Analysis of a time-dependent fluid-structure interaction problem in an optimal control framework over a single time-step

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Abstract

Fluid-structure interaction simulation presents many computational difficulties, particularly when the densities of the fluid and structure are close. A previous report [20] has suggested that recasting the FSI problem in the context of optimal control may significantly reduce computation time. This report introduces a Neumann type control along with detailed analysis for the stability and existence of an optimal solution for a given time-step. The existence of Lagrangian multipliers is proved, and an optimality system is derived. A gradient based optimization algorithm is then presented with numerical results confirming its effectiveness in computing an accurate solution.

Key words  optimal control, fluid-structure interaction, finite element method

1 Introduction

Fluid-structure interaction (FSI) simulation has important applications ranging from blood flow to micromixing [3, 5, 7, 9, 19, 26, 29, 32, 33]. However, there are many difficulties encountered when performing a FSI simulation. One source of difficulty arises from enforcing the tight coupling between the fluid and structure velocities and stresses along the interface. Another difficulty is that the domains for simulations are deforming in time and in a way that is directly determined by the fluid and structure solutions. A physical example that is easy to imagine is the flow of blood through an artery in which the pulsatile flow deforms the vessel wall.

There are a wide variety of methods for simulating the coupled FSI system, but each is limited by factors including computational complexity and stability. Possibilities include a monolithic formulation of the problem [34], which is computationally complex due to requiring many large matrix solves to converge on a solution to the nonlinear system. Additional difficulties with this method include the development of efficient and appropriate preconditioners for the matrix resulting from the discretized system, although this is currently an active area of research [2, 14].

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Fluid-structure interaction is multidisciplinary in the sense that the domain can be decomposed into two domains, each governed by different model equations. Both implicit and explicit approaches exist for decoupling these systems of equations, permitting use of partitioned solvers which may be more attractive and allow the use of legacy codes.

The most common approaches decouple the fluid and structure subsystems, which allows for operations on a smaller matrix for each subsystem solve. For a partitioned method, there are many options for how to transmit boundary condition information back and forth between the two subsystems. Clearly, an implicit method that iterates until convergence before progressing to the next time step will be more stable than an explicit method which only solves the structure and fluid subsystems once per time step.

For areas such as aerospace engineering, the large difference in densities of the subsystems may permit efficient simulation. However, for blood flow modeling, often the density of the vessel and fluid are nearly identical, causing problems with stability of decoupling algorithms known as the added mass effect [16].

In the presence of the added mass effect in blood flow modeling, explicit iterations fail unless great care is taken to stabilize the algorithm by penalizing spurious pressure oscillations as in [6]. Even in the case of implicit iterations, generally many nonlinear subsystem solves are required and even then stability can not be guaranteed [24]. By relaxing the update to the structure solve in each implicit iteration [8, 21], stability can be achieved. There are methods for dynamically changing the relaxation parameter in order to speed up convergence [16]. The largest problem with using relaxation schemes is that the closer the two densities are in magnitude, the greater the increase in computational complexity because of the additional nonlinear subsystem solves needed.

Ideally, we desire an algorithm that will decouple the fluid and structure subsystems and quickly converge to a solution having an error within some given tolerance. In [20], the FSI problem was recast as an optimal control problem. The optimization problem used a control to attempt to enforce continuity of velocity on the interface while always enforcing continuity of stress. This work modifies that control by attempting to simultaneously enforce continuity of velocity and stress on the interface. This modified control will allow for a rigorous analysis of the optimization problem.

Section 2 describes the model equations and interface conditions, section 2.2 defines notation used in the rest of the report as well as some important bounds, and the Arbitrary Lagrangian Eulerian (ALE) method is introduced in section 2.3. Time discretized weak formulations in section 2.4 are reformulated in section 3 as a constrained optimization problem using a penalized functional. The existence of an optimal solution to a penalized and unpenalized functional, the existence of Lagrange multipliers, and the derivation of an optimality are also presented in section 3. A steepest descent algorithm is presented in subsection 3.8 followed by a numerical experiment section 4. A summary of our findings are in section 5.

2 Model Equations and Notation

2.1 Navier-Stokes - Linear Elastic Model

The fluid-structure interaction we will consider is an incompressible Newtonian fluid and an isotropic linear elastic structure.
Let $\Omega^f_t$ be a bounded moving fluid domain at time $t$ in $\mathbb{R}^2$ with the boundary $\Gamma^f_t$ such that $\Gamma^f_t = \Gamma^f_N \cup \Gamma^f_D \cup \Gamma^f_I$, where $\Gamma^f_I$ is a moving boundary. Also let $\Omega^s$ be a fixed structure domain with the boundary $\Gamma^s$ such that $\Gamma^s = \Gamma^s_N \cup \Gamma^s_D \cup \Gamma^s_I$, where $\Gamma^s_I$ is the movable fluid boundary at time 0. Consider the system of fluid and structure equations

\begin{align*}
\rho_f \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] - 2\nu_f \nabla \cdot D(\mathbf{u}) + \nabla p &= f_f \quad \text{in } \Omega^f_t, \quad (1) \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega^f_t, \quad (2) \\
\rho_s \frac{\partial^2 \eta}{\partial t^2} - 2\nu_s \nabla \cdot D(\eta) - \lambda \nabla (\nabla \cdot \eta) &= f_s \quad \text{in } \Omega^s, \quad (3)
\end{align*}

where $\mathbf{u}$ denotes the velocity vector of fluid, $p$ the pressure of fluid, $\rho_f$ the density of the fluid, $\nu_f$ the fluid viscosity, $\eta$ the displacement of structure, and $\rho_s$ the structure density. In (1) and (3), $D(\cdot)$ is the rate of the strain tensor, i.e., $D(\mathbf{v}) := (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2$. The Lamé parameters are denoted by $\nu_s$ and $\lambda$, and the body forces are denoted by $f_f$ and $f_s$. Initial and boundary conditions for $\mathbf{u}$ and $\eta$ are given as follows:

\begin{align*}
2\nu_f D(\mathbf{u}) \mathbf{n}_f - p \mathbf{n}_f &= \mathbf{u}_N \quad \text{on } \Gamma^f_N, \quad (4) \\
\mathbf{u} &= \mathbf{u}_D \quad \text{on } \Gamma^f_D, \quad (5) \\
2\nu_s D(\eta) \mathbf{n}_s + \lambda (\nabla \cdot \eta) \mathbf{n}_s &= \eta_N \quad \text{on } \Gamma^s_N, \quad (6) \\
\eta &= 0 \quad \text{on } \Gamma^s_D, \quad (7) \\
\mathbf{u}(x, 0) &= \mathbf{u}_0 \quad \text{in } \Omega^f_0, \quad (8) \\
\eta(x, 0) &= \eta_0 \quad \text{in } \Omega^s, \quad (9) \\
\eta_t(x, 0) &= \dot{\eta}_0 \quad \text{in } \Omega^s, \quad (10)
\end{align*}

where $\dot{\eta}_0 = \dot{\eta}_0$. For brevity, we use $\mathbf{u} = 0$ on $\Gamma^f_D$ but all our results hold for the case where $\mathbf{u} = \mathbf{u}_D \neq 0$ on $\Gamma^f_D$ with a simple modification. The moving boundary $\Gamma^f_I$ is determined by the displacement $\eta$ at time $t$ (Fig. 1). The interface conditions between the fluid and the structure are obtained by enforcing continuity of the velocity and the stress force:

\begin{align*}
\frac{\partial \eta}{\partial t} &= \mathbf{u} \quad \text{on } \Gamma^f_I, \quad (11) \\
2\nu_f D(\mathbf{u}) \mathbf{n}_f - p \mathbf{n}_f &= -2\nu_s D(\eta) \mathbf{n}_s + \lambda (\nabla \cdot \eta) \mathbf{n}_s \quad \text{on } \Gamma^f_I. \quad (12)
\end{align*}
2.2 Notation

We use the Sobolev spaces $W^{m,p}_D$ with norms $\| \cdot \|_{m,p,D}$ if $p < \infty$, $\| \cdot \|_{m,\infty,D}$ if $p = \infty$. Denote the Sobolev space $W^{m,2}_D$ by $H^m_D$ with the norm $\| \cdot \|_{m,D}$. The corresponding space of vector-valued or tensor-valued functions is denoted by $H^m_D$.

For the variational formulation of the flow equations (26)-(27) in the ALE framework, described in section 2.3, we define the function space for the reference domain:

$$H^1_D(\Omega^f_{t_0}) := \{ v \in H^1(\Omega^f_{t_0}) : v = 0 \text{ on } \Gamma^f_{t_0} \}. $$

The function spaces for $\Omega^f_t$ are then defined as

$$H^1_D(\Omega^f_t) := \{ v : \Omega^f_t \times [0,T] \rightarrow \mathbb{R}^2, v = \nabla \circ \Psi^{-1}_t \text{ for } \nabla \in H^1_D(\Omega^f_{t_0}) \},$$

$$L^2(\Omega^f_t) := \{ q : \Omega^f_t \times [0,T] \rightarrow \mathbb{R}, q = \tilde{q} \circ \Psi^{-1}_t \text{ for } \tilde{q} \in L^2(\Omega^f_{t_0}) \}.$$

where $\Psi^{-1}_t$ is the inverse ALE mapping described in section 2.3.

For the structure displacement $\eta$, define the function space

$$H^1_D(\Omega^s) := \{ \xi \in H^1(\Omega^s) : \xi = 0 \text{ on } \Gamma^s_D \}. $$

We use $\langle \cdot, \cdot \rangle_{\Omega^f_t}$, $\langle \cdot, \cdot \rangle_{\Gamma^f_t}$, $\langle \cdot, \cdot \rangle_{\Omega^s}$, and $\langle \cdot, \cdot \rangle_{\Gamma^s_{t_0}}$ to denote the $L^2$ inner product over $\Omega^f_t$, $\Gamma^f_t$, $\Omega^s$, and $\Gamma^s_{t_0}$, respectively.

In the moving fluid domain, we define the bilinear and trilinear forms

$$a(u,v)_{\Omega^f_t} = \frac{1}{4} \int_{\Omega^f_t} (\nabla u + (\nabla u)^T) : (\nabla v + (\nabla v)^T) \, d\Omega^f_t \quad \forall \, u, v \in H^1_D(\Omega^f_t),$$

$$b(v,q)_{\Omega^f_t} = -\int_{\Omega^f_t} q(\nabla \cdot v) \, d\Omega^f_t \quad \forall \, v \in H^1_D(\Omega^f_t), \, q \in L^2(\Omega^f_t),$$

and

$$c(u,v,w)_{\Omega^f_t} = \frac{1}{2} \int_{\Omega^f_t} u \cdot \nabla v \cdot w - u \cdot \nabla w \cdot v \, d\Omega^f_t.$$

For the stationary structure domain, we define the bilinear forms

$$d(\eta,\gamma)_{\Omega^s} = \frac{1}{4} \int_{\Omega^s} (\nabla \eta + (\nabla \eta)^T) : (\nabla \gamma + (\nabla \gamma)^T) \, d\Omega^s \quad \forall \, \eta, \gamma \in H^1_D(\Omega^s),$$

and

$$e(\eta,\gamma)_{\Omega^s} = \int_{\Omega^s} (\nabla \cdot \eta)(\nabla \cdot \gamma) \, d\Omega^s \quad \forall \, \eta, \gamma \in H^1_D(\Omega^s).$$

It is noteworthy that

$$c(u,v,w)_{\Omega^f_t} = 0 \quad \forall \, u,v \in H^1_D(\Omega^f_t). \quad (13)$$
Throughout this report, \( C \) represents a positive constant independent of time. As is well known, \( a(\cdot, \cdot), b(\cdot, \cdot), c(\cdot, \cdot, \cdot) \) and \( d(\cdot, \cdot) \) are continuous and there exist constants \( C_1, C_2, C_3, C_4 \) and \( C_5 \) such that

\[
|a(u, v)_{\Omega_t^f}| \leq C_1 \|u\|_{1,\Omega_t^f} \|v\|_{1,\Omega_t^f} \quad \forall u, v \in H^1_0(\Omega_t^f),
\]

\[
|b(v, q)_{\Omega_t^f}| \leq C_2 \|v\|_{1,\Omega_t^f} \|q\|_{0,\Omega_t^f} \quad \forall v \in H^1_0(\Omega_t^f), \forall q \in L^2(\Omega_t^f),
\]

\[
|c(u, v, w)_{\Omega_t^f}| \leq C_3 \|u\|_{1,\Omega_t^f} \|v\|_{1,\Omega_t^f} \|w\|_{1,\Omega_t^f} \quad \forall u, v, w \in H^1_0(\Omega_t^f),
\]

\[
|d(\eta, \gamma)_{\Omega^*}| \leq C_4 \|\eta\|_{1,\Omega^*} \|\gamma\|_{1,\Omega^*} \quad \forall \eta, \gamma \in H^1_0(\Omega^*),
\]

and

\[
|e(\eta, \gamma)_{\Omega^*}| \leq C_5 \|\eta\|_{1,\Omega^*} \|\gamma\|_{1,\Omega^*} \quad \forall \eta, \gamma \in H^1_0(\Omega^*).
\]

There exist positive coercivity constants \( C_6 \) and \( C_7 \) such that

\[
a(u, u)_{\Omega_t^f} \geq C_6 \|u\|^2_{1,\Omega_t^f} \quad \forall u \in H^1_0(\Omega_t^f),
\]

\[
d(\eta, \eta)_{\Omega^*} \geq C_7 \|\eta\|^2_{1,\Omega^*} \quad \forall \eta \in H^1_0(\Omega^*),
\]

and \( b(\cdot, \cdot) \) has the inf-sup condition

\[
\sup_{0 \neq q \in H^1_0(\Omega_t^f)} \frac{b(v, q)_{\Omega_t^f}}{\|v\|_{1,\Omega_t^f}} \geq C_8 \|q\|_{0,\Omega_t^f} \quad \forall q \in L^2(\Omega_t^f)
\]

where \( C_8 \) is a positive constant.

### 2.3 Arbitrary Lagrangian Eulerian Framework

The Arbitrary Lagrangian Eulerian (ALE) [10] method is one of the most widely used numerical schemes in simulating fluid flows in a moving domain. In the ALE formulation, a one-to-one coordinate transformation is introduced for the fluid domain, and the fluid equations can be rewritten with respect to a fixed reference domain. Specifically, we define the time-dependent bijective mapping \( \Psi_t \) which maps the reference domain \( \Omega_0 \) to the physical domain \( \Omega_t \):

\[
\Psi_t : \Omega_0 \rightarrow \Omega_t, \quad \Psi_t(y) = x(y, t),
\]

where \( y \) and \( x \) are the spatial coordinates in \( \Omega_0 \) and \( \Omega_t \), respectively. The coordinate \( y \) is often called the ALE coordinate. Using \( \Psi_t \), the weak formulation of the flow equations in \( \Omega_t \) can be recast into a weak formulation defined in the reference domain \( \Omega_0 \). Thus, the model equations in the reference domain can be considered for numerical simulation and the transformation function \( \Psi_t \) needs to be determined at each time step as a part of computation.

For a function \( \phi : \Omega_t \times [0, T] \rightarrow \mathbb{R} \), its corresponding function \( \overline{\phi} = \phi \circ \Psi_t \) in the ALE setting is defined as

\[
\overline{\phi} : \Omega_0 \rightarrow \mathbb{R}; \quad \overline{\phi}(y, t) = \phi(\Psi_t(y), t).
\]

The time derivative in the ALE frame is also given as

\[
\frac{\partial \phi}{\partial t} |_{y} : \Omega_t \times [0, T] \rightarrow \mathbb{R}, \quad \frac{\partial \phi}{\partial t} \mid_{y} (x, t) = \frac{\partial \overline{\phi}}{\partial t}(y, t).
\]
Using the chain rule, we have
\[ \frac{\partial \phi}{\partial t} \bigg|_y = \frac{\partial \phi}{\partial t} \bigg|_x + z \cdot \nabla_x \phi, \] (25)
where \( z := \frac{\partial x}{\partial t} \bigg|_y \) is the domain velocity. In (25) \( \frac{\partial \phi}{\partial t} \bigg|_y \) is the so-called ALE derivative of \( \phi \).

The flow equations (1)-(2) can then be written in ALE formulation as
\[ \rho_f \left[ \frac{\partial \mathbf{u}}{\partial t} \bigg|_y + (\mathbf{u} - z) \cdot \nabla_x \mathbf{u} \right] - 2 \nu_f \nabla \cdot D_x(\mathbf{u}) + \nabla_x p = \mathbf{f}_f \quad \text{in } \Omega_T^f, \] (26)
\[ \nabla_x \cdot \mathbf{u} = 0 \quad \text{in } \Omega_T^f, \] (27)
where \( D_x(\mathbf{u}) = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T)/2 \). Note that all spatial derivatives involved in (26)-(27), including the divergence operator, are with respect to \( \mathbf{x} \). Throughout the report we will use \( D_x(\cdot) \) and \( \nabla_x \) only when they need to be clearly specified. Otherwise, \( D(\cdot), \nabla \) will be used as \( D_x(\cdot), \nabla_x \), respectively.

The variational formulation for \((\mathbf{u}, p)\) in ALE framework is given by: find \((\mathbf{u}, p)\) such that
\[ \rho_f \left( \frac{\partial \mathbf{u}}{\partial t} \bigg|_y + (\mathbf{u} - z) \cdot \nabla_x \mathbf{u}, \mathbf{v} \right)_{\Omega_T^f} + 2 \nu_f a(\mathbf{u}, \mathbf{v})_{\Omega_T^f} + b(\mathbf{v}, p)_{\Omega_T^f} - (2 \nu_f D(\mathbf{u}) \cdot n_f - p n_f, \mathbf{v})_{\Gamma_T^f} \]
\[ = (\mathbf{f}_f, \mathbf{v})_{\Omega_T^f} + (\mathbf{u}_N, \mathbf{v})_{\Gamma_N^f} \quad \forall \mathbf{v} \in H^1_D(\Omega_T^f), \] (28)
\[ b(\mathbf{u}, q)_{\Omega_T^f} = 0 \quad \forall q \in L^2(\Omega_T^f). \] (29)

Also, we consider the variational formulation for the linear elastic structure:
\[ \rho_s \left( \frac{\partial^2 \eta}{\partial t^2}, \xi \right)_{\Omega_T^s} + 2 \nu_s d(\eta, \xi)_{\Omega_T^s} + \lambda e(\eta, \xi)_{\Omega_T^s} - (2 \nu_s D(\eta) \cdot n_s + \lambda (\nabla \cdot \eta)n_s, \xi)_{\Gamma_{I_0}} = (\mathbf{f}_s, \xi)_{\Omega_T^s} + (\eta_N, \xi)_{\Gamma_N^s} \quad \forall \xi \in H^1_D(\Omega_T^s), \] (30)
where the interface conditions (11)-(12) are not yet imposed.

In order to define the ALE mapping \( \Psi_t \), we consider the boundary position function \( h : \Gamma_{I_0} \times [0, T] \rightarrow \Gamma_T \). The ALE mapping may be then determined by solving the Laplace equation
\[ \Delta_y \mathbf{x}(y) = 0 \quad \text{in } \Omega_0, \]
\[ \mathbf{x}(y) = h(y) \quad \text{on } \Gamma_{I_0}. \] (31)

This method is called the harmonic extension technique, where the boundary position function \( h \) is extended onto the whole domain [12].

Consider using the Reynolds’ Transport formula [27]
\[ \frac{d}{dt} \int_{V(t)} \phi(x, t) \, dV = \int_{V(t)} \frac{\partial \phi}{\partial t} \bigg|_y + \phi \nabla_x \cdot z \, dV \] (32)
for a function \( \phi : V(t) \rightarrow \mathbb{R} \), where \( V(t) \subset \Omega_t \) such that \( V(t) = \Psi_t(V_0) \) with \( V_0 \subset \Omega_0 \). If \( v \) is a function from \( \Omega_t \) to \( \mathbb{R} \) and \( \mathbf{v} = \nabla \circ \Psi_t^{-1} \) for \( \nabla : \Omega_t \rightarrow \mathbb{R} \), we have that \( \frac{\partial \mathbf{v}}{\partial t} \bigg|_y = 0 \) and therefore
\[ \frac{d}{dt} \int_{\Omega_t} \phi \mathbf{v} \, d\Omega_t = \int_{\Omega_t} \left[ \frac{\partial \phi}{\partial t} \bigg|_y + \phi \nabla_x \cdot z \right] \mathbf{v} \, d\Omega_t. \] (33)
Using (33), (28)-(29) become

\[
\rho_f \frac{d}{dt} (u, v)_{\Omega_t^f} + ((u - z) \cdot \nabla u, v)_{\Omega_t^f} - ((\nabla \cdot z) u, v)_{\Omega_t^f} + 2 \nu_f a(u, v)_{\Omega_t^f} + b(v, p)_{\Omega_t^f} - (2 \nu_f D(u) \cdot n_f - p n_f, v)_{\Gamma_{t^f}} = (f_f, v)_{\Omega_t^f} + (u_N, v)_{\Gamma_{\infty}^N} \forall v \in H^1_D(\Omega_t^f),
\]

(34)

\[
b(u, q)_{\Omega_t^f} = 0 \forall q \in L^2(\Omega_t^f).
\]

(35)

### 2.4 Semi-Discrete Weak Formulations

We will implicitly define \( V(\cdot) \) in the following way:

\[
(f, V(v))_{\Omega_t^f} = (f, v \circ \Psi_t \circ \Psi_t^{-1} \circ \Omega_t^f), (f, V(v))_{\Gamma_{t^f}} = (f, v \circ \Psi_t \circ \Psi_t^{-1} \circ \Gamma_{t^f}),
\]

where \( f \) is a function defined on the domain \( \Omega_t^f \) (or \( \Gamma_{t^f} \)) and \( v \) is a function defined on the domain \( \Omega_t^f \) (or \( \Gamma_{t^f} \)).

Temporal discretization of (34) and (35) by implicit Euler yields

\[
\rho_f \left[ (u^n, v)_{\Omega_t^{n+1}} - (u^{n-1}, v)_{\Omega_t^{n-1}} \right] + \Delta t \rho_f \left[ ((u^n - z^n) \cdot \nabla u^n, v)_{\Omega_t^{n+1}} - ((\nabla \cdot z^n) u^n, v)_{\Omega_t^{n+1}} + 2 \nu_f a(u^n, v)_{\Omega_t^{n+1}} + b(v^n, p^n)_{\Omega_t^{n-1}} \right]
- \Delta t (2 \nu_f D(u^n) \cdot n_f - p^n n_f, v)_{\Gamma_{t^f}}
= \Delta t \left[ (f^n_f, v)_{\Omega_t^{n+1}} + (u^n_N, v)_{\Gamma_{\infty}^N} \right] \forall v \in H^1_D(\Omega_t^{n+1}),
\]

(36)

\[
b(u^n, q)_{\Omega_t^{n+1}} = 0 \forall q \in L^2(\Omega_t^{n+1}).
\]

(37)

A second order time discretization of the structure problem given by

\[
\rho_s \left[ (\eta^n - \eta^{n-1}, \xi)_{\Omega_s} \right] + \Delta t \left[ \nu_s d (\eta^n + \eta^{n-1}, \xi)_{\Omega_s} + \frac{\lambda}{2} c (\eta^n + \eta^{n-1}, \xi)_{\Omega_s} \right] - \Delta t \left( \nu_s (D(\eta^n + \eta^{n-1}) \cdot n_s) + \frac{\lambda}{2} (\nabla \cdot (\eta^n + \eta^{n-1})) n_s, \xi \right)_{\Gamma_{t^f}}
= \frac{\Delta t}{2} \left[ (f_s^n + f_n^{n-1}, \xi)_{\Omega_s} + (\eta_n^n + \eta_n^{n-1}, \xi)_{\Gamma_{\infty}^N} \right] \forall \xi \in H^1_D(\Omega_s),
\]

(38)

\[
\frac{\Delta t}{2} (\eta^n - \eta^{n-1}, \gamma)_{\Omega_s} = (\eta^n - \eta^{n-1}, \gamma)_{\Omega_s} \forall \gamma \in L^2(\Omega_s).
\]

(39)

will be considered in section 3.

The overall order of the time discretization is only first order accurate because of using implicit Euler for the fluid discretization. However, the analytical results can be easily extended to use the Crank-Nicolson time discretization scheme for the fluid instead of implicit Euler, but this was not done in order to keep the fluid equations simpler. Analysis for the structure
subsystem will be developed using the second order time discretization due to a necessity for higher order accuracy, which will be needed later and is explained in Remark 3.6.

Since we have now discretized with respect to time, we define

\[
\Psi_t(y) = \frac{t-t_{n-1}}{\Delta t}\Psi_{t_n}(y) + \frac{t_n-t}{\Delta t}\Psi_{t_{n-1}}(y) \forall t \in [t_{n-1}, t_n].
\]

(40)

Also, please note that as a consequence of (40),

\[
\frac{\partial \Psi_t}{\partial t} = \frac{\Psi_{t_n} - \Psi_{t_{n-1}}}{\Delta t} = z^n \circ \Psi_{t_n} \forall t \in [t_{n-1}, t_n].
\]

(41)

Let us denote \( J_t(y) := \det[\frac{\partial}{\partial y} \Psi_t(y)] \). In the next section, we will make assumptions (14) and (15), as in [4], that gives us \( J_t \) is bounded below by a positive constant \( \kappa_{min} \) and above by a constant \( \kappa_{max} \), for all \( t \in [0, t_n] \). For further information on the mapping regularity condition, please see pp. 19-21 of [25].

3 Optimization Problem

3.1 Weak Formulation of the Constraints

Assuming \( f^n \in H^{-1}(\Omega^f_{t_n}) \), \( g^n \in L^2(\Gamma_{t_n}) \), \( u^n \in L^2(\Omega^f_N) \), \( \eta^n \in L^2(\Gamma^f_N) \), \( u^{n-1} \in H^1_D(\Omega^f_{t_{n-1}}) \), and \( \eta^{n-1} \in H^1_D(\Omega^f_N) \), (36)-(37) can be rewritten as

\[
\rho^f[(u^n, v)_{\Omega^f_{t_n}} - (u^{n-1}, v)_{\Omega^f_{t_{n-1}}}] + \Delta t \rho^f[c(u^n, u^n, v)_{\Omega^f_{t_n}} + \frac{1}{2}((u^n \cdot \eta f)u^n, v)_{\Gamma^f_{t_n}}
\]

\[
+ \frac{1}{2}((u^n \cdot \eta f)u^n, v)_{\Gamma^f_{t_n}} - \frac{1}{2}((z^n \cdot \eta f)u^n, v)_{\Gamma^f_{t_n}} - \frac{1}{2}((\nabla \cdot z^n)u^n, v)_{\Omega^f_{t_n}}
\]

\[
- c(z^n, u^n, v)_{\Omega^f_{t_n}}] + \Delta t 2\nu_f a(u^n, v)_{\Omega^f_{t_n}} + \Delta t [b(v, p^n)_{\Omega^f_{t_n}}
\]

\[
= \Delta t \left( f^n, v \right)_{\Omega^f_{t_n}} + \Delta t \left( u^n, v \right)_{\Gamma^f_{t_n}}
\]

\[
+ \Delta t \left( 2\nu_f D(u^n) \cdot n_f - p n_f, v \right)_{\Gamma^f_{t_n}} \forall v \in H^1_D(\Omega^f_{t_n}),
\]

(1)

\[
b(u^n, q) = 0 \quad q \in L^2(\Omega^f_{t_n})
\]

(2)

using

\[
(u^n \cdot \nabla u^n, v)_{\Omega^f_{t_n}} = -(u^n \cdot \nabla v, u^n)_{\Omega^f_{t_n}} - ((\nabla \cdot u^n) v, u^n)_{\Omega^f_{t_n}} + ((u^n \cdot \eta f) v, u^n)_{\Gamma^f_{t_n} \cup \Gamma^f_N}
\]

and

\[
(z^n \cdot \nabla u^n, v)_{\Omega^f_{t_n}} = -(z^n \cdot \nabla v, u^n)_{\Omega^f_{t_n}} - ((\nabla \cdot z^n) v, u^n)_{\Omega^f_{t_n}} + ((z^n \cdot \eta f) v, u^n)_{\Gamma^f_{t_n}}
\]

by Green’s Theorem.

We set \( g^n := \left[ 2\nu_f D(u^n) \cdot n_f - (u^n \cdot n_f) u^n \right] \big|_{\Gamma^f_{t_n}} \) as our control on the interface to minimize the jump in velocities for the interface condition (11). At an optimal solution, \( \frac{1}{2}((u^n - z^n) \cdot n_f) u^n \) will be approximately zero, since at an optimal solution \( u^n \approx \tilde{\eta}^n \) and \( \tilde{\eta}^n = z^n \). Therefore, \(-g^n \) can be used as the stress for the structure subsystem, which will approximately enforce continuity of stress with \(- (g^n \circ \Psi_{t_{n-1}}) J_t \) representing \[2\nu_s D(\eta^n) \cdot n_s + \lambda(\nabla \cdot \eta^n) n_s \big|_{\Gamma_0} \].
Making this substitution, (1) and (2) become

\[
\rho f [ (u^n, v)_{\Omega_n^f} - (u^{n-1}, v)_{\Omega_{n-1}^f} ] + \Delta t \rho f [ c(u^n, u^n, v)_{\Omega_n^f} + \frac{1}{2} ((u^n \cdot n_f)u^n, v)_{\Gamma_n^f} \\
- \frac{1}{2} (\nabla \cdot z^n, v)_{\Omega_n^f} - c(z^n, u^n, v)_{\Omega_n^f} ] \\
+ \Delta t \nu f g (u^n, v)_{\Omega_n^f} + \Delta t b(v, p^n)_{\Omega_n^f} \\
= \Delta t (f^n, v)_{\Omega_n^f} + \Delta t (g^n, v)_{\Gamma_n^f} + \Delta t (g^n, v)_{\Gamma_{Itn}} \quad \forall v \in H^1_D(\Omega_n^f),
\]

(3)

\[b(u^n, q) = 0 \quad q \in L^2(\Omega_{Itn}).\]

(4)

Also, (38) and (39) can be rewritten as

\[
\rho f [ (\dot{\eta}^n, \xi)_{\Omega^n} - (\dot{\eta}^{n-1}, \xi)_{\Omega^n} ] + \Delta t \nu s d(\eta^n + \eta^{n-1}, \xi)_{\Omega^n} + \frac{\Delta t}{2} \lambda (\eta^n + \eta^{n-1}, \xi)_{\Omega^n} \\
= \frac{\Delta t}{2} (f^n_s + f^{n-1}_s, \xi)_{\Omega^n} + \frac{\Delta t}{2} (\eta^n_N + \eta^{n-1}_N, \xi)_{\Gamma_n^N} \\
- \frac{\Delta t}{2} (\nu (g^n_s), t_n + \nu (g^{n-1}_s), t_{n-1}, \xi)_{\Gamma_{It}} \quad \forall \xi \in H^1_D(\Omega^n),
\]

(5)

\[
\frac{\Delta t}{2} [(\dot{\eta}^n, \gamma)_{\Omega^n} + (\dot{\eta}^{n-1}, \gamma)_{\Omega^n}] = (\eta^n, \gamma)_{\Omega^n} - (\eta^{n-1}, \gamma)_{\Omega^n} \quad \forall \gamma \in L^2(\Omega^n).
\]

(6)

### 3.2 Description of the Optimization Problem

With \(g^n\) chosen arbitrarily, the solution of (3)-(4) and (5)-(6) are not solutions of (36)-(37) and (38)-(39) subject to boundary conditions (11)-(12). Therefore, we will use the stress function \(g^n\) in (3) and (5) as a control in each time step to attempt to enforce the continuity of velocity (11) and continuity of stress (12) along the interface, i.e., we wish to minimize the penalized functional

\[
J^\delta_n (u^n, p^n, \eta^n, \dot{\eta}^n, g^n) = \frac{1}{2} \int_{\Gamma_{Itn}} |u^n - v(\dot{\eta}^n)|^2 d\Gamma_{Itn} + \frac{\delta}{2} \int_{\Gamma_{Itn}} |g^n|^2 d\Gamma_{Itn},
\]

(7)

subject to (3)-(4) and (5)-(6), where \(\Gamma_{Itn}\) denotes the interface in the \(n\)-th time step and \(\delta\) is a positive constant penalty parameter that is chosen to dictate the importance of the last term in (7).

We anticipate that it will not be possible to get a stability estimate for \(\dot{\eta}^n\) in \(H^1_D(\Omega^n)\) and the existence of an optimal \(\dot{\eta}^n\) can be shown only in \(L^2(\Omega^n)\). In this case, the functional (7) is not well-defined since the trace of an optimal \(\dot{\eta}^n\) on \(\Gamma_{Itn}\) is not well-defined. In order to avoid this difficulty, we will replace \(\dot{\eta}^n\) with its first order approximation, \(\frac{\eta^n - \eta^{n-1}}{\Delta t}\). As will be seen in Remark 3.6, using a first order approximation in the functional will cause no greater loss in accuracy than using a higher order approximation. Minimizing

\[
J^\delta_n (u^n, p^n, \eta^n, \dot{\eta}^n, g^n) = \frac{1}{2} \int_{\Gamma_{Itn}} |u^n - v(\dot{\eta}^n) - \frac{v(\eta^n) - v(\eta^{n-1})}{\Delta t}|^2 d\Gamma_{Itn} + \frac{\delta}{2} \int_{\Gamma_{Itn}} |g^n|^2 d\Gamma_{Itn},
\]

(8)

subject to (3)-(4) and (5)-(6) enforces continuity of velocity and stress along the interface (11)-(12).
Now, our optimization problem to be solved is:

\[
\text{find } u^n, p^n, \eta^n, \hat{\eta}^n, \text{ and } g^n \text{ such that (8) is minimized subject to (3)-(4) and (5)-(6).} \tag{9}
\]

Define

\[
S := \{(u^n, p^n, \eta^n, \hat{\eta}^n, g^n) \in H^1_D(\Omega^f_{t_n}) \times L^2(\Omega^f_{t_n}) \times H^1_D(\Omega^s) \times L^2(\Omega^s) \times L^2(\Gamma_{t_n}) : \nonumber \]

\[
J_n^δ(u^n, p^n, \eta^n, \hat{\eta}^n, g^n) < \infty \text{ and (3)-(4) and (5)-(6) are satisfied.} \}
\tag{10}
\]

Then \((\hat{u}^n, \hat{p}^n, \hat{\eta}^n, \hat{\eta}^n, \hat{g}^n) \in S\) is called an optimal solution if there exists \(\epsilon > 0\) such that

\[
\forall (u^n, p^n, \eta^n, \hat{\eta}^n, g^n) \in S \text{ satisfying } \|g^n - \hat{g}^n\|_{0, \Gamma_{t_n}} \leq \epsilon. \tag{11}
\]

### 3.3 A Priori Estimates

We make the following assumptions to obtain a priori bounds for solutions to the weak formulations (3)-(6) and for analysis throughout the rest of the report:

- the Neumann boundary \(\Gamma^f_{t_n}\) is an outflow boundary, \(\tag{12}\)
- \(u^i \cdot n_f \in L^\infty(\Gamma^f_N)\) for \(i = 1, \ldots, n, \tag{13}\)
- \(\Omega^f_t = \Psi_t(\Omega^f_{t_0})\) is bounded and Lipschitz continuous, \(\tag{14}\)
- \(\Psi_t \in W^{1,\infty}(\Omega^f_{t_0})\) and \(\Psi_t^{-1} \in W^{1,\infty}(\Omega^f_{t}) \forall t \in [0, t_n], \tag{15}\)
- \(z, \frac{\partial z}{\partial t} \in W^{1,\infty}(\Omega^f_t) \forall t \in [0, t_n]. \tag{16}\)

Assumptions (12)-(13) are necessary for stability of any Navier-Stoke flow with nonhomogeneous Neumann boundary conditions. Assumptions (14)-(16) are reasonable for the movement and shape of the moving domain [12, 15].

As a result of (14) and (15), proposition 2.1 of [12] further gives

\[
\exists \kappa_{\text{min}}, \kappa_{\text{max}} \in \mathbb{R}^+ \text{ such that } 0 < \kappa_{\text{min}} \leq J_t \leq \kappa_{\text{max}} < \infty \forall t \in [0, t_n]. \tag{17}\]

The following estimates will be used for analysis of the optimal control problem.

**Theorem 3.1 Stability of \(u^n\)**

If \(\Delta t^2 C_9 \|\nabla \cdot z^i\|_{L^\infty, \Omega^f_t} < 1\) for \(i = 1, \ldots, n\) where

\[
C_9 = \|\nabla_y \varphi(z')\|_{L^\infty, \Omega^f_{t_0}} \|\nabla_y \Psi_t\|_{L^\infty, \Omega^f_t}, \text{ then}
\]

\[
\rho^f \|u^n\|^2_{0, \Omega^f_n} + 2\Delta t \sum_{i=1}^{n} \nu F C_0 \|u^i\|^2_{1, \Omega^f_t} \leq C \left[ \Delta t \sum_{i=0}^{n} \left[ \|f^n_j\|^2_{0, \Omega^f_t} + \|u^i\|^2_{0, \Omega^f_N} + \|g^i\|^2_{0, \Gamma_{t_n}} + \rho^f \|u^0\|^2_{0, \Omega^f_n} \right] \right]. \tag{18}\]
Proof: Let \((v, q) = (u^n, p^n)\) in (3) - (4). From this,

\[
\rho_f \left[ \|u^n\|^2_{0, \Omega_{t_n}^f} - (u^{n-1}, \mathcal{V}(u^n))_{\Omega_{t_n-1}^f} \right] + \Delta t \rho_f \left[ c(u^n, u^n, u^n)_{\Omega_{t_n}^f} + \frac{1}{2}((u^n \cdot n_f), |u^n|^2)_{\Gamma_N} - \frac{1}{2}((\nabla \cdot z^n), |u^n|^2)_{\Omega_{t_n}^f} \right] = \Delta t (f^f_t, u^n)_{\Omega_{t_n}^f} + \Delta t (u^n, u^n)_{\Gamma_{t_n}^f} + \Delta t (g^n, u^n)_{\Gamma_{t_n}^f},
\]

(19)

\[
b(u^n, p^n) = 0.
\]

(20)

Using \(c(u^n, u^n, u^n) = 0, c(z^n, u^n, u^n)_{\Omega_{t_n}^f} = 0\), dropping \(\frac{1}{2}((u^n \cdot n_f), |u^n|^2)_{\Gamma_N}\) by the assumption \(u^n \cdot n_f > 0\) (12), and also using \(b(u^n, p^n) = 0\) in (19) gives

\[
\rho_f \|u^n\|^2_{0, \Omega_{t_n}^f} - \Delta t \rho_f \frac{1}{2}((\nabla \cdot z^n), |u^n|^2)_{\Omega_{t_n}^f} + \Delta t 2 \nu_f a(u^n, u^n)_{\Omega_{t_n}^f} \leq \Delta t (f^f_t, u^n)_{\Omega_{t_n}^f} + \Delta t (u^n, u^n)_{\Gamma_{t_n}^f} + \Delta t (g^n, u^n)_{\Gamma_{t_n}^f} + \rho_f (u^{n-1}, \mathcal{V}(u^n))_{\Omega_{t_n-1}^f}.
\]

(21)

Combining terms and using (19) and Cauchy-Schwarz inequality in (21) yields

\[
\rho_f \|u^n\|^2_{0, \Omega_{t_n}^f} - \Delta t \rho_f \frac{1}{2}((\nabla \cdot z^n), |u^n|^2)_{\Omega_{t_n}^f} + \Delta t 2 \nu_f C_6 \|u^n\|^2_{1, \Omega_{t_n}^f} \leq \Delta t \|f^n\|^2_{-1, \Omega_{t_n}^f} + \Delta t \|u^n\|^2_{0, \Gamma_{t_n}^f} + \Delta t \|g^n\|^2_{0, \Gamma_{t_n}^f} + \rho_f \|u^{n-1}\|^2_{0, \Omega_{t_n-1}^f} \|\mathcal{V}(u^n)\|^2_{0, \Omega_{t_n-1}^f}.
\]

(22)

The Trace theorem followed by Young’s inequality gives

\[
\rho_f \|u^n\|^2_{0, \Omega_{t_n}^f} - \Delta t \rho_f \frac{1}{2}((\nabla \cdot z^n), |u^n|^2)_{\Omega_{t_n}^f} + \Delta t 2 \nu_f C_6 \|u^n\|^2_{1, \Omega_{t_n}^f} \leq C \Delta t \left[ \|f^n\|^2_{-1, \Omega_{t_n}^f} + \|u^n\|^2_{0, \Gamma_{t_n}^f} + \|g^n\|^2_{0, \Gamma_{t_n}^f} \right] + \Delta t \nu_f C_6 \|u^n\|^2_{1, \Omega_{t_n}^f} + \rho_f \left[ \frac{1}{2} \|u^{n-1}\|^2_{0, \Omega_{t_n-1}^f} + \frac{1}{2} \|\mathcal{V}(u^n)\|^2_{0, \Omega_{t_n-1}^f} \right].
\]

(23)

We rewrite and combine like terms in (23) to get

\[
\frac{\rho_f}{2} \left[ \|u^n\|^2_{0, \Omega_{t_n}^f} - \|u^{n-1}\|^2_{0, \Omega_{t_n-1}^f} \right] + \frac{\rho_f}{2} \left[ \|u^n\|^2_{0, \Omega_{t_n}^f} - \|\mathcal{V}(u^n)\|^2_{0, \Omega_{t_n-1}^f} \right] - \Delta t \rho_f \frac{1}{2}((\nabla \cdot z^n), |u^n|^2)_{\Omega_{t_n}^f} + \Delta t \nu_f C_6 \|u^n\|^2_{1, \Omega_{t_n}^f} \leq C \Delta t \left[ \|f^n\|^2_{-1, \Omega_{t_n}^f} + \|u^n\|^2_{0, \Gamma_{t_n}^f} + \|g^n\|^2_{0, \Gamma_{t_n}^f} \right].
\]

(24)
Since \( u^n \) is not time dependent, we can integrate both sides of (32) from \( t_{n-1} \) to \( t_n \) where 
\[
\phi(x, t) = (u^n(x))^2
\] as in [4], using 
\[
\int_{t_{n-1}}^{t_n} \frac{d}{dt} \left| \nabla \cdot z \right|^2_{\Omega_t} \ dt = \int_{t_{n-1}}^{t_n} \left( \frac{\partial V(u^n)^2}{\partial t} \right)_{y} + (\nabla_x \cdot z(x, t)) |V(u^n)|^2 \ dt
\]
with \( z(x, t) = z^n(x) \ \forall t \in [t_{n-1}, t_n] \) by (41) and \( \frac{\partial V(u^n)^2}{\partial t} |_y = 0 \), to get 
\[
\| u^n \|^2_{\Omega_{t_{n-1}}} - \| V(u^n) \|^2_{\Omega_t} - \Delta t \left( (\nabla_x \cdot z^n), |u^n|^2 \right)_{\Omega_t} \\
= \int_{t_{n-1}}^{t_n} \left( (\nabla_x \cdot z^n), |V(u^n)|^2 \right)_{\Omega_t} \ dt - \Delta t \left( (\nabla_x \cdot z^n), |u^n|^2 \right)_{\Omega_t}.
\]
Because \( u^n \) and \( V(u^n) \) have the same values for corresponding points on the moving domain, and \( |J_t - J_{t_n}| \leq \Delta t \ C_9 \) where \( C_9 = C \| \nabla_x V(z^n) \|_{L^\infty(\Omega_{t_0})} \| \nabla_y \Psi \|_{L^\infty(\Omega_{t_0})} \) and \( C \) is a positive constant that does not depend on \( \Delta t \) or the ALE mapping (see [4]),
\[
\| u^n \|^2_{\Omega_{t_{n-1}}} - \| V(u^n) \|^2_{\Omega_t} - \Delta t \left( (\nabla_x \cdot z^n), |u^n|^2 \right)_{\Omega_t} \\
\leq \int_{t_{n-1}}^{t_n} \left( (\nabla_x \cdot z^n), |V(u^n)|^2 \right)_{\Omega_t} \ dt \leq \Delta t^2 C_9 \left( |\nabla \cdot z^n|, |u^n|^2 \right)_{\Omega_t} \\
\leq \Delta t^2 C_9 \| \nabla \cdot z^n \|_{L^\infty(\Omega_t)} \| u^n \|^2_{\Omega_t} \tag{25}
\]
Substituting (25) into (24),
\[
\frac{\rho f}{2} \left[ \| u^n \|^2_{\Omega_t} - \| u^{n-1} \|^2_{\Omega_{t_{n-1}}} \right] + \Delta t \nu f C_0 \| u^n \|^2_{\Omega_{t_{n-1}}} \\
\leq C \Delta t \left[ \| f_{n} \|^2_{\Omega_{t_{n-1}}} + \| u^N \|^2_{\Omega_{\Gamma_{t_{n-1}}}} \right] + \Delta t^2 \frac{\rho f}{2} \| \nabla \cdot z^n \|_{L^\infty(\Omega_{t_{n-1}})} \| u^n \|^2_{\Omega_{t_{n-1}}}.
\]
Please observe that with \( \nabla \cdot z^n \in L^\infty(\Omega_{t_{n-1}}) \), an implication of (16), we can derive two stability results. The first is for boundedness at a single time step:
\[
\frac{\rho f}{2} \left[ 1 - \Delta t^2 C_9 \| \nabla \cdot z^n \|_{L^\infty(\Omega_{t_{n-1}})} \right] \| u^n \|^2_{\Omega_{t_{n-1}}} \\
+ \Delta t \nu f C_0 \| u^n \|^2_{\Omega_{t_{n-1}}} \\
\leq C \Delta t \left[ \| f_{n} \|^2_{\Omega_{t_{n-1}}} + \| u^N \|^2_{\Omega_{\Gamma_{t_{n-1}}}} \right] + \| u^{n-1} \|^2_{\Omega_{t_{n-1}}}. 
\]
The second is for boundedness over all time steps using a discrete Gronwall lemma after first
multiplying by 2 and summing over time steps.

\[
\rho^f \|u^n\|_{0,\Omega_{t_n}}^2 + 2\Delta t \sum_{i=1}^{n} \nu_f C_6 \|u^i\|_{1,\Omega'_t}^2 \\
\leq C\Delta t \sum_{i=0}^{n} \left[ \|f_j^0\|_{0,\Omega'^i_t}^2 + \|u_N^i\|_{0,\Gamma_N^i}^2 + \|g^i\|_{0,\Gamma_{t_i}}^2 \right] \\
+ \Delta t \sum_{i=0}^{n} \Delta t C_9 \|\nabla \cdot z^i\|_{L^\infty,\Omega^i_t} \rho^f \|u^i\|_{0,\Omega'^i_t}^2 + \rho^f \|u^0\|_{0,\Omega^0_t}^2.
\]

This yields

\[
\rho^f \|u^n\|_{0,\Omega_{t_n}}^2 + 2\Delta t \sum_{i=1}^{n} \nu_f C_6 \|u^i\|_{1,\Omega'_t}^2 \\
\leq C \left[ \Delta t \sum_{i=0}^{n} \left[ \|f_j^0\|_{0,\Omega'^i_t}^2 + \|u_N^i\|_{0,\Gamma_N^i}^2 + \|g^i\|_{0,\Gamma_{t_i}}^2 \right] + \rho^f \|u^0\|_{0,\Omega^0_t}^2 \right]
\]

when \(\Delta t^2 C_9 \|\nabla \cdot z^i\|_{L^\infty,\Omega^i_t} < 1\) for \(i = 1, \ldots, n\). Therefore, \(\Delta t\) must be chosen sufficiently small. \(\square\)

**Theorem 3.2** Stability of \(p^n\)

If \(\Delta t^2 C_9 \|\nabla \cdot z^i\|_{L^\infty,\Omega^i_t} < 1\) for \(i = 1, \ldots, n\), then

\[
\|p^n\|_{0,\Omega_{t_n}} \leq C P(\|f_j^0\|_{0,\Omega^0_t}, \|g_0^0\|_{0,\Gamma_{t_0}}, \|u_N^0\|_{0,\Gamma_{t_0}}, \ldots, \|f_j^n\|_{0,\Omega_{t_n}}, \|g_n^n\|_{0,\Gamma_{t_n}}, \|u_N^n\|_{0,\Gamma_{t_n}}), \quad (26)
\]

where \(P(\ldots, \cdot, \cdot)\) is a quadratic polynomial.

**Proof:** Rearranging (3), we get

\[
b(v, p^n)_{\Omega_{t_n}} = -\frac{\rho^f}{\Delta t} \left[ (u^n, v)_{\Omega_{t_n}} - (u^{n-1}, \nabla(v))_{\Omega_{t_n-1}} \right] + \rho^f \left[ -c(u^n, u^n, v)_{\Omega_{t_n}} + c(z^n, u^n, v)_{\Omega_{t_n}} \right. \\
+ \frac{1}{2}((\nabla \cdot z^n)u^n, v)_{\Omega_{t_n}} - \frac{1}{2}((u^n \cdot n_f)u^n, v)_{\Gamma_N} \right] - 2\nu_f a(u^n, v)_{\Omega_{t_n}} + (f_j^n, v)_{\Omega_{t_n}} \\
+ (u_N^n, v)_{\Gamma_N} + (g^n, v)_{\Gamma_{t_n}} \quad \forall v \in H_D^1(\Omega_{t_n}). \quad (27)
\]
In (27),
\[
(u^{n-1}, 
\mathcal{V}(v))_{\Omega_{t_{n-1}}^f} = \int_{\Omega_{t_{n-1}}^f} u^{n-1} \cdot \mathcal{V}(v) \, d\Omega_{t_{n-1}}^f
\]
\[
= \int_{\Omega_{t_0}^f} J_{t_{n-1}} \mathcal{V}(u^{n-1}) \cdot \mathcal{V}(v) \, d\Omega_{t_0}^f
\]
\[
\leq \int_{\Omega_{t_0}^f} J_{t_{n-1}} \mathcal{V}(u^{n-1}) \cdot \mathcal{V}(v) \frac{J_{t_n}}{\kappa_{\min}} \, d\Omega_{t_0}^f
\]
\[
\leq \frac{1}{\kappa_{\min}} \left\| J_{t_n}^\frac{1}{2} \right\|_{L^\infty(\Omega_{t_0}^f)} \left\| J_{t_{n-1}}^\frac{1}{2} \right\|_{L^\infty(\Omega_{t_0}^f)} \left( \int_{\Omega_{t_0}^f} \mathcal{V}(u^{n-1})^2 \, d\Omega_{t_0}^f \right)^\frac{1}{2} \left( \int_{\Omega_{t_0}^f} \mathcal{V}(v)^2 \, d\Omega_{t_0}^f \right)^\frac{1}{2}
\]
\[
\leq \frac{1}{\kappa_{\min}} \left\| u^{n-1} \right\|_{0,\Omega_{t_{n-1}}^f} \left\| v \right\|_{1,\Omega_{t_n}^f}
\]
\[
\leq C \left\| u^{n-1} \right\|_{0,\Omega_{t_{n-1}}^f} \left\| v \right\|_{1,\Omega_{t_n}^f}.
\] (28)

Applying Cauchy-Schwarz as well as using (14)-(16) and (28) in (27) gives
\[
\frac{b(v, p^n)_{\Omega_{t_n}^f}}{\|v\|_{1,\Omega_{t_n}^f}} \leq \frac{C}{\Delta t} \left[ \| u^n \|_{1,\Omega_{t_n}^f} + \| u^{n-1} \|_{0,\Omega_{t_{n-1}}^f} \right] + \rho^f \| u^n \|_{1,\Omega_{t_n}^f} \left[ C_3 \| u^n \|_{1,\Omega_{t_n}^f} + C_3 \| z^n \|_{1,\Omega_{t_n}^f} \right] + \frac{1}{2} \| \nabla \cdot z^n \|_{L^\infty(\Omega_{t_n}^f)} + \| u^{n} \cdot n_f \|_{L^\infty(\Gamma_N)} + 2\nu_f C_1 \| u^n \|_{1,\Omega_{t_n}^f} + \| f_f^n \|_{0,\Omega_{t_n}^f} + \| u_N^n \|_{0,\Gamma_{t_n}} + \| g^n \|_{0,\Gamma_{t_n}} \quad \forall v \in H_D(\Omega_{t_n}^f).
\] (29)

Therefore, using (18) with \( \Delta t \) sufficiently small, the inf-sup condition (21), the assumption \( u^{n} \cdot n_f \in L^\infty(\Gamma_N^f) \) (13), and \( z^n \in H_D^1(\Omega_{t_n}^f) \) by (16),
\[
\| p^n \|_{0,\Omega_{t_n}^f} \leq \sup_{0 \neq v \in \Omega_{t_n}^f} \frac{b(v, p^n)}{\| v \|_{1,\Omega_{t_n}^f}} \leq C P(\| f_f^n \|_{0,\Omega_{t_n}^f}, \| g_0^n \|_{0,\Gamma_{t_n}}, \| u_N^n \|_{0,\Gamma_N^n}, \cdots, \| f_f^n \|_{0,\Omega_{t_n}^f}, \| g^n \|_{0,\Gamma_{t_n}}, \| u_N^n \|_{0,\Gamma_N^n}),
\]
where \( P(\cdot, \cdots, \cdot) \) is a quadratic polynomial. \( \square \)
Theorem 3.3 Stability of $\eta^n$ and $\dot{\eta}^n$

\[
\Delta t \left[ \sum_{i=1}^{n} \left| \frac{\dot{\eta}^i + \dot{\eta}^{i-1}}{2} \right|_{0,\Omega^s}^2 + \sum_{i=1}^{n} \left| \frac{\eta^i + \eta^{i-1}}{2} \right|_{1,\Omega^s}^2 \right] 
\leq \Delta t C \sum_{i=0}^{n} \left[ \left\| f_i^s \right\|_{-1,\Omega^s}^2 + \left\| \dot{\eta}_N \right\|_{-\frac{1}{2},\Gamma_N^s}^2 + \left\| g_i \right\|_{-\frac{1}{2},\Gamma_{t_0}}^2 + \left\| \dot{\eta}_0 \right\|_{0,\Omega^s}^2 + \left\| \eta_0 \right\|_{1,\Omega^s}^2 \right]. \quad (30)
\]

Proof: Letting $\xi = \frac{\eta^n - \eta^{n-1}}{\Delta t}$ in (5), $\gamma = \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\Delta t}$ in (6), and substituting,

\[
\rho^s \left( \frac{\dot{\eta}^n - \dot{\eta}^{n-1}}{\Delta t} , \frac{\eta^n + \eta^{n-1}}{2} \right)_{\Omega^s} 
\]

\[
+ \nu_s d(\eta^n + \eta^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} + \frac{\lambda}{2} e(\eta^n + \eta^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} \]

\[
= \frac{1}{2} (f_s^n + f_s^{n-1}, \eta^n - \eta^{n-1})_{\Omega^s} + \frac{1}{2} (\eta_N^n + \eta_N^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Gamma_N^s} 
\]

\[
- \frac{1}{2} (\nabla (g^n) J_{t_n} + \nabla (g^{n-1}) J_{t_{n-1}}, \eta^n - \eta^{n-1})_{\Gamma_{t_0}}.
\]

Adding $\rho^s \left( \frac{2\eta^{n-1}}{\Delta t} , \frac{\eta^n + \eta^{n-1}}{2} \right)_{\Omega^s}$, $\nu_s d(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s}$, and $\frac{\lambda}{2} e(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s}$ to both sides of the equation,

\[
\rho^s \left( \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{\Delta t} , \frac{\eta^n + \eta^{n-1}}{2} \right)_{\Omega^s} 
\]

\[
+ \nu_s d(\eta^n + \eta^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} + \frac{\lambda}{2} e(\eta^n + \eta^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} \]

\[
= \frac{1}{2} (f_s^n + f_s^{n-1}, \eta^n - \eta^{n-1})_{\Omega^s} + \frac{1}{2} (\eta_N^n + \eta_N^{n-1} - \frac{\eta^n - \eta^{n-1}}{\Delta t})_{\Gamma_N^s} 
\]

\[
- \frac{1}{2} (\nabla (g^n) J_{t_n} + \nabla (g^{n-1}) J_{t_{n-1}}, \eta^n - \eta^{n-1})_{\Gamma_{t_0}} 
\]

\[
+ \rho^s \left( \frac{2\eta^{n-1}}{\Delta t} , \frac{\eta^n + \eta^{n-1}}{2} \right)_{\Omega^s} + \nu_s d(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} 
\]

\[
+ \frac{\lambda}{2} e(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s}.
\]

(32)

Using (20), dropping a positive term, multiplying by $\Delta t$, and simplifying, (32) becomes

\[
2 \rho^s \left[ \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{2} \right]_{0,\Omega^s}^2 + 4 \nu_s C_7 \left[ \frac{\eta^n + \eta^{n-1}}{2} \right]_{1,\Omega^s}^2 
\]

\[
= \frac{1}{2} \left[ (f_s^n + f_s^{n-1}, \eta^n - \eta^{n-1})_{\Omega^s} + (\eta_N^n + \eta_N^{n-1} - \eta^n - \eta^{n-1})_{\Gamma_N^s} 
\]

\[
- (\nabla (g^n) J_{t_n} + \nabla (g^{n-1}) J_{t_{n-1}}, \eta^n - \eta^{n-1})_{\Gamma_{t_0}} 
\]

\[
+ \rho^s \left( \frac{\dot{\eta}^{n-1} + \dot{\eta}^{n-1}}{\Delta t} \right)_{\Omega^s} + 2 \nu_s d(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s} 
\]

\[
+ \frac{\lambda}{2} e(\eta^n + \eta^{n-1} - \frac{2\eta^n - \eta^{n-1}}{\Delta t})_{\Omega^s}.
\]
Applying Cauchy-Schwarz inequality, the trace theorem, \(0 < J_t < \kappa_{\text{max}} < \infty\), and then Young’s inequality,

\[
2\rho^s \left\| \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{2} \right\|_{0,\Omega^s}^2 + 4\nu_s C_7 \left\| \frac{\eta^n + \eta^{n-1}}{2} \right\|_{1,\Omega^s}^2 \leq C \left[ \left( \| f^n_s \|_{1,\Omega^s}^2 + \| f^{n-1}_s \|_{1,\Omega^s}^2 \right) + \left( \| \eta^n_N \|_{\frac{1}{2},\Gamma_N^s}^2 + \| \eta^{n-1}_N \|_{\frac{1}{2},\Gamma_N^s}^2 \right) \right]
\]

\[
+ \left( \| g^n \|_{\frac{1}{2},\Gamma_{t_n}}^2 + \| g^{n-1} \|_{\frac{1}{2},\Gamma_{t_n}^{-1}}^2 \right) \right] + \frac{\nu_s C_7}{4} \left\| \frac{\eta^n - \eta^{n-1}}{2} \right\|_{1,\Omega^s}^2
\]

\[
+ \rho^s \left\| \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{2} \right\|_{0,\Omega^s}^2 + \rho^s \left\| \frac{\dot{\eta}^{n-1}}{2} \right\|_{0,\Omega^s}^2
\]

\[
+ 3\nu_s C_7 \left\| \frac{\eta^n + \eta^{n-1}}{2} \right\|_{1,\Omega^s}^2 + \left[ \frac{2\nu_s C_4^2}{C_7} + \frac{\lambda^2 C_5^2}{\nu_s C_7} \right] \left\| \frac{\eta^{n-1}}{2} \right\|_{1,\Omega^s}^2.
\] (33)

Adding and subtracting terms with the triangle inequality for norms, Cauchy-Schwarz inequality, and combining like terms,

\[
\rho^s \left\| \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{2} \right\|_{0,\Omega^s}^2 + \nu_s C_7 \left\| \frac{\eta^n + \eta^{n-1}}{2} \right\|_{1,\Omega^s}^2 \leq C \left[ \left( \| f^n_s \|_{1,\Omega^s}^2 + \| f^{n-1}_s \|_{1,\Omega^s}^2 \right) + \left( \| \eta^n_N \|_{\frac{1}{2},\Gamma_N^s}^2 + \| \eta^{n-1}_N \|_{\frac{1}{2},\Gamma_N^s}^2 \right) \right]
\]

\[
+ \left( \| g^n \|_{\frac{1}{2},\Gamma_{t_n}}^2 + \| g^{n-1} \|_{\frac{1}{2},\Gamma_{t_n}^{-1}}^2 \right) \right] + 4n \rho^s \left( \sum_{j=1}^{n-1} \left\| \frac{\dot{\eta}^j + \dot{\eta}^{j-1}}{2} \right\|_{0,\Omega^s}^2 \right)
\]

\[
+ 4n \left[ \frac{\nu_s C_7}{2} + \frac{2\nu_s C_4^2}{C_7} + \frac{\lambda^2 C_5^2}{\nu_s C_7} \right] \sum_{j=1}^{n-1} \left\| \frac{\eta^j + \eta^{j-1}}{2} \right\|_{1,\Omega^s}^2
\]

\[
+ C \left[ \| \eta^0 \|_{0,\Omega^s}^2 + \| \eta^0 \|_{1,\Omega^s}^2 \right].
\] (34)

Let \(\alpha = \min \{ \rho^s, \frac{\nu_s C_7}{2} \}\) and \(\beta = \max \{ \rho^s, \frac{\nu_s C_7}{2} + \frac{2\nu_s C_4^2}{C_7} + \frac{\lambda^2 C_5^2}{\nu_s C_7} \}\). With the discrete Gronwall lemma [22] and no restriction on \(\Delta t\), (34) is bounded by

\[
\left\| \frac{\dot{\eta}^n + \dot{\eta}^{n-1}}{2} \right\|_{0,\Omega^s}^2 + \left\| \frac{\eta^n + \eta^{n-1}}{2} \right\|_{1,\Omega^s}^2 \leq C \alpha \exp \left( \frac{4n^2 \beta}{\alpha} \right) \left[ \left( \| f^n_s \|_{1,\Omega^s}^2 + \| f^{n-1}_s \|_{1,\Omega^s}^2 \right) + \left( \| \eta^n_N \|_{\frac{1}{2},\Gamma_N^s}^2 + \| \eta^{n-1}_N \|_{\frac{1}{2},\Gamma_N^s}^2 \right) \right]
\]

\[
+ \left( \| g^n \|_{\frac{1}{2},\Gamma_{t_n}}^2 + \| g^{n-1} \|_{\frac{1}{2},\Gamma_{t_n}^{-1}}^2 \right) \right] + \left\| \frac{\eta^0}{2} \right\|_{0,\Omega^s}^2 + \left\| \eta^0 \right\|_{1,\Omega^s}^2.
\]
Multiplying by $\Delta t$ and summing (34) over time steps,

$$
\Delta t \left[ \sum_{i=1}^{n} \left\| \frac{\dot{\eta}^i + \dot{\eta}^{i-1}}{2} \right\|_{0, \Omega^{s}}^2 + \sum_{i=1}^{n} \left\| \frac{\eta^i + \eta^{i-1}}{2} \right\|_{1, \Omega^{s}}^2 \right]
\leq \Delta t C \sum_{i=0}^{n} \left[ \left\| \mathbf{f}^i \right\|_{-1, \Omega^{s}}^2 + \left\| \eta^i \right\|_{-\frac{1}{3}, \Gamma_N^{s}}^2 + \left\| \mathbf{g}^i \right\|_{-\frac{1}{2}, \Gamma_{T_i}}^2 + \left\| \eta^0 \right\|_{0, \Omega^{s}}^2 + \left\| \eta^0 \right\|_{1, \Omega^{s}}^2 \right].
$$

\[\square\]

### 3.4 The Existence of an Optimal Solution

**Theorem 3.4** There exists a $(\mathbf{u}^n, p^n, \dot{\eta}^n, \ddot{\eta}^n, \mathbf{g}^n) \in S$ such that (9) is minimized.

**Proof:** Clearly, the set $S$ is nonempty. Let $\{(\mathbf{u}^n_{(k)}, p^n_{(k)}, \eta^n_{(k)}, \dot{\eta}^n_{(k)}, \mathbf{g}^n_{(k)})\}$ be a sequence in $S$ such that

$$
\lim_{k \to \infty} \mathcal{J}^n(\mathbf{u}^n_{(k)}, p^n_{(k)}, \eta^n_{(k)}, \dot{\eta}^n_{(k)}, \mathbf{g}^n_{(k)}) = \inf_{(\mathbf{u}^n, p^n, \eta^n, \dot{\eta}^n, \mathbf{g}^n) \in S} \mathcal{J}^n(\mathbf{u}^n, p^n, \eta^n, \dot{\eta}^n, \mathbf{g}^n).
$$

By the definition of (10), $\mathbf{g}^n_{(k)}$ is uniformly bounded in $L^2(\Gamma_{T_n})$.

From (18), (26), and (30), we have,

$$
\rho^f \left\| \mathbf{u}^n_{(k)} \right\|_{0, \Omega^{t_0}}^2 + \Delta t \nu_f C_6 \left[ \left\| \mathbf{u}^n_{(k)} \right\|_{1, \Omega^{t_0}}^2 + \sum_{i=1}^{n-1} \left\| \mathbf{u}^i \right\|_{1, \Omega^{t_i}}^2 \right]
\leq C \left[ \Delta t \left( \left\| \mathbf{g}^n_{(k)} \right\|_{0, \Gamma_{T_n}}^2 + \sum_{i=0}^{n-1} \left\| \mathbf{g}^i \right\|_{0, \Gamma_{T_i}}^2 + \sum_{i=0}^{n} \left[ \left\| \mathbf{f}^i \right\|_{0, \Omega^{t_i}}^2 + \left\| \mathbf{g}^i_{(k)} \right\|_{0, \Gamma_{T_i}}^2 \right] \right) + \rho^f \left\| u^0 \right\|_{0, \Omega^{t_0}}^2 \right],
$$

$$
\left\| p^n \right\|_{0, \Omega^{t_n}} \leq C P \left( \left\| \mathbf{f}^0 \right\|_{0, \Omega^{t_0}}, \left\| \mathbf{g}^0 \right\|_{0, \Omega^{t_0}}, \left\| \mathbf{u}^0 \right\|_{0, \Gamma_{T_0}}, \left\| \mathbf{f}^0 \right\|_{0, \Omega^{t_n}}, \left\| \mathbf{g}^0 \right\|_{0, \Omega^{t_n}}, \left\| \mathbf{u}^0 \right\|_{0, \Gamma_{T_n}} \right),
$$

and

$$
\left\| \dot{\eta}^n_{(k)} \right\|_{0, \Omega^s}^2 + \left\| \ddot{\eta}^n_{(k)} \right\|_{0, \Omega^s}^2
\leq C \left[ \sum_{i=0}^{n} \left\| \mathbf{f}^i_{(k)} \right\|_{-1, \Omega^s}^2 + \left\| \eta^i \right\|_{-\frac{1}{3}, \Gamma_N^{s}}^2 + \left\| \eta^0 \right\|_{0, \Omega^s}^2 + \left\| \eta^0 \right\|_{1, \Omega^s}^2 \right] + \sum_{i=0}^{n-1} \left\| \mathbf{g}^i \right\|_{-\frac{1}{2}, \Gamma_{T_i}}^2 + \left\| \mathbf{g}^n_{(k)} \right\|_{-\frac{1}{2}, \Gamma_{T_n}}^2 \right].
$$

Therefore, $(\mathbf{u}^n_{(k)}, p^n_{(k)}, \eta^n_{(k)}, \dot{\eta}^n_{(k)}, \mathbf{g}^n_{(k)}) \in H^1_D(\Omega_{T_{n}}) \times L^2(\Omega_{T_{n}}) \times H^1_D(\Omega^{s}) \times L^2(\Omega^{s}) \times L^2(\Gamma_{T_{n}})$ is uniformly bounded. Then, there must exist subsequences such that
The last two statements with strong convergence are a result of the compact embeddings
for some \((\hat{u}, \hat{p}, \hat{\eta}, \hat{n}) \in S\).

In this section, we show that as \(\Delta t \to 0\), the optimal solution to (9) converges to a solution
that satisfies (36)-(37) and (38)-(39) and satisfies the interface conditions (11)-(12) within a
tolerance on the order of \(\Delta t^2\).

Additionally, since \(J^\delta_n\) is lower semicontinuous,
\[
J^\delta_n(\hat{u}, \hat{p}, \hat{\eta}, \hat{n}, \hat{\eta}^n, \hat{g}^n) = \inf_{(u^n, p^n, \eta^n, \hat{n}, \eta^\delta, g^n) \in S} J^\delta_n(u^n, p^n, \eta^n, \hat{n}, \eta^\delta, g^n),
\]
so there exists a solution to the optimal control problem, although we can not show in general
that \((\hat{u}, \hat{p}, \hat{\eta}, \hat{n}, \hat{\eta}^n, \hat{g}^n)\) is unique.

\section{Convergence of Vanishing Penalty Parameter}

In this section, we show that as \(\delta \to 0\), the optimal solution to (9) converges to a solution
that satisfies (36)-(37) and (38)-(39) and satisfies the interface conditions (11)-(12) within a
tolerance on the order of \(\Delta t^2\).

In addition to (12)-(16), we make the following assumptions on regularity of the strong solution
satisfying (1)-(12) that are needed for the theorem that follows: \(u, \frac{du}{dt} |_{y} \in L^2(0, t_n; H^2(\Omega_f^t))\),
Theorem 3.5 Let $(u^n_3, p^n_3, \eta^n_3, \tilde{\eta}^n_3, g^n_3)$ denote an optimal solution satisfying (3)-(4) and (5)-(6) for $\delta > 0$, where $u^{n-1} = u(t_{n-1}) \in H^1_D(\Omega^f_{t_{n-1}})$, $\eta^{n-1} = \eta(t_{n-1}) \in H^1_D(\Omega^s)$, and $u(t_{n-1})$, $\eta(t_{n-1})$, and $\eta(t_{n-1})$ are solutions to (1)-(12) at time $t = t_{n-1}$.

Then, the solution $(\tilde{u}^n, \tilde{p}^n, \tilde{\eta}^n, \tilde{\eta}^n)$ exists as a solution of (3)-(6) at $t = t_n$ such that

$$
\|u^n_3 - u^n_3\|_{1, \Omega^f_{t_n}} + \|p^n_3 - p^n_3\|_{0, \Omega^f_{t_n}} + \|\eta^n_3 - \eta^n_3\|_{1, \Omega^s} + \|\tilde{\eta}^n - \tilde{\eta}^n\|_{0, \Omega^s} \to 0 \quad \text{as} \quad \delta \to 0. 
$$

Also, in the limit, $J_n(\tilde{u}^n, \tilde{p}^n, \tilde{\eta}^n, \tilde{\eta}^n) \leq C \Delta t^3$. 

Proof: Consider the solution to the semi-discrete weak formulations (36)-(37) and (38)-(39) satisfying the boundary conditions (11)-(12), and denote this $(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \tilde{\eta}^n_{sd})$. Letting $g^n_{sd} = 2\nu f D(u^n_{sd}) \cdot n_f - p^n_{sd} n_f - \frac{1}{2} (u^n_{sd} - z^n_{sd} \cdot n_f) u^n_{sd} | \Gamma_{t_n}$, we see that $(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \tilde{\eta}^n_{sd})$ is also a solution to (3)-(6). Also, $u^n_{sd}$, $\eta^n_{sd}$, and $\tilde{\eta}^n_{sd}$ are bounded in $H^1_D(\Omega^f_{t_n})$, $H^1_D(\Omega^s)$, and $H^1_D(\Omega^s)$, respectively, so

$$
J_n(\eta^n_{sd}, \tilde{\eta}^n_{sd}, \eta^n_{sd}) = \frac{1}{2} \left[ \int_{\Gamma_{t_n}} \left( \eta^n_{sd} - \frac{\nu \eta^n_{sd} - \nu \tilde{\eta}_n^{n-1}}{\Delta t} \right)^2 d\Gamma_{t_n} + \frac{\delta}{2} \int_{\Gamma_{t_n}} |g^n_{sd}|^2 d\Gamma_{t_n} \right] \leq C \left[ \|u^n_{sd}\|_{1, \Omega^f_{t_n}} + \|\eta^n_{sd}\|_{1, \Omega^s} + \|\tilde{\eta}^n_{sd}\|_{1, \Omega^s} + \|\tilde{\eta}^n_{sd}\|_{1, \Omega^s} \right] < \infty.
$$

Therefore, we can conclude that $(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \tilde{\eta}^n_{sd}, g^n_{sd}) \in S$.

By the definition of an optimal solution in (11) and since $(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \tilde{\eta}^n_{sd}, g^n_{sd}) \in S$,

$$
J_n(\eta^n_{sd}, \tilde{\eta}^n_{sd}, \eta^n_{sd}) = \inf_{(u, p, \eta, \tilde{\eta}, g) \in S} J_n(u^n, p^n, \eta^n_{sd}, \tilde{\eta}^n_{sd}, g^n_{sd}).
$$

In (36),

$$
J_n(\eta^n_{sd}, \tilde{\eta}^n_{sd}, \eta^n_{sd}) = \frac{1}{2} \left[ \int_{\Gamma_{t_n}} \left( \eta^n_{sd} - \frac{\nu \eta^n_{sd} - \nu \tilde{\eta}_n^{n-1}}{\Delta t} \right)^2 d\Gamma_{t_n} + \frac{\delta}{2} \int_{\Gamma_{t_n}} |g^n_{sd}|^2 d\Gamma_{t_n} \right] \leq C \left[ \|u^n_{sd} - u(t_{n})\|_{0, \Gamma_{t_n}} + \|u(t_{n}) - \nu \eta(t_{n})\|_{0, \Gamma_{t_n}} + \|\eta(t_{n}) - \eta(t_{n})\|_{0, \Gamma_{t_n}} \right] + \frac{\delta}{2} \|g^n_{sd}\|_{0, \Gamma_{t_n}}^2.
$$

where $u(t_{n})$ and $\eta(t_{n})$ are solutions to (1)-(12) at time $t = t_{n}$. Since $u(t_{n})$ and $\eta(t_{n})$ satisfy (11), $\|u(t_{n}) - \nu \eta(t_{n})\|_{0, \Gamma_{t_n}} = 0$. 


By adding and subtracting $\frac{\eta(t_n)}{\Delta t}$ to $\left\| \eta(t_n) - \frac{\eta(t_n) - \eta(t_{n-1})}{\Delta t} \right\|_{0, \Gamma_{t_{n}}}^2$, making the substitution based on our assumption that $\eta^{n-1}_{sd} = \eta(t_{n-1})$, and using the triangle inequality for
\[
\mathcal{J}^\delta_n(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \hat{\eta}^n_{sd}, g^n_{sd}) \leq \frac{\delta}{2} \left\| g^n_{sd} \right\|_{0, \Gamma_{t_{n}}}^2 + C \left[ \left\| u^n_{sd} - u(t_n) \right\|_{0, \Gamma_{t_{n}}}^2 + \left\| \frac{\eta(t_n) - \eta(t_{n-1})}{\Delta t} \right\|_{0, \Gamma_{t_{n}}}^2 \right].
\]
(38)

Since $\Omega^f_t$ is Lipschitz continuous by (14), we use the trace theorem to get
\[
\left\| u^n_{sd} - u(t_n) \right\|_{0, \Gamma_{t_{n}}} \leq C \left\| u^n_{sd} - u(t_n) \right\|_{0, \Omega^f_t} \left\| u^n_{sd} - u(t_n) \right\|_{1, \Omega^f_t}.
\]
(39)

With the assumptions on $u$ made in the statement of the theorem and using the error estimate derived in [23],
\[
\left\| u^n_{sd} - u(t_n) \right\|_{0, \Gamma_{t_{n}}}^2 \leq C \Delta t \frac{3}{2}.
\]
(40)

The second order time discretization [25, 31] for the linear elasticity equations gives
\[
\left\| \eta(t_n) - \eta^n_{sd} \right\|_{0, \Gamma_{t_{n}}}^2 \leq C \left\| \eta(t_n) - \eta^n_{sd} \right\|_{0, \Omega^f_t} \left\| \eta(t_n) - \eta^n_{sd} \right\|_{1, \Omega^f_t} \leq C \Delta t \frac{7}{2}.
\]
(41)

Using (41),
\[
\left\| \frac{\eta(t_n) - \eta^n_{sd}}{\Delta t} \right\|_{0, \Gamma_{t_{n}}}^2 \leq C \Delta t \frac{3}{2}.
\]
(42)

Substituting (40) and (42) into (38) gives
\[
\mathcal{J}^\delta_n(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \hat{\eta}^n_{sd}, g^n_{sd}) \leq \frac{\delta}{2} \left\| g^n_{sd} \right\|_{0, \Gamma_{t_{n}}}^2 + C \left[ \Delta t \frac{3}{2} + \left\| \frac{\eta(t_n) - \eta(t_{n-1})}{\Delta t} \right\|_{0, \Gamma_{t_{n}}}^2 \right].
\]
(43)

Using Taylor series expansion along with $\left\| \eta(t) \right\|_{L^\infty(t_{n-1}, t_n; H^1(\Omega^f))} < \infty$ gives
\[
\left\| \eta(t_n) - \frac{\eta(t_n) - \eta(t_{n-1})}{\Delta t} \right\|_{0, \Gamma_{t_{n}}}^2 \leq C \Delta t \frac{2}{2} \left\| \eta(t) \right\|_{L^\infty(t_{n-1}, t_n; H^1(\Omega^f))} \leq C \Delta t^2.
\]
(44)

We now substitute (44) into (43) and (43) into (36) to get
\[
\mathcal{J}^\delta_n(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \hat{\eta}^n_{sd}, g^n_{sd}) \leq \mathcal{J}^\delta_n(u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \hat{\eta}^n_{sd}, g^n_{sd}) \leq C \Delta t \frac{3}{2} + \frac{\delta}{2} \left\| g^n_{sd} \right\|_{0, \Gamma_{t_{n}}}^2.
\]
(45)

From (45), we can see that $g^n_{sd}$ is uniformly bounded. Combining this with (18), (26), and (30), we know that $u^n_{sd}, p^n_{sd}, \eta^n_{sd}, \hat{\eta}^n_{sd}$ are uniformly bounded as well.

Then, there must exist subsequences such that
\[
\begin{align*}
\mathbf{u}_n^\delta & \rightarrow \tilde{u}^n \text{ in } H^1(\Omega^f_t), \quad \mathbf{\eta}_n^\delta \rightarrow \tilde{\eta}^n \text{ in } H^1(\Omega^s), \\
\mathbf{p}_n^\delta & \rightarrow \tilde{p}^n \text{ in } L^2(\Omega^f_t), \quad \dot{\mathbf{\eta}}_n^\delta \rightarrow \tilde{\dot{\eta}}^n \text{ in } L^2(\Omega^s), \\
\mathbf{u}_n^\delta & \rightarrow \tilde{\mathbf{u}}^n \text{ in } L^2(\Omega^f_t), \quad \mathbf{g}_n^\delta \rightarrow \tilde{\mathbf{g}}^n \text{ in } L^2(\Gamma_{I_n}), \\
\text{and } \mathbf{u}_n^\delta |_{\Gamma_{F_N} \cup \Gamma_{I_n}} & \rightarrow \tilde{\mathbf{u}}^n \text{ in } L^2(\Gamma_{F_N} \cup \Gamma_{I_n}),
\end{align*}
\]
as \( \delta \rightarrow 0 \) for some \((\tilde{\mathbf{u}}^n, \tilde{\mathbf{p}}^n, \tilde{\mathbf{\eta}}^n, \tilde{\dot{\mathbf{\eta}}}^n, \tilde{\mathbf{g}}^n) \) \( \in S \), such that

\[
\mathcal{J}_n(\tilde{\mathbf{u}}^n, \tilde{\mathbf{p}}^n, \tilde{\mathbf{\eta}}^n, \tilde{\dot{\mathbf{\eta}}}^n, \tilde{\mathbf{g}}^n) = \left\| \frac{\mathbf{u}^n - \mathcal{V}(\tilde{\mathbf{\eta}}^n) - \mathcal{V}(\mathbf{\eta}(t_{n-1}))}{\Delta t} \right\|^2_{0, \Gamma_{I_n}} \leq C\Delta t^3. \tag{46}
\]

Remark 3.6 For any choice of finite difference formula in the functional, note from (42) that we will lose two powers of \( \Delta t \) from (41). This is the reason that, from an analytical perspective, we should choose to use a second order time discretization for the structure despite only using a first order time formula in the functional.

Additionally, with respect to (42), we see that by using a second order time discretization for the structure, there would be no improvement by increasing the order of the finite difference formula used in the functional.

3.6 The Existence of Lagrange Multipliers

We use the same approach to show the existence of Lagrange multipliers as was used in [11, 13].

Lemma 3.7 Let \( X \) and \( Y \) be two real Banach spaces, \( \mathcal{J} \) a functional on \( X \), and \( M \) a mapping from \( X \) to \( Y \). Assume \( u \) is a solution of the following constrained minimization problem:

\[
\text{find } u \in X \text{ such that } \mathcal{J} = \inf \{ \mathcal{J}(v) | v \in X, M(v) = y_0 \},
\]

where \( y_0 \) is some fixed element of \( Y \). Additionally, assume the following three conditions are satisfied:

- \( M \) is Fréchet differentiable in an open neighborhood of \( u \) and its Fréchet derivative \( M' \) is continuous at \( u \).
- \( \mathcal{J} : \text{Neighborhood}(u) \subset X \rightarrow \mathbb{R} \) is Fréchet differentiable at \( u \) with Fréchet derivative \( \mathcal{J}' \).
- \( M'(u) \) maps onto \( Y \).

Then there exists a \( \mu \in Y^* \) satisfying

\[
-\mathcal{J}'(u) \cdot w + \langle \mu, M'(u) \cdot w \rangle = 0 \quad \forall \ w \in X.
\]

Proof: See [17].
Let \( X = H^1_D(\Omega^f_{\text{top}}) \times L^2(\Omega^f_{\text{top}}) \times H^1_D(\Omega^s) \times L^2(\Omega^s) \times L^2(\Gamma_{\text{top}}), Y = H^1_D(\Omega^f_{\text{top}})^* \times L^2(\Omega^f_{\text{top}})^* \times H^1_D(\Omega^s)^* \times L^2(\Omega^s)^* \), and \( M : X \to Y \) be the constraint equations: \( M(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) = (f^n_j, \phi_1, f^n_s, \phi_2) \) for all \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \in X \) and \((f^n_j, \phi_1, f^n_s, \phi_2) \in Y\), if and only if,

\[
\begin{align*}
\rho^f(u^n, v)_{\Omega^f_{\text{top}}} &+ \Delta t \rho^f(c(u^n, u^n, v)_{\Omega^f_{\text{top}}} + \frac{1}{2}((u^n \cdot n_f) u^n, v)_{\Gamma^f_N} - c(z^n, u^n, v)_{\Omega^f_{\text{top}}}) \\
&+ \Delta t 2 \nu_f d(u^n, v)_{\Omega^f_{\text{top}}} + \Delta t b(v, p^n)_{\Omega^f_{\text{top}}} - \Delta t (\dot{g}^n, v)_{\Gamma_{\text{top}}} \\
&= (f^n_j, v)_{\Omega^f_{\text{top}}} \quad \forall v \in H^1_D(\Omega^f_{\text{top}}), \\
b(u^n, q) &= (\phi_1, q) \quad q \in L^2(\Omega^f_{\text{top}}),
\end{align*}
\]  

\( (47) \)

\[
\begin{align*}
\rho^s(\dot{\eta}^n, \xi)_{\Omega^s} &+ \Delta t \nu_s d(\eta^n + \eta^{n-1}, \xi)_{\Omega^s} + \frac{\Delta t}{2} \lambda c(\eta^n + \eta^{n-1}, \xi)_{\Omega^s} + \frac{\Delta t}{2} (V(g^n) J_{\text{top}}, \xi)_{\Gamma_{\text{top}}} \\
&= (f^n_s, \xi)_{\Omega^s} \quad \forall \xi \in H^1_D(\Omega^s), \\
\frac{\Delta t}{2} (\dot{\eta}^n, \gamma)_{\Omega^s} - (\eta^n, \gamma)_{\Omega^s} &= (\phi_2, \gamma)_{\Omega^s} \quad \forall \gamma \in L^2(\Omega^s).
\end{align*}
\]  

\( (49) \)

\( (50) \)

**Theorem 3.8** Let \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n)\) be an optimal solution to \((9)\) and \( M \) be defined as in \((47)-(50)\).

The Fréchet derivative of \( M \), denoted \( M' \), exists in an open neighborhood of \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n)\) and \( M' \) is continuous at \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n)\). \( M' \) is defined by

\[
\begin{align*}
\rho^f(w^n, v)_{\Omega^f_{\text{top}}} &+ \Delta t \rho^f(c(u^n, w^n, v)_{\Omega^f_{\text{top}}} + c(w^n, u^n, v)_{\Omega^f_{\text{top}}} + \frac{1}{2}((u^n \cdot n_f) w^n, v)_{\Gamma^f_N} \\
&+ \frac{1}{2}((w^n \cdot n_f) u^n, v)_{\Gamma^f_N} - c(z^n, w^n, v)_{\Omega^f_{\text{top}}}) \\
&+ \Delta t 2 \nu_f d(w^n, v)_{\Omega^f_{\text{top}}} + \Delta t b(v, r^n)_{\Omega^f_{\text{top}}} - \Delta t (h^n, v)_{\Gamma_{\text{top}}} \\
&= (f^n_j, v)_{\Omega^f_{\text{top}}} \quad \forall v \in H^1_D(\Omega^f_{\text{top}}), \\
b(w^n, q) &= (\phi_1, q) \quad q \in L^2(\Omega^f_{\text{top}}),
\end{align*}
\]  

\( (51) \)

\[
\begin{align*}
\rho^s(\varphi^n, \xi)_{\Omega^s} &+ \Delta t \nu_s d(\theta^n, \xi)_{\Omega^s} + \frac{\Delta t}{2} \lambda c(\theta^n, \xi)_{\Omega^s} + \frac{\Delta t}{2} (V(h^n) J_{\text{top}}, \xi)_{\Gamma_{\text{top}}} \\
&= (f^n_s, \xi)_{\Omega^s} \quad \forall \xi \in H^1_D(\Omega^s), \\
\frac{\Delta t}{2} (\varphi^n, \gamma)_{\Omega^s} - (\theta^n, \gamma)_{\Omega^s} &= (\phi_2, \gamma)_{\Omega^s} \quad \forall \gamma \in L^2(\Omega^s).
\end{align*}
\]  

\( (53) \)

\( (54) \)
Proof: Let $\epsilon > 0$.
\[
(M(u^n, p^n, \eta^n, \dot{\eta}^n, g^n)) - M(u^n, p^n, \eta^n, \dot{\eta}^n, g^n)
- M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \cdot (u_1^n - u_2^n, p_1^n - p_2^n, \eta_1^n - \eta_2^n, \dot{\eta}_1^n - \dot{\eta}_2^n, g_1^n - g_2^n) (v, q, \xi, \gamma))
\]
\[
= c(u^n_1, u^n_1, v) + \frac{1}{2}((u^n_1 \cdot n_f) u^n_1, v)_{\Gamma_N} - c(u^n_2, u^n_2, v) + \frac{1}{2}((u^n_2 \cdot n_f) u^n_2, v)_{\Gamma_N}
- \frac{1}{2}(((u^n_1 - u^n_2) \cdot n_f) u^n_1, v)_{\Gamma_N}
- \frac{1}{2}(((u^n_1 - u^n_2) \cdot n_f) u^n_2, v)_{\Gamma_N}
\]
\[
= c(u^n_1 - u^n_2, u^n_1, v)_{\Omega_{\text{in}}} + \frac{1}{2}(((u^n_1 - u^n_2) \cdot n_f) u^n_1, v)_{\Gamma_N}
+ c(u^n_2, u^n_1 - u^n_2, v)_{\Omega_{\text{in}}} + \frac{1}{2}((u^n_2 \cdot n_f) (u^n_1 - u^n_2), v)_{\Gamma_N}
- \frac{1}{2} (c(u^n_1 - u^n_2, u^n_1, v)_{\Omega_{\text{in}}} - 1/2) ((u^n_2 \cdot n_f) (u^n_1 - u^n_2), v)_{\Gamma_N}.
\]
Therefore, \[\|M(u^n_1, p^n_1, \eta^n_1, \dot{\eta}^n_1, g^n_1) - M(u^n_2, p^n_2, \eta^n_2, \dot{\eta}^n_2, g^n_2)\]
\[\|M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \cdot (u^n_1 - u^n_2, p^n_1 - p^n_2, \eta^n_1 - \eta^n_2, \dot{\eta}^n_1 - \dot{\eta}^n_2, g^n_1 - g^n_2)\|_Y \leq C \|u^n_1 - u^n_2\|_{1, \Omega_{\text{in}}} \|v\|_{1, \Omega_{\text{in}}}.
\]
It is straightforward to show that $\mathcal{J}_n^\delta$ is Frechét differentiable at $(u^n, p^n, \eta^n, \dot{\eta}^n, g^n)$ with Frechét derivative $(\mathcal{J}_n^\delta)'$.

**Theorem 3.9** The operator $M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n)$ is onto $Y$, for a sufficiently small $\Delta t$.

**Proof:** With $h^n = 0$, (51)-(52) are the linearized Navier-Stokes equations with four extra terms: ($(\nabla \cdot z^n) w^n, v)_{\Omega_{\text{in}}} + (c z^n, w^n, v)_{\Omega_{\text{in}}}$, $(w^n \cdot n_f) u^n, v)_{\Gamma_N}$, and $(u^n \cdot n_f) w^n, v)_{\Gamma_N}$. Because $z^n \in W^{1,\infty}(\Omega_{\text{in}})$ by assumption (16), the proof of well-posedness closely follows the proof of well-posedness for linearized Navier-Stokes with nonhomogeneous Neumann boundary conditions, given sufficiently small $\Delta t$. Similarly, with $h^n = 0$, (53)-(54) have the same properties of well-posedness as (49)-(50).

This means that a solution to (51)-(54) exists for all $(f^n_1, \phi^n_1, f^n_2, \phi^n_2) \in Y$. Therefore, we can find a $(w^n, r^n, \eta^n, \dot{\eta}^n, h^n) \in X$ that satisfies (51)-(54). Additionally, we choose $h^n = 0 \in L^2(\Gamma_{\text{in}})$ to see that $M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \cdot (w^n, r^n, \theta^n, \phi^n, h^n) = (f^n_1, \phi^n_1, f^n_2, \phi^n_2)$, so $M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n)$ is onto $Y$.

**Theorem 3.10** Let $(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \in X$ denote an optimal solution to (9). Then there exists a nonzero Lagrange multiplier $(\tilde{u}^n, \tilde{p}^n, \tilde{\eta}^n, \tilde{\eta}^n, h^n) \in Z$ such that
\[
-(\mathcal{J}_n^\delta)'((u^n, p^n, \eta^n, \dot{\eta}^n, g^n)) \cdot (w^n, r^n, \theta^n, \phi^n, h^n)
+
(M'(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \cdot (w^n, r^n, \theta^n, \phi^n, h^n), (u^n, p^n, \eta^n, h^n)) = 0
\forall (w^n, r^n, \theta^n, \phi^n, h^n) \in X,
\]
(55)
where \( Z = H_D^1(\Omega_{t_n}^f) \times L^2(\Omega_{t_n}^f) \times H_D^1(\Omega^s) \times L^2(\Omega^s) \).

**Proof:** We have shown that \( M' \) is onto \( Y \), that \( M' \) exists and is continuous in a neighborhood about \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n) \in X\), and we also know that \( J_n^\delta \) is Fréchet differentiable. We now apply Lemma 3.7 to see that there exists a solution \((\bar{u}^n, \bar{p}^n, \bar{\eta}^n, \bar{\dot{\eta}}^n) \in Z\) satisfying (55). \( \Box \)

### 3.7 Lagrange Multiplier Rule

The optimality system will now be derived using the Lagrange multiplier rule. Let us begin by defining the Lagrangian

\[
\mathcal{L}(u^n, p^n, \eta^n, \dot{\eta}^n, g^n, \bar{u}^n, \bar{p}^n, \bar{\eta}^n, \bar{\dot{\eta}}^n) =
- J_n^\delta(u^n, p^n, \eta^n, \dot{\eta}^n, g^n) + \rho f'[u^n, \bar{u}^n]_{\Omega_{t_n}^f} - (u^{n-1}, \mathcal{V}(\bar{u}^n))_{\Omega_{t_n}^f} -
\]

\[
+ \Delta t \rho f'[c(u^n, v, \bar{u}^n)] + \frac{1}{2} \left( \| \bar{u} \| \cdot n \right)_{\Omega_{t_n}^f} - \frac{1}{2} \left( \| \nabla \cdot z \| \right)_{\bar{u}^n} + \Delta t b(\bar{u}^n, p^n)_{\Omega_{t_n}^f} - \Delta t (f^n, \bar{u}^n)_{\Omega_{t_n}^f}
\]

\[
- \Delta t (u^n, \bar{u}^n)_{\Omega_{t_n}^f} - \Delta t (g^n, \bar{u}^n)_{\Omega_{t_n}^f} + \Delta t b(u^n, p^n) + \rho s[\| \bar{\eta}^n \| \bar{\eta}^n]_{\Omega^s} - \bar{\eta}^n_{n-1} \bar{\eta}^n_{n} \bar{\eta}^n_{n}
\]

\[
+ \Delta t \nu s d(\bar{\eta}^n)_{\Omega^s} - \frac{\Delta t}{2} (\bar{\eta}^n_{n-1} \bar{\eta}^n_{n})_{\Omega^s} - \frac{\Delta t}{2} (f^n s + f^n s_{n-1}, \bar{\eta}^n)_{\Omega^s}
\]

for any \((u^n, p^n, \eta^n, \dot{\eta}^n, g^n, \bar{u}^n, \bar{p}^n, \bar{\eta}^n, \bar{\dot{\eta}}^n) \in X \times Z\). We will now seek to find stationary points of \( \mathcal{L} \) over the product space \( X \times Z \). Variations in the Lagrange multipliers \( \bar{u}^n, \bar{p}^n, \bar{\eta}^n, \) and \( \bar{\dot{\eta}}^n \) yield the state equations (3)-(6). Variations in the state variables \( u^n, p^n, \eta^n, \) and \( \dot{\eta}^n \) yield the adjoint equations

\[
\rho f'(u^n, v)_{\Omega_{t_n}^f} + \Delta t \rho f'[c(u^n, v, \bar{u}^n)]_{\Omega_{t_n}^f} + c(v, u^n, \bar{u}^n)_{\Omega_{t_n}^f} + \frac{1}{2} \left( \| u^n \cdot n \| \right)_{\bar{u}^n} + \frac{1}{2} \left( \| \nabla \cdot z \| \right)_{\bar{u}^n} + \Delta t b(v, p^n)_{\Omega_{t_n}^f} + \Delta t (f^n, v)_{\Omega_{t_n}^f}
\]

\[
= \left( u^n - \frac{\mathcal{V}(\bar{\eta}^n) - \mathcal{V}(\bar{\eta}^n_{n-1})}{\Delta t}, v \right)_{\Omega_{t_n}^f} \quad \forall v \in H_D^1(\Omega_{t_n}^f),
\]

\[
b(\bar{u}^n, q) = 0 \quad q \in L^2(\Omega_{t_n}^f),
\]

\[
- (\bar{\eta}^n, \xi)_{\Omega^s} + \Delta t \nu s d(\bar{\eta}^n, \xi)_{\Omega^s} + \frac{\Delta t}{2} \xi_{\bar{\eta}^n} - \frac{\Delta t}{2} (f^n s + f^n s_{n-1}, \bar{\eta}^n)_{\Omega^s}
\]

\[
= - \frac{1}{\Delta t} \left( \mathcal{V}(u^n)_{I_{t_n}^f} - \frac{\eta^n - \eta^n_{n-1}}{\Delta t}, \xi \right)_{I_{t_n}^f} \quad \forall \xi \in H_D^1(\Omega^s),
\]

\[
\frac{\Delta t}{2} (\bar{\eta}^n, \gamma)_{\Omega^s} + \rho s(\bar{\eta}^n, \gamma)_{\Omega^s} = 0 \quad \forall \gamma \in L^2(\Omega^s).
\]
Also, variations in the control \( g^n \) yield the necessary condition

\[
\delta(g^n, c)_{\Gamma_{tn}} = -\Delta t (c, \bar{u}^n)_{\Gamma_{tn}} + \frac{\Delta t}{2} ((\nabla(c)J_{tn}, \bar{\eta}^n))_{\Gamma_{tn}} \quad \forall c \in L^2(\Gamma_{tn}). \tag{60}
\]

The adjoint problem (56)-(59) is well-posed, similar to the the linearized problem (47)-(50). Also, (55) can be rewritten as the adjoint equations (56)-(59) and the necessary condition (60).

It is obvious that an optimal solution to (7) will satisfy the state equations (47)-(50), and therefore we see that an optimal solution satisfies the optimality system (3)-(6), (56)-(59), and (60).

3.8 Steepest Descent Approach

The optimality system is often large in practice, and so it is common practice to decouple the state equations, adjoint equations, and the necessary condition. As in [13], we will use a gradient method for minimizing the penalized function (9).

Accordingly, \( g^n_{(k)} = g^n_{(k-1)} - \rho_k \frac{dJ^n}{dg^n_{(k)}}, \) where \( \rho_k \) is a step-size appropriately chosen. The necessary condition (60) allows us to solve for \( \frac{dJ^n}{dg^n_{(k)}}, \) yielding [18]

\[
\frac{dJ^n}{dg^n_{(k)}} = \delta g^n_{\Gamma_{tn}} + \Delta t \bar{u}^n_{\Gamma_{tn}} - \frac{\Delta t}{2} \bar{\eta}^n \circ (\Psi^n_{(k)})^{-1}\big|_{\Gamma_{tn}}.
\]

Let \( \rho_k \) have the form \( \frac{\alpha_k}{\delta} \), and now the algorithm has the form

\[
g^n_{(k+1)} = (1 - \alpha_k)g^n_{(k)} - \frac{\alpha_k}{\delta} (\Delta t \bar{u}^n_{(k)}|_{\Gamma_{tn}} - \frac{\Delta t}{2} \bar{\eta}^n \circ (\Psi^n_{(k)})^{-1}|_{\Gamma_{tn}}).
\] \tag{61}

Algorithm 3.11 Steepest Descent Algorithm

1. Choose an initial control \( g^n_{(0)} \)
2. For \( k = 0, 1, \ldots \)
   a) Solve (3)-(6) for \( (u^n_{(k)}, p^n_{(k)}, \eta^n_{(k)}, \dot{\eta}^n_{(k)}) \)
   b) If \( \int_{\Gamma_{tn}} \left| u^n_{(k)} - \frac{v(\eta^n_{(k)}) - v(\eta^{n-1}_{(k)})}{\Delta t} \right|^2 < \epsilon_{tol}, \) then break
   c) Solve (56)-(59) for \( (\bar{u}^n_{(k)}, \bar{p}^n_{(k)}, \bar{\eta}^n_{(k)}, \bar{\dot{\eta}}^n_{(k)}) \)
   d) Update the control using (61)

4 Numerical Results

An FSI problem using the ALE formulation for the moving fluid domain, reported in [24] and subsequently reproduced in [20], uses parameters that are consistent with blood flow in a human body.

A force \( b(t) \) is applied to the left fluid boundary (Fig. 2) at \( t \) s where

\[
b(t) = \begin{cases} (-10^3(1 - \cos 2\pi t/0.025), 0) \text{ dyne/cm}^2, & t \leq 0.025 \\ (0, 0), & 0.025 < t < T. \end{cases}
\]
The function $b(t)$ defines the stress on the inlet denoted by $u_N$ in (4). For numerical tests, we impose the Neumann condition on both the inflow and outflow boundaries in order to use the same conditions and parameters as in the literature. The volume force for the fluid and structure are $f(t) = (0, 0)$ dyne/cm$^2$. The other boundary conditions on the domain configuration are homogeneous Dirichlet or Neumann (Fig. 2), and the simulation begins at rest.

The reference domain for the fluid subsystem has height 1 cm and length 6 cm. The density of the fluid, $\rho_f$, is 1 g/cm$^3$ and the viscosity of the fluid, $\nu_f$, is 0.035 g/cm-s. The structure domain has height 0.1 cm and length 6 cm. The density of the structure, $\rho_s$, is 1.1 g/cm$^3$. The Young’s Modulus of the structure, $E$, is $3 \times 10^6$ dyne/cm$^2$ and its Poisson ratio, $\nu$, is 0.3. The Lamé parameters $\lambda$ and $\nu_s$ are defined as follows:

$$
\lambda = \frac{\nu E}{(1 - 2\nu)(1 + \nu)} \text{ dyne/cm}^2, \quad \nu_s = \frac{E}{2(1 + \nu)} \text{ dyne/cm}^2.
$$

Here we compare the vertical displacement over time of three points along the interface. Comparison is made between using the solution found using Aitken’s relaxation [8] and using the optimization Algorithm 3.11. Please see [8] for more details on Aitken’s relaxation.

The result using Aitken’s relaxation is reliable and useful as a reference solution with which to compare (Fig. 3). Spatial discretization was made in the $x$ direction with $h_x = 0.2$ cm and in the $y$ direction with $h_y = 0.1$ cm for both fluid and structure domains on a uniform mesh. The simulation was performed with $\Delta t = 1e-4$ s from $T = 0$ s to $T = 0.1$ s. Computations were performed in FreeFEM++ [28] using the triangular ($P_1 + \text{bubble}$, $P_1$) finite element pair for the fluid and triangular $P_1$ elements for the structure. The stopping criteria used for Aitken’s relaxation was $(\int_{\Gamma_0} (\eta^k_n - \eta^{k-1}_n)^2 \, d\Gamma)^{1/2} < 10^{-7}$, while $\delta = 1e-15$ and $\epsilon_{tol} = 10^{-4}$ for Algorithm 3.11.

There is strong agreement between the solution computed by Aitken’s relaxation and the two solutions computed using Algorithm 3.11. The optimization algorithm has been implemented for both the first and second order formulations of the structure subsystem to demonstrate the similarity in result and that requiring a second order structure subsystem formulation is needed for analysis, but not necessarily for computation.

5 Conclusion

We have introduced a control that has allowed the fluid-structure interaction problem to be decoupled into subsystems that may be solved in parallel. This control allowed us to demon-
strate the uniform boundedness of each optimization variable in order to show the existence of an optimal solution for a given $\delta$. Then, it was shown that as $\delta \to 0$, the optimal solutions for each given $\delta$ converge to an optimal solution with no penalty parameter, and that this optimal solution satisfies the constraint equations and minimizes the functional to within a $C\Delta t^3$ target tolerance.

The existence of Lagrange multipliers was proved, allowing us to derive the optimality system and to show that an optimal solution satisfies the optimality system. The steepest descent algorithm was then introduced for updating the control in order to decouple the optimality system.

The numerical results confirm that Algorithm 3.11 accurately simulates the fluid-structure interaction for a blood flow problem of great computational difficulty, due to the added mass effect. This computation was made over 1000 time-steps while maintaining the correct solution profile, despite the analysis supporting only a single-time step. In future work, we hope to provide an analytical framework for the optimal control problem over all time-steps and then to extend this work to other fluid-structure configurations.

References


