General Directions: Show all work analytically. When determining the convergence or divergence of a series, state the test (one on the handout) you use and show in detail how you use it. Summarize your conclusion in sentence form.

For problems 1-5, let \( a_n = \frac{2}{2n+5} - \frac{2}{2n+7} \) for \( n \geq 1 \).

1. List the first four terms of \( \{a_n\} \). Do not simplify. (4 points)
   
   \[
   a_1 = \frac{2}{7} - \frac{2}{9} \\
   a_2 = \frac{2}{9} - \frac{2}{11} \\
   a_3 = \frac{2}{11} - \frac{2}{13} \\
   a_4 = \frac{2}{13} - \frac{2}{15}
   \]

2. Analytically determine if \( \{a_n\} \) converges. (6 points)
   
   \[
   \lim_{n \to \infty} \frac{2}{2n+5} - \frac{2}{2n+7} = \lim_{n \to \infty} \frac{2}{2n+5} - \frac{2}{2n+7} = 0 - 0 = 0,
   \]
   
   So \( \{a_n\} \) converges to 0.

3. Determine a formula for \( s_n = a_1 + a_2 + a_3 + \cdots + a_n \) that holds for \( n \geq 3 \). (5 points)
   
   \[
   s_n = \frac{2}{7} - \frac{2}{2n+7}
   \]

4. Analytically determine if \( \{s_n\} \) converges. (5 points)
   
   \[
   \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{2}{7} - \frac{2}{2n+7} = \frac{2}{7}
   \]
   
   So \( \{s_n\} \) converges to \( \frac{2}{7} \).

5. Does \( \sum_{n=1}^{\infty} a_n \) converge? Explain why or why not. (5 points)
   
   Yes. \( \sum_{n=1}^{\infty} a_n = 5 \) means \( \lim_{n \to \infty} s_n = 5 \), and in this case \( s_n \to \frac{2}{7} \), so \( \sum_{n=1}^{\infty} a_n = \frac{2}{7} \).
6. Given \( b_n = \frac{n}{3+n^2} \), analytically determine whether the series \( \sum_{n=1}^{\infty} (-1)^n b_n \) is absolutely convergent, conditionally convergent or divergent. (12 points)

\[
\lim_{n \to \infty} \frac{n}{3+n^2} \cdot \frac{1}{n} = \lim_{x \to \infty} \frac{x^2}{3+4x^2} = 0, \quad \text{so by the limit comparison test,} \quad \sum_{n=1}^{\infty} \frac{n}{3+n^2} \text{ diverges (because the harmonic series diverges).}
\]

If \( f(x) = \frac{x}{3+x^2} \), then \( f(x) > 0 \) for \( x > 0 \).

Also, \( f'(x) = \frac{-(x^2-3)}{(x^2+3)^2} < 0 \) for \( x \geq 2 \), and \( \frac{n}{3+n^2} \to 0 \).

So by the alternating series test, \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges.

\[\therefore \sum_{n=1}^{\infty} (-1)^n b_n \text{ converges conditionally.}\]
For problems 7-9, state if the expression is a sequence or series and determine if it converges or diverges. For a series, state the test (one on the handout) you use and show in detail how you use it. If the series converges and it is possible, analytically find its exact sum. Show all work analytically. Summarize your conclusion in sentence form. (10 points each)

7. \[ \sum_{n=1}^{\infty} \frac{5n^2}{2n^2 + 1} \]

\[ \lim_{n \to \infty} \frac{5n^2}{2n^2 + 1} = \frac{5}{2} \neq 0, \text{ so the series diverges (by the test for divergence).} \]

8. \[ \left\{ \frac{\ln 1}{3}, \frac{\ln 2}{5}, \frac{\ln 3}{7}, \frac{\ln 4}{9}, \frac{\ln 5}{11}, \ldots \right\} \]

\[ a_n = \frac{\ln (n)}{2n+1} \]

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\ln n}{2n+1} = \lim_{n \to \infty} \frac{\frac{1}{n}}{2} = 0, \]

so the sequence converges to 0.

9. \[ \sum_{n=1}^{\infty} \frac{3^n}{4^{n+2}} \]

\[ = \frac{3}{4^3} + \frac{3^2}{4^4} + \frac{3^3}{4^5} + \ldots \]

\[ = \frac{3}{4^3} \cdot \frac{1}{1 - \frac{3}{4}} = \frac{3}{4^3} \cdot 4 = \frac{3}{16} \]

This is a geometric series with \( a = \frac{3}{4^3}, r = \frac{3}{4}, \) so it converges because \( |\frac{3}{4}| < 1. \)
10. Use the Integral Test (if applicable) to determine if the series is convergent or divergent. Show all work analytically. (12 points)

\[ \sum_{n=1}^{\infty} \frac{2}{4n^2 + 1} \]

Let \( f(x) = \frac{2}{4x^2 + 1} \), we have \( f'(x) = \frac{-16x}{(4x^2 + 1)^2} \).

Since \( f(x) > 0 \) and \( f'(x) < 0 \) for all \( x > 1 \), we see \( f \) is positive, continuous, and decreasing for \( x \geq 1 \), and the integral test applies.

\[
\int_{1}^{\infty} \frac{2}{4x^2 + 1} \, dx = \lim_{b \to \infty} \left[ \arctan u \right]_{2}^{b} = \frac{\pi}{2} - \arctan 2 \quad (\text{converges})
\]

Since the integral converges, the series converges.
For problems 11-13 consider the power series \( \sum_{n=1}^{\infty} \frac{(x+3)^n}{n4^n} \)

11. What is the center of the power series? (2 points)

The center is \( x = -3 \) (or \( a = -3 \)).

12. Analytically find the radius of convergence. (10 points)

\[
\lim_{n \to \infty} \frac{(x+3)^n}{(n+1)4^{n+1}} = \lim_{n \to \infty} \frac{(x+3)^n}{4^n} = \frac{x+3}{4}
\]

For \( \left| \frac{x+3}{4} \right| < 1 \) we require \( -4 < x+3 < 4 \) or \( -7 < x < 1 \).

The radius of convergence is \( R = 4 \).

13. Determine the interval of convergence. Be sure to include a check for convergence at the endpoints of the interval. (6 points)

If \( x = -7 \), we have \( \sum \frac{(x+3)^n}{n4^n} = \sum \frac{(-1)^n}{n} \) which converges (alt. harmonic series).

If \( x = 2 \), we have \( \sum \frac{(x+3)^n}{n4^n} = \sum \frac{1}{n} \) which diverges (harmonic series).

The interval of convergence is \( [-7, 1) \).
For problems 14-15 let \( b_n = \frac{1}{(5)^n(n+1)!} \), where the alternating series \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges to \( s \).

14. Use your calculator to find the value of \( s_4 \). This should be written as an exact value or to 6 decimal places. (4 points)

\[
S_4 = \frac{1}{5} \sum_{n=1}^{4} \frac{(-1)^n}{(n+1)!} = \frac{-878}{9375} \approx -0.093533
\]

15. Give a numerical estimate of the value of \( R_4 = s - s_4 \). That is, \(|R_4| \leq ____\)? This should be written as an exact value or to 8 decimal places. (4 points)

*Since this is an alternating series with decreasing terms, \(|R_n| \leq b_{n+1}\)*

So \(|R_4| \leq \frac{1}{5^{5.6!}} = \frac{1}{225000} \approx .0000004*

\[\text{Assume } n \geq 1\]

\[5^{n+1} > 5^n, \text{ and } (n+1)! > n!, \text{ so } 5^{n+1}(n+1)! > 5^n n!\]

and \( \frac{1}{5^n n!} > \frac{1}{5^{n+1}(n+1)!} \)