PhD Qualifying Examination

Solid Mechanics

Spring 2009

<u>IMPORTANT:</u>

- **TWO hours are allotted for the exam**

The exam is closed book and notes except for the attached information sheets.

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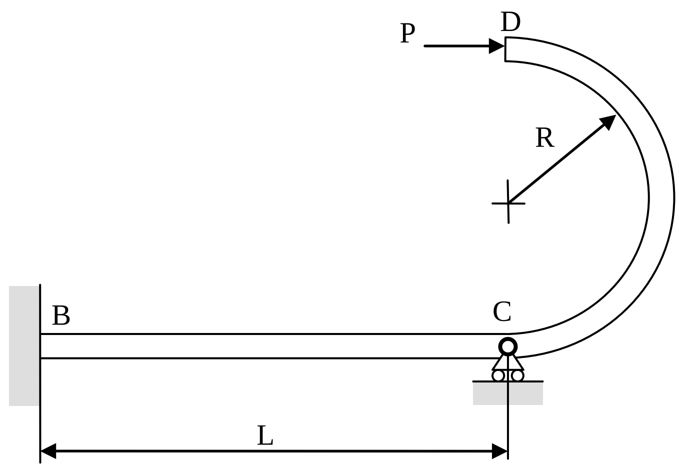
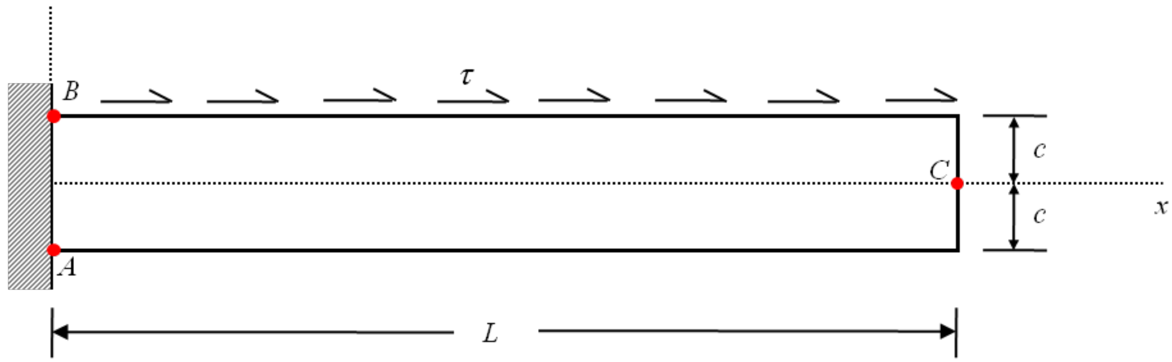
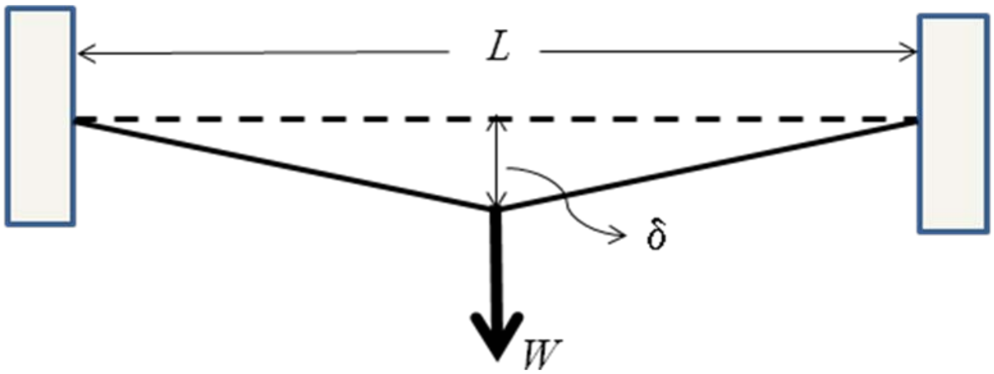
Problem #1. (34%) Member BCD shown in the figure has a uniform square cross section of area A. The member is subjected to a horizontal force P. Assuming the effect of shear is negligible, determine the support reaction at C and the horizontal displacement at D in terms of applied force P, modulus of elasticity E, radius of curvature R, length L, cross-sectional area A and the moment of inertia of the cross section I.



Problem #2. (33%) A cantilever beam with rectangular cross-section is loaded by a uniformly distributed shear stress τ applied to its upper surface only, as shown. Obtain expressions for the x-direction normal stress at *A* and at *B*. Neglect stress concentration effects. Also determine the deflection components at point *C*.



Problem #3. (33%) A wire of cross-sectional area *A*, length *L* and Young's modulus *E* is initially straight between two supports as shown below by the dashed line. It can be assumed that there is no force in the wire in this initial configuration. If a weight *W* is applied to the center of the wire, determine the relationship between this weight and the vertical displacement, δ, that is defined in the figure, where now the wire is represented by the solid line. Assume linear material behavior for the wire.



Fundamental Equations of Mechanics of Materials

Axial Load

Normal Stress

$$\sigma = \frac{P}{A}$$

Displacement

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E}$$

$$\delta = \Sigma \frac{PL}{AE}$$

$$\delta_T = \alpha \, \Delta T L$$

Torsion

Shear stress in circular shaft

$$\tau = \frac{T\rho}{J}$$

where

$$J = \frac{\pi}{2} c^4 \text{ solid cross section}$$

$$J = \frac{\pi}{2} (c_o^4 - c_i^4) \text{ tubular cross section}$$

Power

$$P = T\omega = 2\pi f T$$

Angle of twist

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G}$$

$$\phi = \Sigma \frac{TL}{JG}$$

Average shear stress in a thin-walled tube

(closed section)

$$\tau_{avg} = \frac{T}{2tA_m}$$

Shear Flow

$$q = \tau_{avg} t = \frac{T}{2A_m}$$

$$\tau = \frac{Tt}{\sum_{i=1}^{n} \frac{1}{3} b_i t_i^3}$$

(open section)

Bending

Normal stress

$$\sigma = \frac{My}{I}$$

Unsymmetric bending

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}, \qquad \tan \alpha = \frac{I_z}{I_y} \tan \theta$$

Shear

Average direct shear stress

$$\tau_{avg} = \frac{V}{A}$$

Transverse shear stress

$$\tau = \frac{VQ}{It}$$

Shear flow

$$q = \tau t = \frac{VQ}{I}$$

Stress in Thin-Walled Pressure Vessel

Cylinder

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_2 = \frac{pr}{2t}$$

Sphere

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Stress Transformation Equations

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Principal Stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum in-plane shear stress

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Absolute maximum shear stress

$$\tau_{\substack{abs \\ max}} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

$$\sigma_{avg} = \frac{\sigma_{max} + \sigma_{min}}{2}$$

Material Property Relations

Poisson's ratio

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}$$

Generalized Hooke's Law

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy},\ \gamma_{yz} = \frac{1}{G}\tau_{yz},\ \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

where

$$G = \frac{E}{2(1+\nu)}$$

Relations Between w, V, M

$$\frac{dV}{dx} = -w(x), \qquad \frac{dM}{dx} = V$$

Elastic Curve

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$EI\frac{d^4v}{dx^4} = -w(x)$$

$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

Buckling

Critical axial load

$$P_{\text{cr}} = \frac{\pi^2 EI}{(KL)^2}$$

Critical stress

$$\sigma_{\text{cr}} = \frac{\pi^2 E}{(KL/r)^2},\ r = \sqrt{I/A}$$

Secant formula

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$

Energy Methods

Conservation of energy $\quad U_e = U_i$

Strain energy

$$U_i = \frac{N^2L}{2AE} \qquad \textit{constant axial load}$$

$$U_i = \int_0^L \frac{M^2dx}{EI} \qquad \textit{bending moment}$$

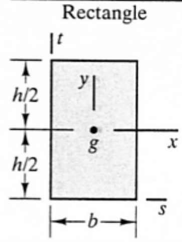
$$U_i = \int_0^L \frac{f_s V^2dx}{2GA} \qquad \textit{transverse shear}$$

$$U_i = \int_0^L \frac{T^2dx}{2GJ} \qquad \textit{torsional moment}$$

Properties of plane areas

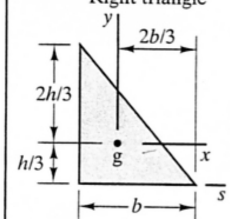
Point g is the centroid

Rectangle



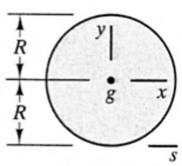
$A = bh$, $I_x = \frac{bh^3}{12}$, $I_s = \frac{bh^3}{3}$, $I_{st} = \frac{b^2h^2}{4}$

Right triangle



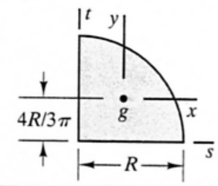
$A = \frac{bh}{2}$, $I_x = \frac{bh^3}{36}$, $I_s = \frac{bh^3}{12}$, $I_{xy} = -\frac{b^2h^2}{72}$

Complete circle



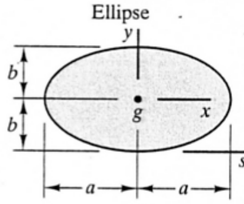
$A = \pi R^2$, $I_x = \frac{\pi R^4}{4}$, $I_s = \frac{5\pi R^4}{4}$, $J_g = \frac{\pi R^4}{2}$

Quarter circle



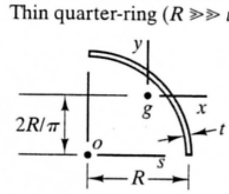
$A = \frac{\pi R^4}{4}$, $I_x = 0.0549\,R^4$, $I_s = \frac{\pi R^4}{16}$, $I_{st} = \frac{R^4}{8}$

Ellipse



$A = \pi ab$, $I_x = \frac{\pi ab^3}{4}$, $I_s = \frac{5\pi ab^3}{4}$, $J_g = \frac{\pi ab}{4}(a^2 + b^2)$

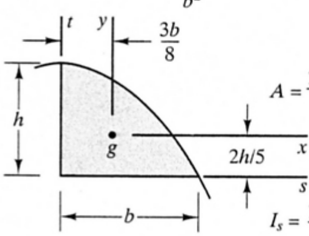
Thin quarter-ring ($R \gg t$)



$A = \frac{\pi Rt}{2}$, $I_x \approx 0.149\,R^3t$, $I_s \approx \frac{\pi R^3 t}{4}$, $J_o \approx \frac{\pi R^3 t}{2}$

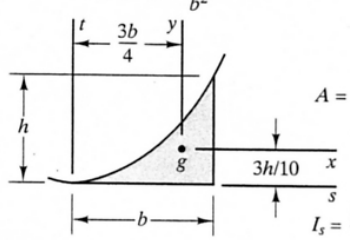
R = mean radius

Parabola $t = h - \frac{hs^2}{b^2}$



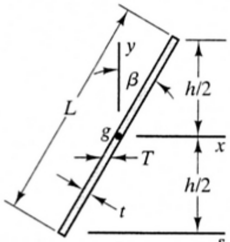
$A = \frac{2bh}{3}$, $I_s = \frac{16bh^3}{105}$

Parabola $t = \frac{hs^2}{b^2}$



$A = \frac{bh}{3}$, $I_s = \frac{bh^3}{21}$

Narrow rectangle ($L \gg t$)



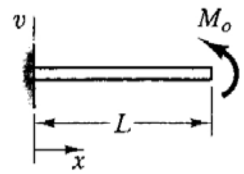
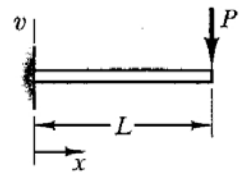
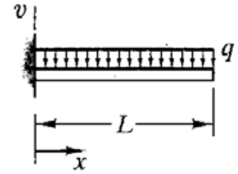
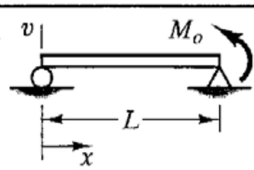
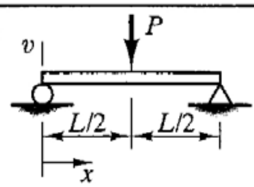
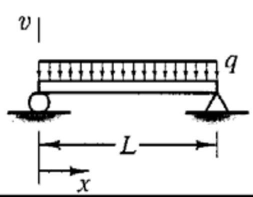
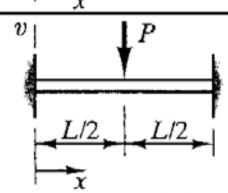
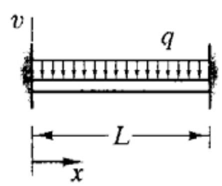
$T = \frac{t}{\cos\beta}$

$A = Lt$ or $A = hT$

$I_x \approx \frac{tL^3}{12}\cos^2\beta$ or $I_x \approx \frac{Th^3}{12}$

$I_s \approx \frac{tL^3}{3}\cos^2\beta$ or $I_s \approx \frac{Th^3}{3}$

Deflections and slopes of uniform beams

Beam and loading	Deflection (+ up)	Slope (+ CCW)	Equations
1 	$v = \frac{M_o L^2}{2EI}$ at $x = L$	$\theta = \frac{M_o L}{EI}$ at $x = L$	$v = \frac{M_o}{2EI} x^2$ $M = M_o$
2 	$v = -\frac{PL^3}{3EI}$ at $x = L$	$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$v = \frac{P}{6EI}(x^3 - 3Lx^2)$ $M = -P(L - x)$
3 	$v = -\frac{qL^4}{8EI}$ at $x = L$	$\theta = -\frac{qL^3}{6EI}$ at $x = L$	$v = -\frac{q}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$ $M = -\frac{q}{2}(L - x)^2$
4 	$v = -\frac{M_o L^2}{9\sqrt{3}EI}$ at $x = \frac{L}{\sqrt{3}}$	$\theta_o = -\frac{M_o L}{6EI}$ $\theta_L = \frac{M_o L}{3EI}$	$v = \frac{M_o}{6EIL}(x^3 - L^2x)$ $M = \frac{M_o x}{L}$
5 	$v = -\frac{PL^3}{48EI}$ at $x = \frac{L}{2}$	$\theta_o = -\frac{PL^2}{16EI}$ $\theta_L = +\frac{PL^2}{16EI}$	$v = \frac{P}{48EI}(4x^3 - 3L^2x)$ for $0 \leq x \leq \frac{L}{2}$
6 	$v = -\frac{5qL^4}{384EI}$ at $x = \frac{L}{2}$	$\theta_o = -\frac{qL^3}{24EI}$ $\theta_L = +\frac{qL^3}{24EI}$	$v = -\frac{q}{24EI}(x^4 - 2Lx^3 + L^3x)$ $M = \frac{q}{2}(L - x)x$
7 	$v = -\frac{PL^3}{192EI}$ at $x = \frac{L}{2}$	$\theta = \pm\frac{PL^2}{64EI}$ at quarter points	$v = \frac{P}{48EI}(4x^3 - 3Lx^2)$ for $0 \leq x \leq \frac{L}{2}$
8 	$v = -\frac{qL^4}{384EI}$ at $x = \frac{L}{2}$	$\theta = -\frac{qL^3}{124.7EI}$ at $x = 0.2113L$	$v = -\frac{qx^2}{24EI}(L - x)^2$ $M = \frac{q}{2}(L - x)x - \frac{qL^2}{12}$

Basic equations of isotropic linear elastiticy.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0.$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha T, \quad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \alpha T,$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] + \alpha T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{xz} = \frac{1}{G}\tau_{xz}, \quad \gamma_{zy} = \frac{1}{G}\tau_{zy}.$$

Plane stress and strain in polar coordinates.

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} = 0$$

$$\varepsilon_\theta = \frac{1}{2G}\left[\sigma_\theta - \frac{3-\kappa}{4}(\sigma_r + \sigma_\theta)\right], \quad \varepsilon_r = \frac{1}{2G}\left[\sigma_r - \frac{3-\kappa}{4}(\sigma_r + \sigma_\theta)\right], \quad \gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

$\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.