PhD Qualifying Examination

Solid Mechanics

Spring 2010

IMPORTANT:

- **TWO hours are allotted for the exam**

The exam is closed book and notes except for the attached information sheets.

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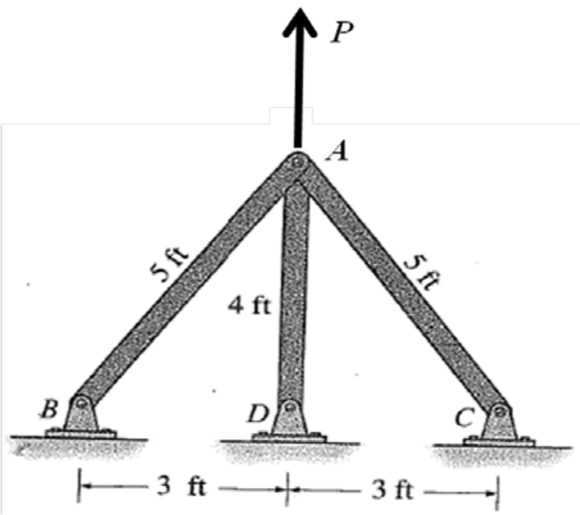
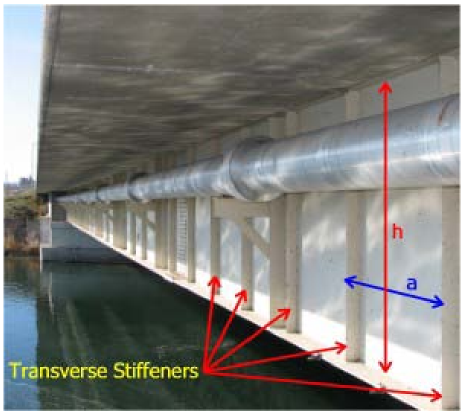
Problem #1. **(33%)** Three bars are made of steel with a Young's modulus of 29×10^6 psi. The bars form a pin-connected truss as shown. For a force of P = 5000 lb, determine the vertical displacement of joint A. Each bar has a cross-sectional area of 2 in^2.



Problem #2. (34%) Long, slender steel beams often have a cross-section in the shape of an "I" with top and bottom flanges and a thin web between them. In the bridge beam pictured below, the top flange of the beam is embedded in a concrete deck and cannot be seen. (Don't be distracted by the piping that happens to be supported on the outside of this bridge beam). Let the flanges have width W and thickness T_f. Let the overall depth (or height) of the I-shape be "h" and the web thickness be T_w.

a) Why is the I-shape a good, i.e. efficient, use of material in most cases?
b) What is the main structural purpose of the flanges?
c) What is the main structural purpose of the web?
d) What determines how wide or narrow and how thick or thin the flanges should be? Are there upper or lower limits to the ratio W/T_f?
e) What determines how deep or shallow and how thick or thin the web should be? Are there upper or lower limits to the ratio h/T_w?
f) Sometimes we see that transverse stiffeners (small vertical steel plates identified in the picture) are added to the web of the beam at certain locations along the beam length. Why are these stiffeners sometimes required? Would their spacing "a" need to be uniform along the whole length of the beam or could this spacing vary along the length? Why or why not?



g) The maximum allowable compression stress in the compression flange is a function of whether or not the flange is braced laterally by connecting it to a floor or deck structure (as in this picture) or connecting it to other parallel beams at certain locations along the length of the beam. Why does the allowable compressive stress depend on the presence and spacing of lateral bracing?
h) Sometimes long slender beams have a cross-section in the shape of a hollow, rectangular tube rather than an I-shape, despite the higher cost per pound of the tube shape. What is the main structural advantage that the hollow tube shape provides compared to the I-shape that might make the tube preferable in certain cases?

Problem #3. (34%)

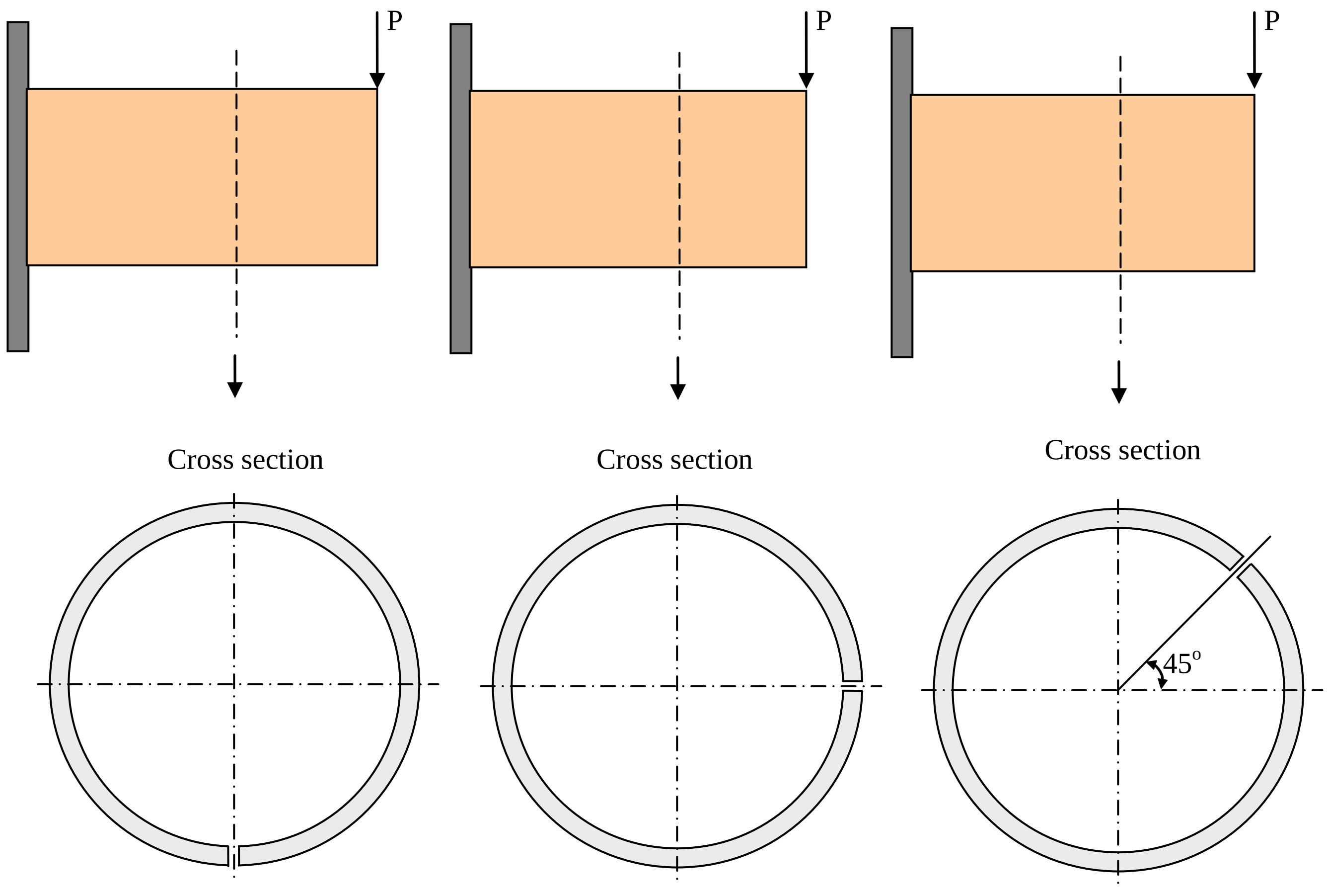
Transverse shear force P in the vertical direction is applied to three cantilever beams with thin-walled, open circular cross sections shown in the figure. The beams have identical length and identical cross sections except the position of the open slit.

(a) Draw the shear flow on the cross sections.

(b) On each cross section, approximately mark the location where the magnitude of the transverse shear stress is the maximum, explain your answer.

(c) Are the magnitudes of the three maximum shear stresses the same? If not, which one is the largest and which one is the smallest? Explain your answer.

(d) Will the cantilever beams be twisted by the loading? If yes, which beam has the largest angle of twist, and which one has the smallest? Explain your answer.



Fundamental Equations of Mechanics of Materials

Axial Load

Normal Stress

$$\sigma = \frac{P}{A}$$

Displacement

$$\delta = \int_0^L \frac{P(x)dx}{A(x)E}$$

$$\delta = \Sigma \frac{PL}{AE}$$

$$\delta_T = \alpha \, \Delta T L$$

Torsion

Shear stress in circular shaft

$$\tau = \frac{T\rho}{J}$$

where

$$J = \frac{\pi}{2} c^4 \text{ solid cross section}$$

$$J = \frac{\pi}{2} (c_o{}^4 - c_i{}^4) \text{ tubular cross section}$$

Power

$$P = T\omega = 2\pi f T$$

Angle of twist

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G}$$

$$\phi = \Sigma \frac{TL}{JG}$$

Average shear stress in a thin-walled tube (closed section)

$$\tau_{\text{avg}} = \frac{T}{2tA_m}$$

Shear Flow

$$q = \tau_{\text{avg}} t = \frac{T}{2A_m}$$

$$\tau = \frac{Tt}{\sum_{i=1}^{n} \frac{1}{3} b_i t_i^3}$$ (open section)

Bending

Normal stress

$$\sigma = \frac{My}{I}$$

Unsymmetric bending

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}, \qquad \tan \alpha = \frac{I_z}{I_y} \tan \theta$$

Shear

Average direct shear stress

$$\tau_{\text{avg}} = \frac{V}{A}$$

Transverse shear stress

$$\tau = \frac{VQ}{It}$$

Shear flow

$$q = \tau t = \frac{VQ}{I}$$

Stress in Thin-Walled Pressure Vessel

Cylinder

$$\sigma_1 = \frac{pr}{t} \qquad \sigma_2 = \frac{pr}{2t}$$

Sphere

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Stress Transformation Equations

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Principal Stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum in-plane shear stress

$$\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$

Absolute maximum shear stress

$$\tau_{\substack{\text{abs}\\ \text{max}}} = \frac{\sigma_{\max} - \sigma_{\min}}{2}$$

$$\sigma_{\text{avg}} = \frac{\sigma_{\max} + \sigma_{\min}}{2}$$

Material Property Relations

Poisson's ratio

$$\nu = -\frac{\epsilon_{\text{lat}}}{\epsilon_{\text{long}}}$$

Generalized Hooke's Law

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy},\ \gamma_{yz} = \frac{1}{G}\tau_{yz},\ \gamma_{zx} = \frac{1}{G}\tau_{zx}$$

where

$$G = \frac{E}{2(1+\nu)}$$

Relations Between w, V, M

$$\frac{dV}{dx} = -w(x), \qquad \frac{dM}{dx} = V$$

Elastic Curve

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$EI\frac{d^4v}{dx^4} = -w(x)$$

$$EI\frac{d^3v}{dx^3} = V(x)$$

$$EI\frac{d^2v}{dx^2} = M(x)$$

Buckling

Critical axial load

$$P_{\text{cr}} = \frac{\pi^2 EI}{(KL)^2}$$

Critical stress

$$\sigma_{\text{cr}} = \frac{\pi^2 E}{(KL/r)^2},\ r = \sqrt{I/A}$$

Secant formula

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)\right]$$

Energy Methods

Conservation of energy

$$U_e = U_i$$

Strain energy

$$U_i = \frac{N^2L}{2AE} \qquad \textit{constant axial load}$$

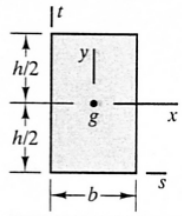
$$U_i = \int_0^L \frac{M^2dx}{EI} \qquad \textit{bending moment}$$

$$U_i = \int_0^L \frac{f_s V^2dx}{2GA} \qquad \textit{transverse shear}$$

$$U_i = \int_0^L \frac{T^2dx}{2GJ} \qquad \textit{torsional moment}$$

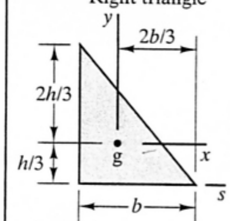
Properties of plane areas *Point g is the centroid*

Rectangle



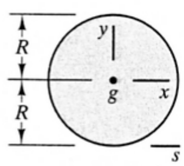
$A = bh$, $I_x = \frac{bh^3}{12}$, $I_s = \frac{bh^3}{3}$, $I_{st} = \frac{b^2h^2}{4}$

Right triangle



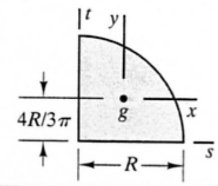
$A = \frac{bh}{2}$, $I_x = \frac{bh^3}{36}$, $I_s = \frac{bh^3}{12}$, $I_{xy} = -\frac{b^2h^2}{72}$

Complete circle



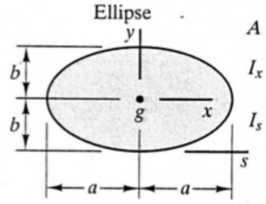
$A = \pi R^2$, $I_x = \frac{\pi R^4}{4}$, $I_s = \frac{5\pi R^4}{4}$, $J_g = \frac{\pi R^4}{2}$

Quarter circle



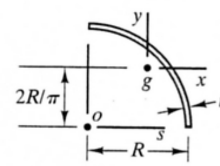
$A = \frac{\pi R^4}{4}$, $I_x = 0.0549\,R^4$, $I_s = \frac{\pi R^4}{16}$, $I_{st} = \frac{R^4}{8}$

Ellipse



$A = \pi ab$, $I_x = \frac{\pi ab^3}{4}$, $I_s = \frac{5\pi ab^3}{4}$, $J_g = \frac{\pi ab}{4}(a^2 + b^2)$

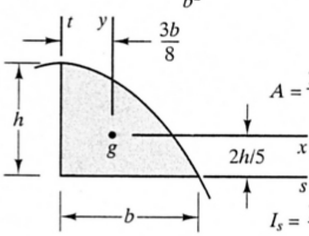
Thin quarter-ring ($R \gg t$)



$A = \frac{\pi Rt}{2}$, $I_x \approx 0.149\,R^3t$, $I_s \approx \frac{\pi R^3t}{4}$, $J_o \approx \frac{\pi R^3t}{2}$

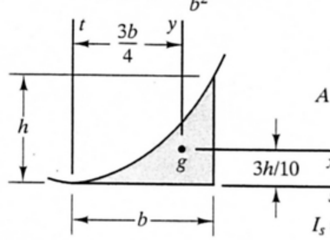
R = mean radius

Parabola $t = h - \frac{hs^2}{b^2}$



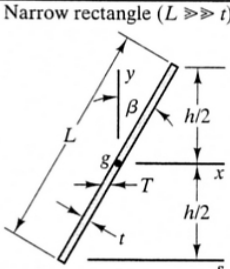
$A = \frac{2bh}{3}$, $I_s = \frac{16bh^3}{105}$

Parabola $t = \frac{hs^2}{b^2}$



$A = \frac{bh}{3}$, $I_s = \frac{bh^3}{21}$

Narrow rectangle ($L \gg t$)



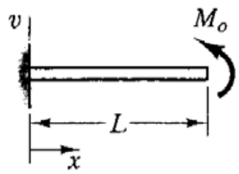
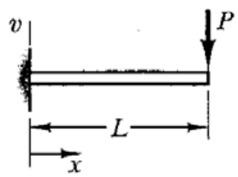
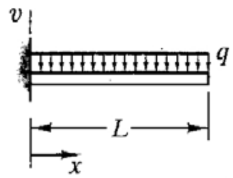
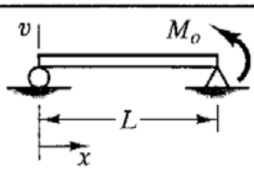
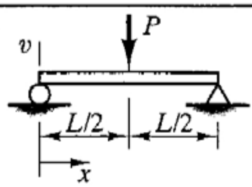
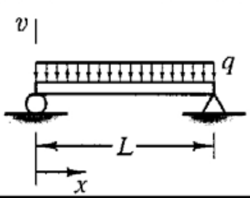
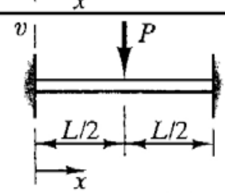
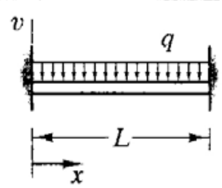
$T = \frac{t}{\cos\beta}$

$A = Lt$ or $A = hT$

$I_x \approx \frac{tL^3}{12}\cos^2\beta$ or $I_x \approx \frac{Th^3}{12}$

$I_s \approx \frac{tL^3}{3}\cos^2\beta$ or $I_s \approx \frac{Th^3}{3}$

Deflections and slopes of uniform beams

Beam and loading	Deflection (+ up)	Slope (+ CCW)	Equations
1 	$v = \frac{M_o L^2}{2EI}$ at $x = L$	$\theta = \frac{M_o L}{EI}$ at $x = L$	$v = \frac{M_o}{2EI} x^2$ $M = M_o$
2 	$v = -\frac{PL^3}{3EI}$ at $x = L$	$\theta = -\frac{PL^2}{2EI}$ at $x = L$	$v = \frac{P}{6EI}(x^3 - 3Lx^2)$ $M = -P(L - x)$
3 	$v = -\frac{qL^4}{8EI}$ at $x = L$	$\theta = -\frac{qL^3}{6EI}$ at $x = L$	$v = -\frac{q}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$ $M = -\frac{q}{2}(L - x)^2$
4 	$v = -\frac{M_o L^2}{9\sqrt{3}EI}$ at $x = \frac{L}{\sqrt{3}}$	$\theta_o = -\frac{M_o L}{6EI}$ $\theta_L = \frac{M_o L}{3EI}$	$v = \frac{M_o}{6EIL}(x^3 - L^2x)$ $M = \frac{M_o x}{L}$
5 	$v = -\frac{PL^3}{48EI}$ at $x = \frac{L}{2}$	$\theta_o = -\frac{PL^2}{16EI}$ $\theta_L = +\frac{PL^2}{16EI}$	$v = \frac{P}{48EI}(4x^3 - 3L^2x)$ for $0 \leqslant x \leqslant \frac{L}{2}$
6 	$v = -\frac{5qL^4}{384EI}$ at $x = \frac{L}{2}$	$\theta_o = -\frac{qL^3}{24EI}$ $\theta_L = +\frac{qL^3}{24EI}$	$v = -\frac{q}{24EI}(x^4 - 2Lx^3 + L^3x)$ $M = \frac{q}{2}(L - x)x$
7 	$v = -\frac{PL^3}{192EI}$ at $x = \frac{L}{2}$	$\theta = \pm\frac{PL^2}{64EI}$ at quarter points	$v = \frac{P}{48EI}(4x^3 - 3Lx^2)$ for $0 \leqslant x \leqslant \frac{L}{2}$
8 	$v = -\frac{qL^4}{384EI}$ at $x = \frac{L}{2}$	$\theta = -\frac{qL^3}{124.7EI}$ at $x = 0.2113L$	$v = -\frac{qx^2}{24EI}(L - x)^2$ $M = \frac{q}{2}(L - x)x - \frac{qL^2}{12}$

Basic equations of isotropic linear elastiticy.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0,$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0.$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha T, \quad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha T,$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha T$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{xz} = \frac{1}{G}\tau_{xz}, \quad \gamma_{zy} = \frac{1}{G}\tau_{zy}.$$

Plane stress and strain in polar coordinates.

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} = 0$$

$$\varepsilon_\theta = \frac{1}{2G}\left[\sigma_\theta - \frac{3-\kappa}{4}\left(\sigma_r + \sigma_\theta\right)\right], \quad \varepsilon_r = \frac{1}{2G}\left[\sigma_r - \frac{3-\kappa}{4}\left(\sigma_r + \sigma_\theta\right)\right], \quad \gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

$\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.