that outcome \( F(x') \) dominates outcome \( F(\tilde{x}) \), a contradiction. \( \square \)

It is worth noticing that if \((x, \beta)\) is robust (strictly) efficient to uncertain multiobjective optimization problem (15), then \( \beta = F(x) \), because otherwise, by the constraints \( f(x, u) - \beta \in \mathbb{R}^p_\geq \) for all \( u \) in \( \mathcal{U} \), outcome \( F(x) \) would dominate outcome \( \beta \). On the other hand, if \((x, \beta)\) is robust weakly efficient to uncertain multiobjective optimization problem (15), the concept of a strictly nondominated outcome, together with the aforementioned constraints, then imply that \( F(x) \leq \beta \) with at most \( p - 1 \) strict inequalities. In Proposition 5, we imply that, without loss of generality, one may assume that the vector-valued objective function in problem (1) is not subject to any uncertainty. From now on, this assumption is always made when referring to an uncertain multiobjective optimization problem. We note here that Proposition 5 does not eliminate the need to study multiobjective problems with uncertainty in the objective functions. Not making use of Proposition 5 or even defining other robust counterpart problems may offer new insight into robust multiobjective optimization (Goberna et al. 2015, Witting et al. 2013, and others).

2.2 Methodology

Soyster (1973) was among the first to consider a robust-counterpart methodology to handle uncertainty in single-objective optimization problems, in what he called set-inclusive constraints.

Definition 5. If uncertainty set \( \mathcal{U} \) is the Cartesian product of \( n \) convex sets of \( \mathbb{R}^m \), that is, \( \mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_n \) where \( \mathcal{U}_j \subseteq \mathbb{R}^m \) is convex for \( j \in \{1, \ldots, n\} \), \( g(x, u) = \sum_{j=1}^n x_j u_j \) where \( u_j \in \mathcal{U}_j \) for \( j \in \{1, \ldots, n\} \), and \( \mathcal{K} \) is a convex set, then the uncertainty is called set-inclusive, now known as column-wise.

The feasible region of the optimization problem Soyster considered is

\[ \mathcal{X}^S = \{x \in \mathbb{R}^n : x \geq 0, x_1 u_1 + \ldots + x_n u_n \subseteq \mathcal{K}\}, \]

where the symbol + represents the Minkowski sum of sets. Soyster showed that \( \mathcal{X}^S \) is a convex set, and that if \( \mathcal{K} \) is a polyhedral set having the form \( \mathcal{K}(b) = \{z \in \mathbb{R}^m : z \leq b\} \), for some \( b \in \mathbb{R}^m \), then

\[ \mathcal{X}^S = \{x \in \mathbb{R}^n : x \geq 0, \overline{G} x \in \mathcal{K}(b)\}, \]

(17)
where $\overline{G} = [\overline{g}_{ij}]$ is the $m \times n$ matrix whose components are $\overline{g}_{ij} = \sup\{u_{ij} : u_j \in U_j\}$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Notice that if some components of $x$ are nonnegative, one then needs to consider the infimum instead of the supremum in the corresponding components of $\overline{G}$. Soyster also pointed out that if, in the definition of $K(b)$, one has $z_i = b_i$ for some $i \in \{1, \ldots, m\}$, then the set $U_i$ can be omitted from the definition of $\Lambda^S$ whenever the $i$th component of $u_j$ takes more than one value. Solving a single-objective optimization problem with column-wise uncertainty and the set $K$ polyhedral therefore reduces to optimizing the objective function over the polyhedron in definition (17). This reduction clearly is independent of the objective function. Consequently, we obtain the following generalization of Soyster’s result to uncertain multiobjective optimization problem (1).

**Theorem 2.** Consider uncertain multiobjective optimization problem (1) with column-wise uncertainty, set $K(b)$ polyhedral, and $\mathcal{X} \subseteq \mathbb{R}^p$. Its robust (strictly/weakly) efficient set is identical to the robust (strictly/weakly) efficient set of the multiobjective optimization problem

$$
\min_{x} f(x) \\
\text{s.t.} \quad \overline{G}x \in K(b), \quad x \in \mathcal{X}.
$$

Soyster’s model is very conservative because it simultaneously combines the worst-case values for all the uncertain parameters. Later, Ben-Tal and Nemirovski (1998, 1999) followed a robust-counterpart-based approach for uncertain single-objective inequality-constrained problems with the so-called row-wise uncertainty in order to provide a less conservative model.

**Definition 6.** If $g(x, u)$ is an $m$-dimensional vector whose components are $g_i(x, u^i)$ with $g_i : \mathbb{R}^n \times \mathbb{R}^{r_i} \to \mathbb{R}$ and $r_i \in \mathbb{Z}_+$ for all $i \in \{1, \ldots, m\}$, $\mathcal{U}$ is the Cartesian product of $m$ closed and convex sets of the spaces of $u^i$’s (i.e., $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_m$ with $\mathcal{U}_i \in \mathbb{R}^{r_i}$ for all $i \in \{1, \ldots, m\}$), and $K = \mathbb{R}^m_+$, then the uncertainty set $\mathcal{U}$ is called row-wise.

With row-wise uncertainty, the robust counterpart of uncertain multiobjective optimization problem (1) then assumes the form

$$
\min_{x} f(x) \\
\text{s.t.} \quad g_i(x, u^i) \leq 0 \quad \text{for all } u^i \in \mathcal{U}_i \text{ and } i \in \{1, \ldots, m\}, \quad x \in \mathcal{X}. \tag{18}
$$

We wish to develop a connection between uncertain multiobjective optimization problem (1) with row-wise uncertainty and its robust counterpart (18). As mentioned earlier, the converse of Proposition 1 is not always true, that is, there might exist some uncertain multiobjective optimization problem with all its instances being feasible but whose robust counterpart is infeasible. Ben-Tal and Nemirovski (1998) provided some conditions for the converse of Proposition 1 to be true as described in the following proposition (see, e.g., Theorem 2.1 in Ben-Tal and Nemirovski (1998)).

**Proposition 6.** Assume that $g_i$ is convex and continuous in $x$ and affine in $u^i$ for all $i \in \{1, \ldots, m\}$ and $\mathcal{X}$ is convex and compact. There exists an infeasible instance of uncertain multiobjective optimization problem (1) with row-wise uncertainty if and only if its robust counterpart (18) is infeasible.

Under the conditions of Proposition 6, row-wise uncertainty is convex robust-counterpart suitable. Moreover, provided the component functions of the objective function $f$ satisfy the quasiconvexity properties stated in Proposition 4, any nondominated outcome to robust counterpart (18) would be realized if some vector $u$ of the uncertainty set $\mathcal{U}$ was considered independently.

**Corollary 2.** Assume that $g_i$ is convex and continuous in $x$ and affine in $u^i$ for all $i \in \{1, \ldots, m\}$, $\mathcal{X}$ is convex and compact, and the functions $f_k$ for all $k \in \{1, \ldots, p\}$ and their sum are quasiconvex. Every (strictly/weakly) efficient point to robust counterpart (18) is a (strictly/weakly) efficient point to at least one instance (2).
Robust counterpart (18) of uncertain multiobjective optimization problem (1) with row-wise uncertainty is a semi-infinite problem and therefore is challenging to solve, in particular, it may be computationally prohibitive. In the remainder of this section, we show how robust counterpart (18) can be transformed into a more tractable problem while preserving the (strictly/weakly) efficient set.

For any $i \in \{1, \ldots, m\}$ and any $x \in \mathbb{R}^n$, let

$$V_i(x) = \max\{g_i(x, u^i) : u^i \in U_i\} \quad (19)$$

be the worst-case value of the left-hand-side of $i$th constraint of robust counterpart (18) associated with $x$. Notice that since $U_i$ is a nonempty subset of a compact set for any $i \in \{1, \ldots, m\}$, $V_i(x)$ is finite for any $x \in X$. Consider the following bilevel optimization problem

$$\begin{align*}
\min_x & \quad f(x) \\
\text{s.t.} & \quad V_i(x) \leq 0 \quad \text{for all } i \in \{1, \ldots, m\} \\
& \quad x \in X,
\end{align*} \quad (20)$$

and let $X_B \subseteq \mathbb{R}^n$ be its set of feasible solutions. From the definition of $V_1(x), \ldots, V_m(x)$, we straightforwardly obtain the following result.

**Proposition 7.** $X_{RC} = X_B$

Bilevel optimization problem (20) has a finite number of lower-level constraints $V_i(x) \leq 0$, namely $m$ of them. Recall that the lower-level variables $u^i$ correspond to the uncertain parameters of uncertain multiobjective optimization problem (1), and the robust (strictly/weakly) efficient solutions to the latter, that is, the upper-level variables $x$, need to be independent of any specific $u^i$-variable values. Therefore we make use of Lagrangian duality theory to reformulate problem (20) into another bilevel multiobjective problem whose lower-level constraints do not explicitly incorporate the variables $u^i$. According to this duality theory, under certain conditions, convex nonlinear programs are equivalent to their dual counterparts in the sense that their optimal objective functions values are equal (Bazaraa et al. 2013). Let $i \in \{1, \ldots, m\}$ and let

$$W_i(x) = \min\{w_i(v^i) : v^i \in D_i(x)\} \quad (21)$$

be the optimal objective value of a dual problem to the left-hand side of each lower-level constraints (i.e., to optimization problem (19)) associated with $i$, where $v^i \in D_i(x) \subseteq \mathbb{R}^n_i$ are the dual variables and $D_i(x)$ is their feasible set. By weak duality, we have $V_i(x) \leq w(v^i)$ for all $v^i \in D_i(x)$. Moreover, if the dual gap is zero (i.e., $V_i(x) = W_i(x)$), problems (19) and (21) form a strong-dual pair. Consider then the following bilevel optimization problem

$$\begin{align*}
\min_x & \quad f(x) \\
\text{s.t.} & \quad W_i(x) \leq 0 \quad \text{for all } i \in \{1, \ldots, m\}, \\
& \quad x \in X,
\end{align*} \quad (22)$$

and let $X_D \subseteq \mathbb{R}^n$ be its set of feasible solutions. A direct consequence of strong duality holding for all the lower-level constraints of bilevel optimization problem (20) clearly is the following.

**Proposition 8.** Assume problems (19) and (21) form a strong-dual pair for all $i \in \{1, \ldots, m\}$. We have $X_{RC} = X_D$.

The final step of our reformulation process for robust counterpart (18) is to eliminate the optimization problem in the lower-level constraints of bilevel optimization problem (22). This leads us to consider the
where $\mathbf{v} = (v^1, \ldots, v^m) \in \mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_m}$. Let $\mathcal{X}_E \subseteq \mathbb{R}^n \times (\mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_m})$ denote the set of feasible solutions to optimization problem (23). We now show that the projection of $\mathcal{X}_E$ onto $\mathbf{x}$, hereafter denoted $\text{proj}_x \mathcal{X}_E$, corresponds exactly to $\mathcal{X}_{RC}$. In other words, any feasible solution to robust counterpart (4) relates to a feasible solution to problem (23) having the same $\mathbf{x}$-variables and vice versa.

**Proposition 9.** Assume problems (19) and (21) form a strong-dual pair for all $i \in \{1, \ldots, m\}$. We have $\mathcal{X}_{RC} = \text{proj}_x \mathcal{X}_E$.

**Proof.** By Propositions 7 and 8, we only have to prove that $\mathcal{X}_D = \text{proj}_x \mathcal{X}_E$. Let $\mathbf{x} \in \mathcal{X}_D$. For all $i \in \{1, \ldots, m\}$, let $\mathbf{v}^i$ denote any optimal solution to problem (21), that is, $w_i(\mathbf{v}^i) = \tilde{w}_i(\mathbf{x}) \leq 0$, the inequality coming from problem (22). Therefore $(\mathbf{x}, \mathbf{v}^1, \ldots, \mathbf{v}^m)$ belongs to $\mathcal{X}_E$. Consequently $\mathcal{X}_D \subseteq \text{proj}_x \mathcal{X}_E$.

Conversely, let $(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}^1, \ldots, \tilde{\mathbf{v}}^m) \in \mathcal{X}_E$. From the definition of problem (21) and the constraints of problem (23), we have $W_i(\tilde{\mathbf{x}}) \leq w_i(\tilde{\mathbf{v}}^i) \leq 0$ for all $i \in \{1, \ldots, m\}$. Because $\tilde{\mathbf{x}} \in \mathcal{X}$, we obtain $\tilde{\mathbf{x}} \in \mathcal{D}$ and therefore $\text{proj}_x \mathcal{X}_E \subseteq \mathcal{X}_D$.

Considering problem (23) instead of robust counterpart (4) therefore guarantees the robust feasibility of the restriction to the decision space of any feasible solution to problem (23). In the next theorem, we examine the relationship between the (strictly/weakly) efficient sets to both robust counterpart (4) and problem (23). We prove that $\tilde{\mathbf{x}}$ is (strictly/weakly) efficient to robust counterpart (4) if and only if $(\tilde{\mathbf{x}}, \tilde{\mathbf{v}})$ is (strictly/weakly) efficient to robust counterpart (23).

**Theorem 3.** Assume problems (19) and (21) form a strong-dual pair for all $i \in \{1, \ldots, m\}$. We have

(i) $E(\mathcal{X}_{RC}) = \text{proj}_x E(\mathcal{X}_E)$,

(ii) $s-E(\mathcal{X}_{RC}) = \text{proj}_x s-E(\mathcal{X}_E)$,

(iii) $w-E(\mathcal{X}_{RC}) = \text{proj}_x w-E(\mathcal{X}_E)$,

where $E(\mathcal{X}_E)$, $s-E(\mathcal{X}_E)$, and $w-E(\mathcal{X}_E)$ are the efficient, strictly efficient, and weakly sets to problem (23), respectively.

**Proof.** By Propositions 7 and 8, we only have to prove that $E(\mathcal{X}_{RC}) = \text{proj}_x E(\mathcal{X}_D)$. Notice that both problems (22) and (23) have the same objective function which is independent of the $\mathbf{v}$-variables. Therefore, the result follows directly from Proposition 9.

In Section 3 we make use of the results developed in this section to find robust efficient solutions to an uncertain multiobjective linear optimization problem arising in Internet traffic engineering.

### 3 An application to Internet traffic engineering

The Internet now appears as one of the main actors in everyone’s life. The omnipresence of network services and Internet applications has generated a massive increase in the Internet traffic which has also become more and more complex and dynamic. To cope with the challenges of Internet traffic growth and nature, network managers have focused on Internet traffic engineering which encompasses on one hand the measurement, modeling, characterization, and control of Internet traffic, and, on the other hand, the simultaneous optimization of network’s resource utilization and traffic performance (Awduche et al. 1999, Awduche 1999).
The architecture of the Internet corresponds to the interconnection of Autonomous Systems (ASes). Each AS is a collection of routers (i.e., nodes) and links owned and controlled by a single administrative entity (e.g., university, company, Internet Service Provider); by contrast, the Internet operates without a central administrative entity. The routing of Internet traffic specifies how the packets of information are routed to reach their destinations, and it is monitored by routing protocols. Many ASes may be involved in the routing paths, so a routing configuration determines how the traffic flow is managed within and between the ASes. The routing through the Internet is then performed hierarchically with each AS implementing its autonomous routing policies, the intradomain routing, and with the routing between ASes using mainly the Border Gateway Protocol (BGP), the interdomain routing. In this section, we only focus on the intradomain routing, because the AS’ administrative entity can determine how to configure its traffic routing with respect to the chosen routing protocol. To learn more about Internet routing optimization, we refer to Rexford (2006).

3.1 Intradomain Robust Routing

Over the last fifteen years, intradomain traffic engineering has received much attention because it contributes to better management and performance of the Internet (Feldmann and Rexford 2001, Applegate and Cohen 2003, Casas 2010). Intradomain traffic engineering is mainly composed of two closely related aspects, the estimation of the traffic that needs to be routed and the design of the routing configurations.

The traffic that needs to be routed across an AS is usually aggregated into Origin-Destination (OD) pairs, where each pair consists of all the traffic having the same origin node and the same destination node (Casas 2010). The volume representation of traffic is described by a traffic matrix, wherein each entry is associated with an OD-pair and it represents the cumulative volume of traffic transmitted from the origin to the destination. The traffic matrix has become extremely difficult to accurately estimate, mainly because of the significant cost of collecting traffic data and of the high complexity of analyzing these data. To bypass this difficulty, AS operators commonly assume that the traffic matrix belongs to a set $U$ which contains a wide range of possible traffic configurations, as initially proposed by Ben-Ameur and Kerivin (2005). The set $U$ can be designed based on traffic measurements and geographical and behavioral information, among others. Network operators favor implementing stable routing plans, that is, routing paths which remain the same despite traffic variations, because they are easier to manage and any routing modification may cause service disruptions and thus QoS deterioration. Intradomain robust routing therefore consists of determining, for each OD-pair, the routing paths that would permit to carry any traffic matrix in $U$ with respect to the network resources.

Several routing protocols, called Interior Gateway Protocols (IGP), can be considered within an AS. Open Shortest Path First (OSPF) and Intermediate System-Intermediate System (IS-IS) route evenly the OD-pair’s traffic on shortest paths based on link weights (Rexford 2006). These IGP are highly distributed and scalable, but one of their main drawbacks is that they do not consider the characteristics of offered traffic and network capacity constraints when making routing decisions (Awduche 1999). This may result in unbalanced utilization of the network resources (e.g., congested versus underutilized links or routers). The Multi-Protocol Label Switch (MPLS) protocol, which appeared in 1999 (Awduche et al. 1999), allows to better control the routing through explicit paths and the specification of the proportions of OD-pair’s traffic being routed along these paths. This more flexible routing protocol is very suitable when performing traffic engineering because it enables a better link capacity management and tends to prevent congestion. Because it has become unrealistic to keep upgrading the link capacities to cope with the faster growing Internet traffic, we consequently focus, in this section, on the intradomain robust routing based on MPLS or on similar routing protocols.

A single robust routing that would be optimal or efficient for all the traffic matrices in $U$ may not generally exist. An alternative, which is most often preferred by network operators, is to seek an oblivious robust routing, that is, a robust routing which has the minimum worst-case performance over the traffic-matrix set $U$. The performance metric most commonly used to evaluate the performance of a routing consists of minimizing the Maximum Link Utilization (MaxLU), that is, the maximum, over all the links, of the ratio of the link’s load to the link’s capacity. One of the benefits of this metric is that the lower the link utilization...
is, the better the network may support sudden traffic variations. This metric may lead to bad distributions of traffic loads, especially with heterogeneous link capacities, and therefore, may deteriorate some important QoS indicator such as the end-to-end delay and packet loss (Casas et al. 2009). From a QoS point of view, a more pertinent metric would be related to the path end-to-end delay which is defined as the sum of the delays on the links composing the path. The link delay is divided into the queuing delay and the propagation delay, the former being dependent on the link load and the latter being constant. Because of the queuing-delay component, dealing with end-to-end delay metrics leads to computationally difficult traffic engineering problems. As proposed and empirically verified by Casas et al. (2009) (see also Casas (2010)), an interesting alternative with respect to QoS performance is to consider the Mean Link Utilization (MeanLU), that is, the average link utilization. The MeanLU metric provides a more global performance and, when combined with a local-performance indicator such as MaxLU, leads to efficient robust routing configurations with respect to end-to-end delay. In their simulations Casas et al. (2009) considered a single-objective robust optimization problem where both MaxLU and MeanLU were aggregated through a weighted sum. In the remainder of this paper, we study the intradomain robust routing problem from a multiobjective point of view by considering simultaneously the MaxLU and MeanLU metrics.

3.2 Robust multiobjective multicommodity flow problem

In this subsection we focus on the intradomain robust routing, yet it is worth mentioning that the upcoming mathematical models remain applicable, or can be easily adapted, to any type of network (or layer in a network), any type of link (i.e., unidirectional versus bidirectional), or any type of traffic (asymmetric versus symmetric), as long as the traffic is distributed throughout the network using an explicit routing protocol such as MPLS.

The AS is represented by a directed graph \( D = (V, A) \) where each vertex in \( V \) corresponds to a router and each arc in \( A \) corresponds to a link. Each arc \( a \) of \( A \) is associated with a positive value \( c_a \) which represents the installed (or available) capacity on the corresponding AS’s link. Each OD-pair is represented by a triplet \( (o, d, t) \), later called commodity, where \( o \in V \) is the origin vertex, \( d \in V \setminus \{o\} \) is the destination vertex, and \( t \in \mathbb{R}_+ \) is the amount of traffic having to be routed from \( o \) to \( d \) across \( D \). Let \( K \) be the set composed of all the commodities. The traffic matrix then is the specification of all the commodity volumes \( t_k \), for \( k \in K \), and the traffic matrix can be represented by a vector \( t \) of \( \mathbb{R}^{|K|} \). The traffic-matrix set \( \mathcal{U} \) therefore is a subset of \( \mathbb{R}^{|K|} \). Recalling that the desired routing configuration must be stable, that is, the selected routing paths must be independent of any traffic matrix in \( \mathcal{U} \), the intradomain robust routing problem then consists of selecting one or more paths in \( D \) for each commodity of \( K \) along which to carry the associated commodity traffic. These paths are chosen such that, whichever traffic matrix of \( \mathcal{U} \) is considered, (i) the flow (i.e., traffic) is always distributed in the same proportions on the commodity’s paths for each commodity of \( K \), and (ii) the total amount of flow carried on an arc never exceeds the arc capacity for each arc of \( A \). In other words, we consider a multicommodity flow problem in \( D \) (Ahuja et al. 1993).

For each commodity \( k \) of \( K \) and each arc \( a \) of \( A \), let \( x^k_a \in [0, 1] \) represent the proportion of flow associated with commodity \( k \) and carried of arc \( a \). For the sake of convenience, let

\[
x^k_a = \begin{bmatrix} x^k_{a_1} & x^k_{a_2} & \cdots & x^k_{a_{|A|}} \end{bmatrix}
\]

be the \(|A|\)-dimensional row vector composed of the flow variables associated with arc \( a \) of \( A \), and let

\[
x^k = \begin{bmatrix} x^k_{a_1} & x^k_{a_2} & \cdots & x^k_{a_{|A|}} \end{bmatrix}^T
\]

be the \(|A|\)-dimensional column vector composed of the flow variables associated with commodity \( k \) of \( K \). The flow-proportion variables \( x^k_a \) can be represented by a \(|K|\)-tuple \( \mathbf{x} = (x^{k_1}, x^{k_2}, \ldots, x^{k_{|K|}}) \) of the \(|K|\)-ary Cartesian power of \( \mathbb{R}^{|A|} \) (i.e., \( (\mathbb{R}^{|A|})^{|K|} \)). For each commodity \( k \) of \( K \), we define the \(|V|\)-dimensional column
vector $b^k$, called the supply/demand vector, as

$$
b^k_v = \begin{cases} 
1 & \text{if } v = o_k, \\
0 & \text{if } v \in V \setminus \{o_k, d_k\}, \\
-1 & \text{if } v = d_k.
\end{cases}
$$

As we deal with flow variables representing traffic proportions instead of traffic volumes, the origin (destination, respectively) vertex of a commodity supplies (demands, respectively) one unit, that is, the entire commodity traffic. For each commodity $k$ of $K$, a vector $x^k$ is an $o_kd_k$-flow proportion if it satisfies the flow-conservation constraints

$$
\sum_{a \in \delta^+(v)} x^k_a - \sum_{a \in \delta^-(v)} x^k_a = b^k_v \quad \text{for } v \in V,
$$

where $\delta^+(v)$ and $\delta^-(v)$ denote the set of arcs of $A$ leaving and entering node $v$, respectively, and the flow-proportion constraints

$$
0 \leq x^k_a \leq 1 \quad \text{for } a \in A.
$$

Constraints (24) may be compactly written in vector and matrix notation as

$$
Mx^k = b^k \quad \text{for } k \in K,
$$

where $M$ is the $|V| \times |A|$ incidence matrix of $D$. For each arc $a$ of $A$, to comply with the capacity requirement (ii) given above, a vector $x_a$ needs to satisfy the capacity constraint

$$
\sum_{k \in K} t_k x^k_a \leq c_a,
$$

where $t = [t_1 \ t_2 \ \ldots \ t_{|K|}]^T$ denotes whichever traffic matrix of $\mathcal{U}$ might be considered.

To model both MaxLU and MeanLU criteria as linear functions, with each arc $a$ of $A$ we associate a variable $p_a \in [0, 1]$ which corresponds to the proportion of the arc’s capacity $c_a$ being used to carry some flow, often called utilization, that is,

$$
p_a = \frac{x_a t}{c_a},
$$

for any given traffic matrix $t$ of $\mathcal{U}$. Let $p$ be the $|A|$-dimensional vector whose components are the $p_a$-variables. For the MaxLU that needs to be minimized, we define the function $f_1 : \mathbb{R}^{|A|} \to \mathbb{R}$ as

$$
f_1(p) = \max_{a \in A} p_a,
$$

whereas for the MeanLU which also needs to be minimized, we define the function $f_2 : \mathbb{R}^{|A|} \to \mathbb{R}$ as

$$
f_2(p) = \frac{1}{|A|} \sum_{a \in A} p_a.
$$

We can now formulate the intradomain robust routing problem as the following uncertain biobjective optimization problem

$$
\begin{align*}
\min_{x, p} & \quad \begin{bmatrix} f_1(p) \\ f_2(p) \end{bmatrix}^T \\
\text{s.t.} & \quad Mx^k = b^k \quad \text{for } k \in K, \\
& \quad x_a t - p_a c_a = 0 \quad \text{for } a \in A, \\
& \quad 0 \leq x^k_a \leq 1 \quad \text{for } k \in K, \\
& \quad 0 \leq p \leq 1,
\end{align*}
$$

for $t \in \mathcal{U}$.

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where \( t \in \mathbb{R}^{|K|} \) and \( \mathbf{0} \) and \( \mathbf{1} \) are the all-zero and all-one vectors, respectively; we do not specify their dimensions because they are always clear from the context. In uncertain multiobjective optimization problem (26), the only uncertain constraints are the proportion equations (25). Because we aim to use the robust-counterpart approach developed in Section 2 to solve problem (26), these proportion equations may appear to be an obstacle, being an equality constraint. In fact, they would require us to find robust feasible vectors \( \mathbf{x} \) and \( \mathbf{p} \) satisfying the restrictive condition that the uncertainty set \( \mathcal{U} \) is contained in the hyperplanes \( \{ \mathbf{t} \in \mathbb{R}^{|K|} : \mathbf{x}_{\alpha} \mathbf{t} = \mathbf{p}_a c_a \} \) associated with all the arcs \( a \) of \( A \). To avoid this, we substitute the capacity constraints
\[
\mathbf{x}_{\alpha} \mathbf{t} - p_a c_a \leq 0 \quad \text{for } a \in A,
\]
for proportion equations (25) in problem (26). As stated in the next proposition, any instance of the resulting uncertain biobjective optimization problem
\[
\min_{\mathbf{x}, \mathbf{p}} \begin{bmatrix} f_1(\mathbf{p}) & f_2(\mathbf{p}) \end{bmatrix}^T
\begin{align*}
\text{s.t.} & \quad M \mathbf{x}^k = \mathbf{b}^k \quad \text{for } k \in K, \\
& \quad \mathbf{x}_{\alpha} \mathbf{t} - p_a c_a \leq 0 \quad \text{for } a \in A, \\
& \quad 0 \leq \mathbf{x}^k \leq \mathbf{1} \quad \text{for } k \in K, \\
& \quad 0 \leq \mathbf{p} \leq \mathbf{1},
\end{align*}
\tag{27}
\]
has the same (strictly/weakly) efficient set as the corresponding instance of problem (26).

**Proposition 10.** Let \( \mathbf{t} \) be any traffic matrix in \( \mathcal{U} \). The (strictly/weakly) efficient set to the instance of problem (27) associated with \( \mathbf{t} \) equals the (strictly/weakly) efficient set to the instance of problem (26) associated with \( \mathbf{t} \).

**Proof.** Let \( (26)_k \) and \( (27)_k \) denote the instance of problems (26) and (27), respectively, associated with \( \mathbf{t} \) in \( \mathcal{U} \). We only prove for the efficient set, the cases of strictly or weakly efficient set are similar and left to the reader.

Consider any efficient point \((\mathbf{x}^*, \mathbf{p}^*)\) to \( (27)_k \), and suppose that there exists an arc, say \( \alpha \), of \( A \) such that \( \mathbf{x}_{\alpha}^* \mathbf{t} < p_{\alpha} c_{\alpha} \). Let \( \epsilon = p_{\alpha} c_{\alpha} - \mathbf{x}_{\alpha}^* \mathbf{t} \). The point \((\mathbf{x}^*, \mathbf{p}^* - \epsilon \mathbf{e}_{\alpha})\) clearly is feasible to \( (27)_k \) and dominates \((\mathbf{x}^*, \mathbf{p}^*)\), where \( \mathbf{e}_{\alpha} \) is the vector whose \( \alpha \)-th component equals 1 and all its other entries are zero. This contradicts the efficiency of \((\mathbf{x}^*, \mathbf{p}^*)\) which therefore is feasible to \( (26)_k \).

Let \((\mathbf{x}, \mathbf{p})\) be an efficient point to \( (26)_k \). This point also is feasible to \( (27)_k \). If \((\mathbf{x}, \mathbf{p})\) was not efficient to \( (27)_k \), it would be dominated by a point \((\mathbf{x}^*, \mathbf{p}^*)\), which is an efficient point to \( (27)_k \). From what we proved above, \((\mathbf{x}^*, \mathbf{p}^*)\) would also be feasible to \( (26)_k \), a contradiction with the efficiency of \((\mathbf{x}, \mathbf{p})\).

Consequently, any efficient point to \( (26)_k \) remains efficient to \( (27)_k \) and vice versa. \( \square \)

So from now on, we only consider uncertain biobjective optimization problem (27) that corresponds to problem (1) where
\[
\mathcal{X} = \left\{ (\mathbf{x}, \mathbf{p}) \in ([0,1]|A|)^{|K|} \times [0,1]|A| : M \mathbf{x}^k = \mathbf{b}^k \text{ for } k \in K \right\},
\]
the vector-valued function \( \mathbf{g} : (\mathbb{R}^{|A|})^{|K|} \times \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|} \) is defined as
\[
\mathbf{g}(\mathbf{x}, \mathbf{p}, \mathbf{t}) = \begin{bmatrix}
\mathbf{x}_{\alpha} \mathbf{t} - p_{\alpha} c_{\alpha} \\
\vdots \\
\mathbf{x}_{a|A|} \mathbf{t} - p_{a|A|} c_{a|A|}
\end{bmatrix},
\]
and set \( \mathcal{K} \) is the non-positive orthant \( \mathbb{R}^{|A|}_- \). The structural elements of problem (27), that is, the deterministic constraints defined by \( \mathcal{X} \), the uncertain constraints defined by \( \mathbf{g} \) and \( \mathcal{K} \), and both objective functions \( f_1 \) and
Proposition 11. Any robust (strictly/weakly) efficient point to problem (26) clearly satisfy Assumption 1. Consequently the robust counterpart of problem (27)

\[
\begin{align*}
\min_{x, p} & \quad [f_1(p) \ f_2(p)]^T \\
\text{s.t.} & \quad Mx^k = b^k \quad \text{for } k \in K, \\
& \quad x_a t - p_a c_a \leq 0 \quad \text{for } a \in A \text{ and for } t \in U, \\
& \quad 0 \leq x^k \leq 1 \quad \text{for } k \in K, \\
& \quad 0 \leq p \leq 1,
\end{align*}
\]

has a feasible set, denoted \(X_{RC}^{IRR} \), that is compact by Proposition 2 and convex by the linearity of the capacity constraints for fixed \(t \). As pointed out by Ben-Tal et al. (2009) (see, e.g., Section 1.2.1, observation C of Ben-Tal et al. (2009)), the set of robust feasible solutions to problem (27) remains intact when the uncertainty set \(U \) is extended to either its convex hull or its closure. From now on, we may therefore assume that the traffic-matrix set \(U \) is closed and convex.

Consider an arc \(a \in A \). Note that the traffic matrices considered in the associated capacity constraints

\[
x_a t - p_a c_a \leq 0 \quad \text{for } t \in U,
\]

are independent of the traffic matrices considered in the capacity constraints associated with the other arcs of \(A \), but they belong to the same set. The uncertainty in the intradomain robust routing problem therefore is row-wise as in Definition 6, although, for the sake of simplicity, we do not differentiate the arcs’ uncertainty sets into \(U_{a_1}, \ldots, U_{a_{|A|}} \). Because the components \(g_i \) of the uncertain vector-valued function \(g \) are linear for all \(i \in \{1, \ldots, |A| \} \) and the deterministic set \(X \) is defined by linear constraints, the convexity assumptions of Proposition 6 are met, and therefore, uncertainty \((g, K, U)\) is convex robust-counterpart suitable. In other words, robust counterpart (28) is feasible (i.e., \(X_{RC}^{IRR} \neq \emptyset \)) if and only if all the instances of problem (27) associated with the traffic matrices in \(U \) are feasible. Moreover, objective functions MaxLU and MeanLU are convex, and thus so is their sum. Beacuse every convex function also is quasiconvex, from Corollary 2 and Proposition 10, we obtain the following.

**Proposition 11.** Any robust (strictly/weakly) efficient point to problem (26) is a (strictly/weakly) efficient point to at least one of the instances of problem (26) for a given traffic matrix \(t \).

Several types of uncertainty sets lead to computationally tractable representations of robust counterpart (28) which, by Theorem 3, enables the generation of the robust (strictly/weakly) efficient solutions to uncertain biobjective optimization problem (27). Ben-Tal et al. (2009) showed that when the uncertainty set \(U \) is represented by a polyhedron, an ellipsoid, or more generally through some conic representation of a perturbation set, then capacity constraints (29) would reduce to explicit systems of linear inequalities, conic quadratic inequalities, or linear matrix inequalities. The type of uncertainty set that appears the most pertinent for the intradomain robust routing problem is the polyhedral uncertainty set, that is,

\[
U = \{t \in \mathbb{R}^{|K|} : Rt \leq d, t \geq 0\},
\]

where \(R \) is an \(s \times |K| \) matrix and \(d \) is an \(s\)-dimensional vector, although the upcoming developments could be derived for ellipsoidal uncertainty. In fact, the polyhedral uncertainty set for the design, operation, and management of virtual private networks was first introduced by Ben-Ameur and Kerivin (2003) (see also Ben-Ameur and Kerivin (2005)) and has since become one of the most common ways to represent traffic uncertainty in the telecommunication industry. The polyhedral uncertainty model of Ben-Ameur and Kerivin (2003) has replaced the long-time considered deterministic traffic matrix \(\bar{t} = [\bar{t}_1 \ \ldots \ \bar{t}_{|K|}]^T \), where for each \(k \in K \), \(\bar{t}_k = \max\{t_k : t \in U\} \). Each component of the traffic matrix \(\bar{t} \) represents the worst-case scenario in terms of volume of traffic for each corresponding OD-pair, and considering \(\bar{t} \) in a robust routing problem would fit within Soyster’s approach described in Section 2. (It is important to note that \(\bar{t} \) does not generally belong to the traffic-matrix set \(U \).)
Let \((\bar{x}, \bar{p})\) be in \((\mathbb{R}^{|A|})^{|K|} \times \mathbb{R}^{|A|}\). For each arc \(a \in A\), the worst-case value of the left-hand-side of capacity constraints (29) is the optimal value of the following linear-optimization problem
\[
V_a(\bar{x}, \bar{p}) = \max\{\bar{s}_a t : R t \leq d, t \geq 0\} - \bar{p}_a c_a,
\]
whose dual problem is
\[
W_a(\bar{x}, \bar{p}) = \max\{z_a d : z_a R \geq \bar{s}_a, z_a \geq 0\} - \bar{p}_a c_a,
\]
where \(z_a\) is the \(s\)-dimensional row vector of dual variables associated with arc \(a\). Both \(V_a(\bar{x}, \bar{p})\) and \(W_a(\bar{x}, \bar{p})\) are the optimal objective values of the linear-optimization problems that form a strong-dual pair. By Propositions 7, 8, and 9, the feasible set \(X_{IR}^{1RR}\) to robust counterpart (28) is the projection onto the original variables \((x, p)\) of the feasible set, denoted \(X_{E}^{1RR}\), to biobjective linear-optimization problem
\[
\min_{x, p, z} \begin{bmatrix} f_1(p) & f_2(p) \end{bmatrix}^T
\text{s.t.} \begin{align*}
M x^k &= b^k & \text{for } k \in K, \\
z_a d - p_a c_a &\leq 0 & \text{for } a \in A, \\
x_a - z_a R &\leq 0 & \text{for } a \in A, \\
z_a &\geq 0 & \text{for } a \in A, \\
0 &\leq x^k & \text{for } k \in K, \\
0 &\leq p & \leq 1.
\end{align*}
\]

Notice that all the constraints of problem (31) are certain, that is, they do not explicitly address the uncertain traffic matrix \(t\). The size of reformulation (31) now is polynomial in \(|A|, |K|\), and \(s\). Applying Theorem 3 enables us to obtain the robust (strictly/weakly) efficient sets to uncertain intradomain robust routing problem (27).

It is well-known that a flow represented by arc values can be decomposed into a flow represented by path values (and possibly some cycle values). This flow decomposition permits us to obtain the (strictly/weakly) efficient robust routing paths for each OD-pair, but it does not allow us to handle routing constraints (e.g., hop constraints) which may be relevant in practice. To model the multicommodity flow problem, an alternative formulation based on path-flow variables, known as the arc-path formulation, could be considered (Ahuja et al. 1993). For each commodity \(k\) of \(K\), let \(P(k)\) be the set of all the elementary paths from \(o_k\) to \(d_k\) in \(D\). Besides the \(x^k_a\)-variables previously introduced, for each path \(p\) of \(P(k)\), let \(x_p \in [0, 1]\) represent the proportion of flow associated with commodity \(k\) and carried on path \(p\). Flow-conservation constraints (24) could be replaced by the constraints
\[
\sum_{p \in P(k)} x_p = 1 & \quad \text{for } k \in K, \\
\sum_{p \in P(k)} x_p - x^k_a = 0 & \quad \text{for } a \in A \text{ and } k \in K, \\
together with
0 &\leq x^p & \text{for } p \in P(k), k \in K.
\]

Note that the flow variables \(x^k_a\) are not necessary in the arc-path formulation, yet we keep them for the sake of clarity and simplicity when dealing with capacity constraints (29). Despite the significantly increased number of variables, capacity constraints (29) remain the only ones with uncertainty. Therefore, as long as some column-generation technique can be applied to efficiently manage the large number of \(x_p\)-variables, an arc-path formulation for the intradomain robust routing problem can be handled similarly to problem (27).

Obviously, problem (27) and its transformation to problem (31) is not limited to the objective functions MaxLU and MeanLU, and we could have considered any other objective functions, even nonlinear or non-convex ones. For instance, we could have minimized the routing cost as in Ben-Ameur and Kerivin (2005) or the end-to-end path queuing delay as in Casas (2010). The strength of our approach lies in the fact that the reformulation of robust counterpart (28) into deterministic problem (31) does not depend on the type or number of objective functions.
3.3 Solution method

Many methods have been proposed for solving multiobjective optimization problems (Ehrgott 2005). For biobjective problems with linear constraints and objectives, one of the most effective and simplest methods consists of applying the parametric simplex algorithm (Ehrgott 2005). To linearize the MaxLU objective, we introduce the variable \( p_{max} \in [0, 1] \), the constraints

\[
p_a - p_{max} \leq 0 \quad \text{for } a \in A,
\]

and substitute \( f_1'(p_{max}) = p_{max} \) for the first objective \( f_1(p) = \max\{p_a : a \in A\} \). We therefore obtain the following biobjective linear optimization problem

\[
\min_{(x, p, z, p_{max})} \left \{ p_{max} \mid \sum_{a \in A} p_a = T : (x, p, z) \text{ feasible to (31), } (p, p_{max}) \text{ satisfies (32), and } p_{max} \in [0, 1] \right \}.
\]

(33)

It is straightforward that the projection of the feasible-solution set to problem (33) onto \((x, p, z)\) is nothing but the feasible-solution set to problem (31). This immediately implies that the (strictly/weakly) efficient solutions to the latter problem can be obtained through solving the former one.

In multiobjective linear optimization problems, all efficient solutions have the same property of being properly efficient. Different definitions of proper efficiency were introduced by Benson, Borwein, Geoffrion, and Kuhn and Tucker (see Ehrgott (2005)), and all these definitions appear to be equivalent when dealing with multiobjective linear optimization problems. We only define proper efficiency in Geoffrion’s sense (Geoffrion 1968) because it carries the essence of avoiding having to deal with “unfair” efficient solutions for which a marginal gain of some objective could be made arbitrarily large with respect to each of the other objective losses.

**Definition 7.** A feasible solution \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{max})\) to problem (33) is called properly efficient in Geoffrion’s sense if

(i) it is efficient

(ii) there exists a real number \( M > 0 \) such that for every feasible solution \((x, p, z, p_{max})\)

\[
(a) \text{ satisfying } p_{max} < \hat{p}_{max}, \text{ one has } f_2(\hat{p}) < f_2(p) \text{ with } \frac{\hat{p}_{max} - p_{max}}{f_2(p) - f_2(\hat{p})} \leq M,
\]

\[
(b) \text{ or satisfying } f_2(p) < f_2(\hat{p}), \text{ one has } \hat{p}_{max} < p_{max} \text{ with } \frac{f_2(\hat{p}) - f_2(p)}{p_{max} - \hat{p}_{max}} \leq M.
\]

Proper efficiency for the intradomain robust routing problem therefore means that the trade-off between the local performance indicator (i.e., \( p_{max} \)) and the global QoS performance indicator (i.e., \( \frac{1}{|A|} \sum_{a \in A} p_a \)) is bounded from above. From a computational point of view, all the properly efficient solutions of a multiobjective convex optimization problem can be obtained by minimizing a weighted sum of the objectives, provided the weights are positive (Geoffrion 1968). Moreover, for multiobjective linear optimization problems, Isermann (1974) proved that all efficient solutions are properly efficient in Geoffrion’s sense. Consequently, to find all the (properly) efficient solutions to problem (33) we can convert it into the following parametric
linear optimization problem

\[
\min_{x,p,z} \lambda p_{\max} + (1 - \lambda) \frac{1}{|A|} \sum_{a \in A} p_a \\
\text{s.t.} \quad Mx^k = b^k \quad \text{for } k \in K, \\
\quad z_a d - p_a c_a \leq 0 \quad \text{for } a \in A, \\
\quad x_a - z_a R \leq 0 \quad \text{for } a \in A, \\
\quad z_a \geq 0 \quad \text{for } a \in A, \\
\quad p_a - p_{\max} \leq 0 \quad \text{for } a \in A, \\
\quad 0 \leq x^k \leq 1 \quad \text{for } k \in K, \\
\quad 0 \leq p \leq 1, \\
\quad p_{\max} \geq 0,
\]

where \(\lambda\) is a scalar parameter in \([0,1]\). The following theorem summarizes the equivalence relationships between the optimal solutions to parametric linear optimization problem (34) and the (strictly/weakly/properly) efficient solutions to biobjective linear optimization problem (33) (Ehrgott 2005).

**Theorem 4.** Let \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\) be a feasible solution to problem (34).

(i) \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\) is (properly) efficient to problem (33) if and only if there exists \(\lambda > 0\) such that \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\) is an optimal solution to problem (34).

(ii) \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\) is strictly efficient to problem (33) if and only if there exists \(\lambda \geq 0\) such that \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\)

(iii) \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\) is weakly efficient to problem (33) if and only if there exists \(\lambda \geq 0\) such that \((\hat{x}, \hat{p}, \hat{z}, \hat{p}_{\max})\)

The parametric simplex algorithm for biobjective linear optimization problems (Ehrgott 2005) can be used to solve problem (34). This algorithm proceeds, for a fixed value of \(\lambda\), as the classical simplex algorithm (Dantzig 1963) and scans the interval \([0,1]\) to find the critical values of \(\lambda\) (i.e., those which would make a current optimal basis no longer optimal) as follows:

(i) look for an optimal basis for problem (34) with \(\lambda = 1\);

(ii) find the largest value of \(\lambda\), lower than the current one, that would lead to a new basis becoming optimal.

This second step is repeated until no new value of \(\lambda\) is found. The algorithm then outputs a sequence of \(\lambda\)-values \(1 = \lambda_1 > \lambda_2 > \ldots > \lambda_\ell \geq 0\), \(\ell \geq 2\), such that the optimal basis found for \(\lambda = \lambda_k\) remains optimal for \(\lambda \in (\lambda_{k+1}, \lambda_k]\), for all \(k \in \{1, \ldots, \ell - 1\}\). The (weakly) nondominated outcomes to problem (34) can be easily found by computing the images of the (weakly) efficient points in the objective space.

Given an optimal basis \(B_k\) for problem (34) with respect to some given critical value \(\lambda^k\), \(k \in \{1, \ldots, \ell - 1\}\), let \(\mathbf{r}^1\) and \(\mathbf{r}^2\) be the reduced-cost vectors associated with objective functions \(f_1\) and \(f_2\), respectively. If \(N_k\) denotes the set of nonbasic variables associated with \(B_k\), the next critical value \(\lambda_{k+1}\) is obtained by solving

\[
\max \left\{ \frac{r_j^2}{r_j^1} : j \in I^k \right\},
\]

where \(I^k = \{ j \in I^k : r_j^2 < 0, r_j^1 \geq 0 \}\).

In the next section, we present the numerical work of solving biobjective intradomain robust routing problem (26) where associated robust counterpart (34) is solved using the preceding parametric simplex algorithm. In particular, we discuss the data, the numerical issues encountered, and the results obtained. We also show how the results can support a decision making process in the telecommunication industry.
4 Computational Experience

4.1 Network Data

In our experiments we use the Abilene Network, the Internet2 high-performance backbone network (Figure 1). Internet2 is a consortium led by more than 200 American universities that collaborate with research, academic, industrial, and governmental institutions in the USA and over 50 countries. Internet2 aims at developing “breakthrough technologies that support the most exacting applications of today—and spark the most essential innovations of tomorrow” (Abilene 2003). The goal of the Abilene Network is to serve as the primary high-bandwidth backbone for Internet2. For our experiments, we consider its topology and capacities as of April 2003. The Abilene Network is composed of 12 Points of Presence (PoPs) across the USA connected by 15 high-speed optical links (i.e., 14 OC-192\(^1\) lines and 1 OC-48\(^2\) line). (Note that Atlanta has two PoPs with the OC-48 line connecting it to Indianapolis.) The Abilene Network then corresponds to a directed graph \(D = (V, A)\), having 12 vertices and 30 arcs.

![Figure 1: Abilene network.](image)

All the possible OD-pairs among the 12 PoPs are considered, which gives a set \(K\) composed of 132 commodities to route through the Abilene Network. Because these commodities represent the aggregation of traffic from each origin to each destination, they allow for the modeling of paths in the network for each origin-destination pair. The volumes for the commodities are not precisely known, yet 24 real observational data sets are compiled by Zhang and are available online (Zhang 2004). This Abilene Network traffic data consists of sampled commodities’ volumes collected by the network protocol Netflow from the Abilene Observatory (Abilene 2003). In the data set, Zhang provides a routing matrix for the Abilene Network which corresponds to the coefficient matrix \(R\) in the linear system defining the uncertainty set \(U\) as in definition (30) (Zhang 2004). The routing matrix has \(|A| = 30\) rows and \(|K| = 132\) columns. The right-hand-side vector \(d\) defining \(U\) is an \(|A|\)-dimensional vector whose components represent the largest observed volume of traffic carried on each arc as reported by Zhang (2004). Because the linear system \(Rt \leq d\) is underdetermined (i.e., has more columns than rows), and so it allows several solutions, a “best” traffic-configuration vector is identified using gravity or tomogravity methods (Casas 2010, Zhang et al. 2003 and the references therein). Considering the uncertainty set \(U\), as in definition (30), our approach allows us to handle all the possible traffic-configuration vectors in the routing problem rather than relying on statistical-based arguments (i.e., to select a “best” traffic-configuration vector).

\(^{1}\)Optical Carrier with a data rate of 9953.28 Mbit/s, that is, 10Gbit/s

\(^{2}\)Optical Carrier with a data rate of 2488.32 Mbit/s, that is, 2.5 Gbit/s
4.2 Numerical difficulties

Several difficulties appeared during the computational work. First, parametric linear optimization problem (34) is a highly degenerate linear program having 3960 \( x \)-variables, 30 \( t \)-variables, and 900 \( z \)-variables for a total of 4890 variables. For some test cases, there were at least 200 bases representing the same basic feasible solution. A high level of numerical precision was required to insure that the degeneracy did not lead to cycling.

Second, note that the decision space of problem (34) has dimension of almost 5000, but the objective space has dimension of 2. In effect, multiple basic feasible solutions of parametric linear optimization problem (34) are mapped to the same Pareto outcomes in the objective space. This phenomenon is known as collapsing between the decision space and the objective space (Dauer 1987, Sebo 1981). Computationally, as the algorithm visits adjacent efficient extreme points in the decision space, it may be stationary in the objective space.

Third, different weights \( \lambda \) in parametric linear optimization problem (34) yield the same Pareto outcomes of biobjective linear optimization problem (31), which we discuss below.

4.3 Results

The routing matrix \( R \) associated with the network and the vectors \( d \) associated with each of the 24 data sets (Zhang 2004) were applied to parametric linear optimization problem (34) and solutions were computed using the CPLEX 12.0 solver (IBM 2012). Figure 2 presents the Pareto outcomes of biobjective linear optimization problem (31) obtained for seven data sets from among the 24 sets that are also robust Pareto outcomes of uncertain biobjective optimization problem (27). For each data set, the robustness of each Pareto outcome is reflected in the fact that this outcome remains feasible for any realizable traffic demand. Note that the utilization on the network is highly dependent on the traffic demands of a given data set. For example, comparing data sets 1 and 4, we see that the largest maximum utilization for the robust Pareto outcomes vary from around 0.45 to 1.

![Figure 2: Robust Pareto outcomes in the objective space for seven data sets.](image)

Table 1 displays the results obtained for selected data sets. The second column of Table 1 presents the unique Pareto outcomes of biobjective linear optimization problem (31). The third and fourth columns present the \( \lambda \)-interval over which that outcome resulted as the image of the optimal solutions of parametric linear optimization problem (34). Note that some of the \( \lambda \)-intervals are much larger than others. Consider,
for example, the intervals † and ‡ associated with data set 4. The large interval $[0.934185, 0.285714]$ and the small interval $[0.276217, 0.25]$ are associated with two adjacent Pareto outcomes. In Subsection 4.4, we explore what changes in the network caused the algorithm to move from a solution that is preferred for the largest $\lambda$-interval, †, to a solution preferred for the smallest $\lambda$-interval, ‡.

<table>
<thead>
<tr>
<th>Data set 1</th>
<th>$(f_1(p_{max}), f_2(p))$</th>
<th>First Generating $\lambda$</th>
<th>Last Generating $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(0.313641, 0.177845)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.313364, 0.168258)</td>
<td>0.838485</td>
<td>0.333333</td>
</tr>
<tr>
<td></td>
<td>(0.34961, 0.150273)</td>
<td>0.332526</td>
<td>0.302326</td>
</tr>
<tr>
<td></td>
<td>(0.36249, 0.144692)</td>
<td>0.299791</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(0.381242, 0.138441)</td>
<td>0.249954</td>
<td>0.142857</td>
</tr>
<tr>
<td></td>
<td>(0.386171, 0.13762)</td>
<td>0.141295</td>
<td>0.0322558</td>
</tr>
<tr>
<td></td>
<td>(0.416566, 0.136597)</td>
<td>0.0321839</td>
<td>0</td>
</tr>
<tr>
<td>Data set 4</td>
<td>(0.859874, 0.485294)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.859874, 0.432658)</td>
<td>0.934185</td>
<td>0.285714†</td>
</tr>
<tr>
<td></td>
<td>(0.950841, 0.396271)</td>
<td>0.276217</td>
<td>0.25‡</td>
</tr>
<tr>
<td></td>
<td>(1.0, 0.379885)</td>
<td>0.249606</td>
<td>0</td>
</tr>
<tr>
<td>Data set 10</td>
<td>(0.725824, 0.368028)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.725824, 0.302472)</td>
<td>0.970195</td>
<td>0.333333</td>
</tr>
<tr>
<td></td>
<td>(0.748441, 0.291163)</td>
<td>0.3325</td>
<td>0.230769</td>
</tr>
<tr>
<td></td>
<td>(0.752553, 0.289929)</td>
<td>0.23011</td>
<td>0.210526</td>
</tr>
<tr>
<td></td>
<td>(0.752621, 0.289911)</td>
<td>0.210317</td>
<td>0.142857</td>
</tr>
<tr>
<td></td>
<td>(0.763265, 0.288137)</td>
<td>0.14282</td>
<td>0.0322524</td>
</tr>
<tr>
<td></td>
<td>(0.789697, 0.287256)</td>
<td>0.0321475</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: Robust Pareto outcomes and the corresponding weights $\lambda$ for selected data sets.

We have assumed, to this point, that all of the arcs are using the full capacity. Now we explore the solutions in which only a fixed proportion of the capacity, but the same proportion across all arcs in the network, is available for all arcs. Figures 3, 4, and 5 show the robust Pareto outcomes when the network capacity has been reduced to 90%, 70%, and 50%, respectively. Note that as the available capacity decreases, the overall utilization of the network increases because the Pareto outcomes move in the upper-right direction. Data sets 4 and 10 are not shown in Figure 4 due to infeasibility. Similarly, data sets 4, 7, 8, and 10 are not shown in Figure 5 due to infeasibility.

![Figure 3: Robust Pareto outcomes with 90% of available capacity.](image-url)
Finally, we explore the solutions in which only a fixed proportion of the capacity, but possibly a different proportion for each arc in the network, is available for each arc. These capacities are generated as uniform random variables in the interval $(0.5, 0.9)$. The same vector of random proportions is used for all data sets to allow for accurate comparisons. Figure 6 shows the robust Pareto outcomes when the network capacity has been reduced by a random amount between 50% and 90%. Data sets 4 and 10 are not shown in Figure 6 due to infeasibility.
Figure 6: Robust Pareto outcomes in the objective space with random installed capacities.

Comparing Figures 4 and 6 for all data sets, note that except for data set 14, randomly decreasing the network capacity by 50% to 90% is approximately the same as reducing the capacity by 70%.

Numerical results for data set 1 with varying proportions of available capacity are presented in Table 2. Figure 7 depicts the robust Pareto outcomes reported in Table 2.

Figure 7: Robust Pareto outcomes in the objective space for data set 1 with varied capacities.

Data sets 4 and 10, which were included in Table 1, are not evaluated here because neither data set is feasible if the available capacity is reduced below 90%. As expected, the efficient link utilizations increase as the arc capacity decreases. It is interesting to note that the λ-intervals are fairly consistent regardless of the available capacity. Further, as expected, the Pareto outcomes for the capacity reduced to 70% are approximately equal to the Pareto outcomes for the randomly reduced capacity.
Table 2: Robust Pareto outcomes and the corresponding weights $\lambda$ for data set 1 with varying available capacity.

<table>
<thead>
<tr>
<th>Capacity</th>
<th>$(f_1(p_{max}), f_2(p))$</th>
<th>First Generating $\lambda$</th>
<th>Last Generating $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90% Capacity</td>
<td>(0.34849, 0.188824)</td>
<td>0.94626</td>
<td>0.342221</td>
</tr>
<tr>
<td></td>
<td>(0.34849, 0.186053)</td>
<td>0.325289</td>
<td>0.302326</td>
</tr>
<tr>
<td></td>
<td>(0.388456, 0.16697)</td>
<td>0.301727</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(0.402766, 0.160769)</td>
<td>0.243981</td>
<td>0.142857</td>
</tr>
<tr>
<td></td>
<td>(0.420797, 0.152911)</td>
<td>0.136704</td>
<td>0.0322581</td>
</tr>
<tr>
<td></td>
<td>(0.463174, 0.151774)</td>
<td>0.0292873</td>
<td></td>
</tr>
<tr>
<td>70% Capacity</td>
<td>(0.448058, 0.243269)</td>
<td>1</td>
<td>0.346684</td>
</tr>
<tr>
<td></td>
<td>(0.448058, 0.240368)</td>
<td>0.960752</td>
<td>0.302326</td>
</tr>
<tr>
<td></td>
<td>(0.499443, 0.214676)</td>
<td>0.329729</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(0.517843, 0.206703)</td>
<td>0.301183</td>
<td>0.142857</td>
</tr>
<tr>
<td></td>
<td>(0.544632, 0.197773)</td>
<td>0.14269</td>
<td>0.0322326</td>
</tr>
<tr>
<td></td>
<td>(0.551673, 0.1966)</td>
<td>0.031855</td>
<td></td>
</tr>
<tr>
<td>50% Capacity</td>
<td>(0.627282, 0.385938)</td>
<td>0.839</td>
<td>0.333333</td>
</tr>
<tr>
<td></td>
<td>(0.627282, 0.336516)</td>
<td>0.298246</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(0.72498, 0.289384)</td>
<td>0.249813</td>
<td>0.142857</td>
</tr>
<tr>
<td></td>
<td>(0.762484, 0.276882)</td>
<td>0.137427</td>
<td>0.0322581</td>
</tr>
<tr>
<td></td>
<td>(0.83712, 0.273194)</td>
<td>0.029753</td>
<td></td>
</tr>
<tr>
<td>Random Capacity</td>
<td>(0.475806, 0.245032)</td>
<td>1</td>
<td>0.369371</td>
</tr>
<tr>
<td></td>
<td>(0.475806, 0.224722)</td>
<td>0.968735</td>
<td>0.308736</td>
</tr>
<tr>
<td></td>
<td>(0.483856, 0.2204)</td>
<td>0.335143</td>
<td>0.248622</td>
</tr>
<tr>
<td></td>
<td>(0.560568, 0.212953)</td>
<td>0.307935</td>
<td>0.149826</td>
</tr>
<tr>
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<td>(0.560786, 0.210806)</td>
<td>0.248378</td>
<td>0.129559</td>
</tr>
<tr>
<td></td>
<td>(0.54096, 0.204873)</td>
<td>0.149536</td>
<td>0.118624</td>
</tr>
<tr>
<td></td>
<td>(0.553137, 0.203061)</td>
<td>0.129451</td>
<td>0.0156654</td>
</tr>
<tr>
<td></td>
<td>(0.562155, 0.201847)</td>
<td>0.118538</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.562793, 0.201837)</td>
<td>0.0152977</td>
<td></td>
</tr>
</tbody>
</table>

4.4 Analysis

Recall the intervals $\dagger$ and $\ddagger$ associated with data set 4 in Table 1, [0.934185, 0.285714] and [0.276217, 0.25], respectively, which are associated with two adjacent robust Pareto outcomes. This change in efficient solutions is characterized in the network by both gains and losses of utilization, not necessarily in equal amounts. Table 3 reports these robust Pareto outcomes and the related changes of arc utilization $p$ associated with each arc in the network. Utilization was lost on the Seattle to Denver cycle (arcs 10 and 27), the Atlanta-1 to Washington D.C. link (arc 5), the Denver to Kansas City link (arc 8), and the Houston to Atlanta-1 link (arc 11). Utilization was gained on the Sunnyvale to Denver cycle (arcs 9 and 24), the Chicago to Indianapolis link (arc 6), and the New York to Washington D.C link (arc 23).
Comparing the arc utilizations for the two robust Pareto outcomes, we observe that for the Pareto outcome associated with the largest \( \lambda \)-interval all utilizations are in the interval \((0.58, 0.86)\). Comparatively, the utilizations of the Pareto outcome associated with the smallest \( \lambda \)-interval are in the interval \((0, 0.96)\). Due to the robustness, these utilizations guarantee that the network will be operational for every realizable traffic demand. However, the first network utilization plan is more balanced and may be preferred over the other.

Consider the robust Pareto outcome \((f'_1(p_{max}), f'_2(p)) = (1, 0.379885)\) obtained from data set 4 from \( \lambda \)-interval \([0.249606, 0]\). Table 4 reports the arc utilization \(p\) associated with each arc in the network.

<table>
<thead>
<tr>
<th>((f'<em>1(p</em>{max}), f'_2(p)))</th>
<th>((0.859874, 0.432658))</th>
<th>((.950841, 0.396271))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p[5]) = 0.727737</td>
<td>(p'[5]) = 0</td>
<td></td>
</tr>
<tr>
<td>(p[6]) = 0.859874</td>
<td>(p'[6]) = 0.950841</td>
<td></td>
</tr>
<tr>
<td>(p[8]) = 0.584289</td>
<td>(p'[8]) = 0.379613</td>
<td></td>
</tr>
<tr>
<td>(p[9]) = 0.859874</td>
<td>(p'[9]) = 0.950841</td>
<td></td>
</tr>
<tr>
<td>(p[10]) = 0.859874</td>
<td>(p'[10]) = 0.746165</td>
<td></td>
</tr>
<tr>
<td>(p[11]) = 0.588257</td>
<td>(p'[11]) = 0.383581</td>
<td></td>
</tr>
<tr>
<td>(p[23]) = 0.859874</td>
<td>(p'[23]) = 0.950841</td>
<td></td>
</tr>
<tr>
<td>(p[24]) = 0.859874</td>
<td>(p'[24]) = 0.950841</td>
<td></td>
</tr>
<tr>
<td>(p[27]) = 0.596986</td>
<td>(p'[27]) = 0.39231</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Arc utilizations for adjacent robust Pareto outcomes for data set 4.

Based on the arc utilization here, the Atlanta-2 to Atlanta-1 link (arc 2), the Atlanta-1 to Washington D.C. link (arc 5), the Denver to Sunnyvale link (arc 9) and the New York to Washington D.C. link (arc 23) emerge as the most important because they are either at their full capacity or totally unused. These arcs are shown in Figure 8. The arcs at full capacity would be critical arcs to maintain because the loss of one due to failure would have a great effect on network quality. On the other hand, arcs not being used can fail with no penalty.
Finally, as our stated goal is to solve the intradomain robust routing problem and to select one or more paths in $D$ for each commodity to travel, we recall that every point $(f_1(p_{\text{max}}), f_2(p))$ in the objective space has a pre-image $(x, p, z, p_{\text{max}})$ in the decision space that represents the Pareto path(s) in the network for each OD-pair. While Table 4 and Figure 8 only report the values of $p$ for the robust Pareto outcome $(1, 0.379885)$, the vector $x$ from the pre-image can be used to construct the exact path(s) on which a given OD-pair will travel.

5 Conclusion

This paper provides a theoretical study of robust multiobjective optimization by verifying that the techniques of robust single objective optimization remain valid in the case of a vector-valued objective function. In particular, we focused on the relationship of the (weakly/strictly) efficient solutions to the uncertain multiobjective optimization problem and those to its related robust counterpart under the column-wise uncertainty of Soyster (1973) and the row-wise uncertainty of Ben-Tal and Nemirovski (1999).

The developed approach was then applied to Internet routing, in particular, to robust intradomain routing in the presence of an explicit routing protocol. It was necessary to apply robust multiobjective optimization here because of the inaccuracy of estimating Internet traffic, which is caused by the cost of data collection and the complexity of data analysis. The robust intradomain routing problem was formulated as a biobjective multicommodity flow problem by minimizing the maximum utilization of any link and the mean utilization of all links. The biobjective optimization problem was then solved using the parametric simplex algorithm. Finally, we presented the results on an existing network, the Abilene network, and discussed the relevance of the results of the parametric simplex algorithm to decision making.

There are many opportunities to continue this work. These opportunities fall into four categories: the uncertainty set, the multiobjective optimization problem, the solution methodology, and the application. A key assumption of our work is that all of the uncertainty occurs in the constraints. However, it may be of interest to maintain the uncertainty at the level of the objective function. Other types of uncertainty sets may be considered to offer other theoretical results. These types of sets include conic quadratic sets or sets of linear matrix inequalities, as both types of sets have been associated with the necessary concept of duality used in this paper. Additionally, we may endeavor to study different sets of objective functions, including the previously mentioned path end-to-end delay. A natural research extension is to study uncertain multiobjective optimization problems with three or more objective functions. Computationally, the work of Dauer (1987) and others could be incorporated into our approach to identify the extreme points and edges of the efficient set (in the decision space) that are necessary to define the nondominated set (in the objective space), potentially decreasing the complexity of the solution technique in terms of reducing the amount of degeneracy (as the dimension of the decision space may still be quite large). We could also engage in
applications in other areas of robust Internet routing such as interdomain routing or routing with an implicit protocol, or, more generally, other domains of human activity.
References


Pareto, V., Cours d’Économie Politique, Rouge, Lausanne, Switzerland (1896).


