To the Graduate School:

This dissertation entitled “Modeling Time-Dependent, Multicomponent, Viscoelastic Fluid Flow” and written by William W. Miles is presented to the Graduate School of Clemson University. I recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy with a major in Mathematical Sciences.

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Accepted by the Graduate School:
Modeling Time-Dependent, Multicomponent, Viscoelastic Fluid Flow

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Mathematical Sciences

by
William W. Miles
December 2002

Advisor: Dr. Vincent J. Ervin
ABSTRACT

In this dissertation we study the modeling of multicomponent, viscoelastic fluid flows. The investigation of multicomponent flows is motivated by innovative mixing processes which create composite materials possessing such desirable characteristics as electrical or thermal conductivity. This work derives the modeling equations for multicomponent, viscoelastic fluid flows for fluids obeying the Oldroyd B constitutive law. Included in these equations is the jump condition which exists at fluid-fluid interfaces. Variational formulations are developed for both single- and multicomponent component flows, and corresponding finite element formulations are presented. Existence of a solution to the approximating system and a priori error estimates are derived for single component flows and then extended to include multicomponent flows. Results of numerical simulations are also presented.
DEDICATION

We dedicate this fine piece of work to all those people who never bothered us with questions about the content of this dissertation. It was always a great relief to be among people that were able think about different things than mathematics.
To my dear friend

Mickey
Some time ago I was born in a little town, where I went to school. Now I am spending my time until retirement studying mathematics.
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I am very thankful to my advisors Donald and Uncle Buck and all the other members of my committee. You were a great help in revealing the secrets of financial mathematics. I also thank my family, my friends and all the other guys.
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CHAPTER 1
INTRODUCTION

The modeling of multicomponent fluid flow is of interest in many engineering and scientific fields: material sciences, chemical engineering, mechanical engineering, medicine and mathematics, to name a few. Examples of situations which involve multicomponent flow include mixing processes, phase-change events, and stefan problems. The research described in this thesis is partially motivated by experiments in the in-situ development of composite materials [45]. Of particular interest is an in-situ process which uses a unique mixing procedure to produce structured materials possessing such characteristics as thermal and electrical conductivity. The advantage of this particular procedure is that it requires less of the specialty material than other more conventional mixing processes. This process involves two differing materials, a major and a minor phase, which are mixed during production. To predict the final structure of the composite material, each phase must be identified. Thus, the problem to be studied is a two-component viscoelastic fluid flow problem. This work focuses on three major aspects with regard to multi-component fluid flows: the existence of solutions to the approximating systems, identifying and locating the interface which exists between two differing materials, and numerically approximating the two-fluid flow system.

True solutions to the modeling equations for physical problems involving multicomponent viscoelastic fluid flow are rarely known, and existence of solutions to such problems has only been shown for special cases. This thesis develops a variational formulation for the solution of multicomponent, viscoelastic fluid flow. We then use the variational formulation to obtain an approximation to the true solution. An error estimate which predicts the error between the true and the approximate solution is also derived. Much of the literature pertaining to existence of solutions and a priori error estimates for fluid flow problems has been directed to the steady-state, single component, Newtonian problem. Newtonian fluids are modeled using the well-known Navier-Stokes equations. There is a rich body of research pertaining
to the Navier-Stokes equations, and the citations are far to numerous to list. The existence of solutions to single-component, non-Newtonian fluid problems has only been studied much more recently. The most significant results have been obtained by M. Renardy [30, 31] and C. Guillope and J. Saut [18]. For the research undertaken in this thesis, we assume that the continuous problem has a unique solution. We show that the approximating system to the discretized variational formulations has a solution and give an error estimate for the approximate solution. Such analysis has been done for the steady state, single-component, viscoelastic case [5, 26, 35]. The time-dependent problem was first addressed in [6]. In Chapter 3, we extend and improve the estimates derived in [6].

The introduction of a second fluid to a flow field presents complexity which is not present in single-component flows. When more than one fluid is present, a fluidic interface exists (assuming the fluids are immiscible) between differing materials. This interface possesses membrane-like properties relating to an imbalance in molecular forces across the interface. While the velocity in a two-component flow field is assumed to be continuous, no such assumption may be made regarding the gradient of the velocity. Thus, the stress satisfies a “jump” condition across the interface. This results in an additional forcing term in the balance of momentum equation. This jump condition is well-known, [7, 22]. A derivation of this condition is presented in section 2.1. This additional forcing term is defined to be the surface or interfacial tension. In much of the existing research, the coefficient of interfacial tension is assumed to be constant and the fluids under consideration Newtonian (i.e. fluids which respond instantaneously to perturbation) [32, 17, 39, 40]. In this work, we allow the surface tension to vary spatially. Also, we assume the fluids involved are viscoelastic in nature, exhibiting both viscous and elastic effects. Thus, the stress is usually defined by a more complicated differential equation (usually involving the velocity and the gradient of the velocity). The ability to track and identify this interface is also of paramount importance in solving multicomponent problems. Tracking methods fall into two broad categories: interface tracking, and interface capturing. Interface tracking techniques attempt to advect the interface in time by dealing with only with points which lie on the interface. The Marker and Cell method [42] is, by far, the most widely used of these methods. This
method identifies several marker points on the interface and moves the markers according to the velocity field. The markers are assumed to be ordered and connecting them gives the interface at any point in time. Interface capturing imbeds the interface within an indicator function which is defined throughout the entire computational domain. Volume of fluid (VOF) methods [20, 44, 32, 17, 38, 24] and level set methods [37, 28] are the most common interface capturing methods. The basic principle of VOF methods is as follows. For each computational cell, the fraction of the cell which is filled by a particular fluid, say fluid 1, is identified. These volume fractions are then advect using the underlying velocity field. The interface is contained in those cells which have volume fractions other than one or zero. These techniques are very popular, and two such methods are described in detail in Appendix A. Level set methods imbed the interface in a distance function such that the zero-level set of the distance function is the interface. The distance function is then advected using the velocity field [37, 28]. The numerical implementation used in this work uses the level set method and is presented in detail in Chapter 2. Further discussion of interface tracking is presented in Chapter 2 of this thesis.

The numerical solution of viscoelastic, multicomponent flow problems is quite challenging. In this work, we focus our attention on fluids whose constitutive equation is described by the Oldroyd B model, which is hyperbolic. The advection equation used to determine the interface location is also hyperbolic. These hyperbolic pde’s must be stabilized in order to avoid nonphysical oscillations in the solutions. Methods of stabilization include:

- the use of lower order approximation schemes,

- the addition of artificial diffusion,

- the addition of streamline upwind diffusion,

- the addition of cross-wind diffusion,

- the modification of boundary conditions.
All of these methods involve a balance between numerical stability, numerical accuracy and computational expense. The method of stabilization may be chosen differently depending on the types of problems to be solved. For examples of these techniques, see [2, 3, 11, 21, 29]. The implementation chosen here makes use of streamline upwinding.

In the remainder of this chapter we present the general balance laws which govern the flow of fluids. Then, the distinction between Newtonian and viscoelastic fluids is explained, followed by a particular model for viscoelasticity called the Oldroyd B model. Chapter 2 describes the issues which arise when a second fluid is introduced to the flow field. An interfacial boundary condition is derived and a scheme to model this boundary condition (the Continuous Surface Force Model) is presented. In Chapter 3, variational formulations for both single and multicomponent flow are derived, and discrete versions of these formulations are presented. Also, in Chapter 3 we present some theoretical results regarding existence of approximating solutions and a priori error estimates for single component fluid flows. Chapter 4 presents analogous results to those in Chapter 3 for multicomponent fluid flows. In Chapter 5 we explain the numerical implementation which is used to solve flows which involve two fluids. Numerical results are then presented. Finally, we make some concluding remarks in Chapter 6 and offer some suggestions for future work.

1.1 The Equations Governing Fluid Flow

The equations which govern the flow of fluids arise from physical laws of nature [8, 7]. The approach taken here follows closely that of a course, Fiber and Film Systems: Modeling and Simulation, given at Clemson University and sponsored by the Center for Advanced Engineering Fibers and Films. We begin by noting that there are two settings in which one might view the study of fluid flows: Lagrangian and Eulerian. The Eulerian approach places the observer at a fixed point from which he observes the flow while the Lagrangian setting places the observer “on a fluid particle” and allows him to be carried by the flow. One might think of the Eulerian observer as standing on a bridge viewing the flow of a river beneath while the Lagrangian observer is in a boat on the river.
The Lagrangian scheme requires an initial labeling of all points in the fluid. Thus, we assign the vector \( \mathbf{X} \) to be the initial fluid orientation. That is, \( \mathbf{X} = (x, 0) \). To formalize these points of view, we let \( F(\mathbf{x}, t) \) denote the value of \( F \) felt by the particle at \((\mathbf{x}, t)\) and \( F(\mathbf{X}, t) \) denote the value of \( F \) felt at time \( t \) by the particle that originated at \( \mathbf{X} \). It is often easier to think of the flow in the Lagrangian sense. However, we are accustomed to working within the Eulerian framework of fixed grids. Thus, it is natural to seek a link between the Eulerian and Lagrangian descriptions of the flow. This is achieved via the material or substantive derivative. The material derivative represents the rate of change of a function with respect to \( t \), following the fluid. We denote the material derivative of \( F \) by \( \frac{dF}{dt} \) and define it as

\[
\frac{dF}{dt} := \frac{\partial F}{\partial t} + \mathbf{u} \cdot \nabla F
\]

where \( \mathbf{u} \) is the velocity of the flow field. From this definition, it is clear that the material derivative accounts for the temporal derivative as well as the change brought about by convection via the flow field.

We define the following vector-tensor operations using the convention of summation over repeated indices. For \( \mathbf{u} = (u_1, u_2, u_3) \) a vector and

\[
T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix},
\]

we define

\[
\nabla \cdot \mathbf{u} := \frac{\partial u_i}{\partial x_i},
\]

\[
(u \cdot T)_i := u_i T_{ji},
\]

\[
(T \cdot u)_i := T_{ij} u_j,
\]

\[
T : P := T_{ij} P_{ij},
\]

\[
(\nabla \mathbf{u})_{ij} := \frac{\partial u_i}{\partial x_j},
\]

\[
\mathbf{I} := \text{the second order identity tensor}.
\]
Useful in establishing the modeling equations is the Reynolds Transport Theorem (RTT)\cite{1}.

**Theorem 1 (RTT)** Given a quantity $F(x,t)$ and an arbitrary volume $V$ moving with the fluid

\[
\frac{d}{dt} \int_V F \, dx = \int_V \left( \frac{dF}{dt} + F \nabla \cdot \mathbf{u} \right) \, dx.
\]

1.1.1 Conservation of Mass

The first equation to be presented represents a conservation of mass. The physics dictate that if we are given an arbitrary volume of fluid, $V$, the mass of fluid in $V$ does not change as $V$ moves in time. Thus,

\[
\frac{d}{dt} \int_V \rho \, dx = 0,
\]

where $\rho$ is the fluid density. Reynolds transport theorem then gives that

\[
\int_V \left( \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} \right) \, dx = 0.
\]

Since $V$ is an arbitrary volume, we obtain the pointwise equation

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega.
\]

(1.1)

This is the equation for the conservation of mass, otherwise known as **continuity equation**. Expanding the material derivative, one sees that (1.1) is equivalent to

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.
\]

(1.2)

Throughout this paper, we will deal with fluids which are said to be **incompressible**. An incompressible fluid is characterized by the idea that a given fluid particle does not experience any change in density as it moves in time. Again, this is a Lagrangian idea so that the material derivative is used. Incompressibility implies that

\[
\frac{d\rho}{dt} = 0.
\]

(1.3)

Thus, the continuity equation becomes

\[
\nabla \cdot \mathbf{u} = 0.
\]

(1.4)
We now turn our attention to the conservation of momentum.

1.1.2 Balance of Momentum

To derive the balance law, we use the following lemma which can be established using Theorem 1.1, (1.3), and (1.4).

**Lemma 1**  Given a quantity $F(x, t)$ and an arbitrary volume $V$ moving with the fluid

$$\frac{d}{dt} \int_V \rho F \, dx = \int_V \rho \frac{dF}{dt} \, dx.$$ 

The physical law governing momentum states that the rate of change of linear momentum of a body must be equal to the sum of the forces acting on the body. In this case, the “body” is the arbitrary control volume of the fluid. There are two types of forces that act on any body: body forces and surface forces. Body forces act on every part of the body of interest whereas surface forces act only as a result of particles contacting other particles on the surface of the body. One might think of body forces as forces which act from a distance, like gravity or the forces generated by a magnetic field, while surface forces result from direct contact of particles with one another. Surface forces may be thought of as “liquid friction” forces. Thus, we have that

$$\frac{d}{dt} \int_V \rho u \, dx = \int_V \rho b \, dx + \int_{\partial V} t \, dS$$

rate of change of momentum = body forces + surface forces

where $S$ is the boundary of $V$. Using Lemma 1, we have that

$$\int_V \rho \frac{du}{dt} \, dx = \int_V \rho b \, dx + \int_{\partial V} t \, dS,$$  \hspace{1cm} (1.5)

where $t$ is the stress vector. Hence, the goal is to express the surface integral which represents the surface forces as a volume integral. A result from Cauchy gives us that the stress vector $t$ and the outward unit normal vector to the surface $n$ are linearly related through a second order tensor called the stress tensor which we denote by $T$. That is to say that

$$t = n \cdot T.$$
We may further assume, as a result of the balance of angular momentum, that the stress
tensor is symmetric. Thus,

\[ t = T \cdot n. \]

Finally, we have that

\[ \int_{\partial V} t \ ds = \int_{\partial V} T \cdot n \ ds \]

\[ = \int_V \nabla \cdot T \ dx \quad \text{ (by the divergence theorem).} \]

Thus, (1.5) may be written as the pointwise equation

\[ \rho \frac{du}{dt} = \rho b + \nabla \cdot T \quad \text{in } \Omega. \]  

(1.6)

Equation (1.6) represents the balance of linear momentum.

1.1.3 The Energy Equation

For a material volume \( V \), the total amount of energy within the volume must be conserved. The components involved in the total energy are: kinetic energy, internal energy, energy flux, energy sources, and work done on the volume. The conversation law states that the rate of change of total energy in \( V \) must equal the work done on \( V \). Thus, we must have that

\[ \frac{d}{dt} \left( \int_V \frac{1}{2} \rho u \cdot u \ dx + \int_V \rho e \ dx \right) + \int_{\partial V} q \cdot n \ ds = \int_{\partial V} t \cdot u \ ds + \int_V \rho b \cdot u \ dx + \int_V \rho r \ dx. \]

kinetic internal energy flux = work done work done energy from
energy energy directed out by surface by body sources or
forces forces sinks

(1.7)

In (1.7), \( e \) denotes the specific energy per unit mass, \( q \) is the heat flux vector, and \( r \) is the energy source per unit mass in the system. Using Lemma 1,

\[ \frac{d}{dt} \int_V \frac{1}{2} \rho u \cdot u \ dx = \int_V \frac{1}{2} \rho \frac{du}{dt} (u \cdot u) \ dx \]

\[ = \int_V \rho u \cdot \frac{du}{dt} \ dx \]
and
\[ \frac{d}{dt} \int_V \rho e \, d\mathbf{x} = \int_V \frac{de}{dt} \, d\mathbf{x}. \]

Then we have
\[ \int_V \rho \left( \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} + \frac{de}{dt} \right) \, d\mathbf{x} = \int_V \nabla \cdot (\mathbf{T} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} + \rho (\mathbf{u} \cdot \mathbf{b} + r) \, d\mathbf{x} \tag{1.8} \]
(by divergence theorem).

Using the fact that \( \mathbf{T} \) is symmetric, one can show that
\[ \nabla \cdot (\mathbf{T} \cdot \mathbf{u}) = \mathbf{u} \cdot (\nabla \cdot \mathbf{T}) + \mathbf{T} : \nabla \mathbf{u}. \tag{1.10} \]

Using (1.8) and (1.10), we obtain from (1.6) the pointwise equation of energy equation
\[ \frac{\rho}{dt} de = \mathbf{T} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \rho r, \quad \text{in } \Omega. \tag{1.11} \]

1.2 Newtonian vs. Viscoelastic Fluids

In this section, we will note some differences between two classes of fluids: Newtonian and viscoelastic. A Newtonian fluid is a fluid which responds instantaneously to perturbations. That is, there is no “delay” in the reaction of the fluid to a force acting on the fluid. Gases and liquids containing small molecules fall into the Newtonian class of fluids. There are no truly Newtonian fluids, although water and isopropyl alcohol are nearly Newtonian. On the other hand, viscoelastic fluids have “memory.” They are characterized by a relaxation time which indicates how long the fluid would take to regain its molecular equilibrium orientation once it has been deformed. Polymeric liquids are prime examples of viscoelastic fluids. They are made up of long polymer chains which entangle. One might think of the phenomena as a plate of spaghetti. Pulling one of the noodles is difficult if it is tangled with other noodles on the plate. This chain entanglement can cause effects in viscoelastic fluids which are quite different from the behavior of Newtonian fluids. We illustrate some of these differences with some experimental examples. Consider mixing a liquid. A rod is placed vertically in the middle of a container of liquid. The rod is then rotated. This is much like stirring one's coffee or tea. We usually notice that a vortex is formed around the rod, pulling the liquid down toward the bottom of the container. However, if the fluid
is viscoelastic, the liquid may actually begin to “climb” the rod. This climbing is a result of the chain entanglement. More formally, in the Newtonian fluid no normal stresses are generated by the spinning of the rod. However, in the viscoelastic case, normal stresses are generated, forcing the fluid in toward the rod. See Figure 1.1.

![Figure 1.1 Viscoelastic Fluid “Climbing the Rod”](image)

Another non-Newtonian characteristic of viscoelastic fluids is that the viscosity of the fluid (a measure of its resistance to flow) is often a function of the shear rate. Most viscoelastic fluids exhibit a shear-thinning effect. That is, as the rate of deformation increases, the fluid viscosity decreases. Thus, if a Newtonian fluid and a viscoelastic fluid were to have the same viscosity at rest, it is likely that the viscoelastic fluid would drain from a tube faster than the Newtonian fluid. There are also some fluids which are shear-thickening although these seem to be much less common [8].

Recall the governing equations for isothermal fluid flow which were discussed in the previous section:

\[
\rho \frac{du}{dt} = \rho b + \nabla \cdot T \\
\nabla \cdot u = 0.
\]  

\[(1.12)\]  
\[(1.13)\]
It is common to decompose the total stress into pressure and extra stress so that $\mathbf{T} = -\rho \mathbf{I} + \tau$.

Then the system of equations takes the form:

$$
\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{b} - \nabla p + \nabla \cdot \tau \quad (1.14)
$$

$$
\nabla \cdot \mathbf{u} = 0. \quad (1.15)
$$

The body forces are assumed to be known. So the unknowns involved are velocity, $\mathbf{u}$, pressure, $p$, and the extra stress, $\tau$. As $\tau$ is a second order, symmetric tensor, the total number of unknowns is six in two dimensions and ten in three dimensions. Note that (1.12), (1.14) represent a system of three equations in 2-D and four equations in 3-D. To complete the system, we require an equation relating the extra stress to the other unknowns. This additional equation is called the constitutive equation.

For Newtonian fluids, the constitutive law is

$$
\tau = 2\eta \mathbf{D}(\mathbf{u}) \quad (1.16)
$$

where $\eta$ is the fluid viscosity. The tensor $\mathbf{D}(\mathbf{u})$ is called the deformation tensor and is defined to be $\mathbf{D}(\mathbf{u}) := \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right)$. Using this expression in (1.14) and expanding the material derivative give the equations which govern the isothermal flow of incompressible Newtonian fluids:

$$
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \Delta \mathbf{u} + \rho \mathbf{b} \quad (1.17)
$$

$$
\nabla \cdot \mathbf{u} = 0.
$$

These equations are called the incompressible Navier-Stokes equations. Equation (1.16) does not predict rod-climbing or shear-thinning effects that we know to exist in non-Newtonian fluids.

For non-Newtonian fluids, the constitutive laws are very different than the Newtonian constitutive law. Many of them are differential laws. A few examples include:

$$
\tau = K |\mathbf{D}(\mathbf{u})|^{n-1} \mathbf{D}(\mathbf{u}) \quad \text{Power Law,}
$$

$$
\tau + \lambda \tau(1) = 2\eta \mathbf{D}(\mathbf{u}) \quad \text{Maxwell},
$$
where \( \tau_{(1)} = \frac{d\tau}{dt} - \nabla \mathbf{u} \cdot \tau - \tau \cdot (\nabla \mathbf{u})^T \)

\[
\tau + D e \tau_{(1)} + \alpha \frac{D e}{(1 - \beta)} \tau \cdot \tau = 2(1 - \beta)D(\mathbf{u}) \quad \text{Giesekus,}
\]

\[
Z(\tau) \tau + \lambda \tau_{(1)} + \frac{\xi}{2} \lambda (D(\mathbf{u}) \cdot \tau + \tau \cdot D(\mathbf{u})) = -\eta D(\mathbf{u}) \quad \text{PTT}
\]

\[
Z(\tau) = \begin{cases} 
1 - \epsilon \lambda \text{Tr}(\tau)/\eta \\
\exp(-\epsilon \lambda \text{Tr}(\tau)/\eta)
\end{cases}
\]

In the \textit{PTT} model, \( \text{Tr}(\tau) \) denotes the trace of \( \tau \) and \( Z \) may be chosen as either of the candidates listed. Different models predict different viscoelastic effects with varying degrees of success, and there is still much ongoing research in the development of constitutive laws. The list above is just to illustrate the flavor of some of the constitutive laws used. A detailed discussion of the constitutive law used in this paper is given later in Section 3.2.
The addition of a second, immiscible fluid to the flow introduces “new” forces within the system. The balance of molecular forces which exists at any point within a single fluid does not exist at points on the interface where two different molecular forces interact at the same location. The fact that fluid particles on the interface between two fluids behave differently than particles within the bulk of either component is well known and easily demonstrated by such classical experiments as the “soap bubble” demonstration and capillary climbing within any container [36]. The interface has membrane-like qualities which are attributed to an imbalance of molecular forces at the interface[7] (see Figure 2.1). The equation which holds on the interface and ways to deal with it numerically are the subjects of this chapter.

In the next section, the boundary or “jump” condition which exists along the interface between two fluids is discussed, and a detailed derivation of the condition is presented. We
note that the derivation presented in this work differs from those found in the literature, which use “energy/area” minimization arguments to achieve the result [7, 22]. A discussion of the “continuum surface force model” is then given in section 2.2. This model, first presented by Brackbill et. al. in[9], gives a method of representing the interfacial forces (which are surface forces) as volume (body) forces which exist in an interfacial region.

2.1 The “Jump” Condition at a Fluid-Fluid Interface

Suppose there is a volume element, $V$, containing a portion of each fluid (see Figure 2.2). The general momentum balance law must still hold:

$$\frac{d}{dt} \int_V \rho u \, dV = \int_V \rho b \, dV + \int_{\partial V} t \, dS$$

(2.1)

where again $\rho \equiv$ density, $u \equiv$ velocity, $b \equiv$ body forces, and $t \equiv$ surface forces. Note that within each fluid $t$ may be written as $t = T \cdot n$, where $T$ is the constitutive stress tensor for the fluid and $n$ is the outward unit normal to the boundary of the volume element under consideration. The volume $V$ may be split into $V^1$, the portion of $V$ containing fluid 1; and $V^2$, the portion of $V$ containing fluid 2, and $I$, the interface between the fluids. So $V = V^1 \cup V^2 \cup I$. See Figure 2.3.
\[ \int_{\partial V} t dS = \int_{\partial V_1} t dS + \int_{\partial V_2} t dS - \int_{I} t^1 dS - \int_{I} t^2 dS \quad (2.2) \]
\[ = \int_{\partial V_1} \mathbf{T}^1 \cdot \mathbf{n} dS + \int_{\partial V_2} \mathbf{T}^2 \cdot \mathbf{n} dS - \int_{I} \mathbf{T}^1 \cdot \mathbf{n} dS + \int_{I} \mathbf{T}^2 \cdot \mathbf{n} dS \quad (2.3) \]
\[ = \int_{V_1} \nabla \cdot \mathbf{T}^1 dV + \int_{V_2} \nabla \cdot \mathbf{T}^2 dV - \int_{I} (\mathbf{T}^1 - \mathbf{T}^2) \cdot \mathbf{n} dS \quad (2.4) \]
\[ = \int_{V} \nabla \cdot \mathbf{T} dV - \int_{I} [\mathbf{T}] \cdot \mathbf{n} dV, \quad (2.5) \]

where \( I \) represents the interface and \( \mathbf{n} \), the unit normal on \( I \), always points into fluid 2. So then the momentum balance becomes

\[ \frac{d}{dt} \int_{V} \rho u dV = \int_{V} \rho \mathbf{b} dV + \int_{V} \nabla \cdot \mathbf{T} - \int_{I} [\mathbf{T}] \cdot \mathbf{n} dS. \quad (2.6) \]

Equation (2.6) differs from the usual balance equation by only the last term on the right side. This “new” force is attributed to an imbalance of molecular forces across the interface and is referred to as surface tension or interfacial tension. Interfacial tension is a cohesive-like property specific to the composition of the interface. The simplest assumption regarding the behavior of this force is that the force acts normal to the curve bounding the interface \( C \) and tangent to the interfacial surface \([36]\) (See Figure 2.4 below). If \( \mathbf{r} \) is tangent to \( C \) and normal to \( \mathbf{n} \) (i.e. tangent to the interfacial surface), surface tension contributing to the
balance of momentum is assumed to act in the direction of the vector $\mathbf{N}$ where

$$\mathbf{N} = \mathbf{r} \times \mathbf{n}.$$  

The magnitude of the interfacial forces contributing to momentum balance, called the \textit{coefficient of surface (interfacial) tension}, is denoted by $\sigma$. Thus, the force due to interfacial tension is given by $\sigma \mathbf{N}$ at each point of $\mathcal{C}$. Hence, the total force due to interfacial tension is

$$\mathbf{F} = \int_{\partial \mathcal{I}} \sigma \mathbf{N} ds.$$  

Since this force is assumed to account for the imbalance of molecular forces at the interface, the following must hold,

$$- \int_\mathcal{I} |\mathbf{T}| \cdot \mathbf{n} dS = \int_{\partial \mathcal{I}} \sigma \mathbf{N} ds \quad (2.7)$$

The following tensor-vector operations are defined [8]

$$(\mathbf{T} \cdot \mathbf{v})_i := \sum_j \tau_{ij} v_j,$$

$$(\mathbf{T} \times \mathbf{v})_{il} := \sum_j \sum_k E_{jkl} \tau_{ij} v_k,$$

where $E_{jkl}$ is a permutation symbol defined by

$$
\begin{align*}
E_{ijk} &= 1, & \text{if } ijk = 123, \text{ 231, or 312} \\
E_{ijk} &= -1, & \text{if } ijk = 321, \text{ 132, or 213} \\
E_{ijk} &= 0, & \text{if any two indices are equal}
\end{align*}
$$
Note that the cross product is equivalent to taking the usual cross product of each column of the tensor with the vector. Recall that $N = r \times n$. Thus,

$$\int_{\partial I} \sigma N ds = \int_{\partial I} \sigma (r \times n) ds$$

$$= \int_{\partial I} \sigma I \cdot (r \times n) ds$$

$$= -\int_{\partial I} (\sigma I \times n) \cdot r ds$$

$$= -\int_I \nabla \times (\sigma I \times n) \cdot n \ dS$$  \hspace{0.5cm} \text{(using Stokes' Theorem)}.

Thus,

$$[T] \cdot n = \nabla \times (\sigma I \times n) \cdot n. \quad (2.8)$$

Now as $\nabla \times (\sigma I \times n)$ is given by

$$\begin{bmatrix}
(s_{n_2})_y + (s_{n_3})_z & -(s_{n_2})_x & -(s_{n_3})_x \\
-(s_{n_1})_y & (s_{n_1})_x + (s_{n_3})_z & -(s_{n_3})_y \\
-(s_{n_1})_z & -(s_{n_2})_z & (s_{n_1})_x + (s_{n_2})_y
\end{bmatrix},$$

then

$$\nabla \times (\sigma I \times n) \cdot n = \begin{bmatrix}
(s (\nabla \cdot n) n_1) - (1 - n_1 n_1) \sigma_x + n_1 n_2 \sigma_y + n_1 n_3 \sigma_z \\
(s (\nabla \cdot n) n_2) - (1 - n_2 n_2) \sigma_y + n_2 n_1 \sigma_x + n_2 n_3 \sigma_z \\
(s (\nabla \cdot n) n_3) - (1 - n_3 n_3) \sigma_z + n_3 n_1 \sigma_x + n_3 n_2 \sigma_y
\end{bmatrix}.$$

From [9], we have that $-\nabla \cdot n = \kappa$ where $\kappa$ is the mean curvature of the interface. Furthermore, the surface gradient of $\sigma$, $\nabla_s \sigma$, may be expressed as $\frac{\partial \sigma}{\partial x_i} = (\delta_{ij} - n_i n_j) \frac{\partial \sigma}{\partial x_i}$. So, (2.9) may be expressed as

$$\nabla \times (\sigma I \times n) \cdot n = -\sigma \kappa n - \nabla_s \sigma.$$

Finally, from (2.8), the equation at the interface is given by

$$[T] \cdot n = -\sigma \kappa n - \nabla_s \sigma. \quad (2.10)$$

### 2.2 Interface Tracking

From equation (2.6), we see that in order to calculate interfacial tension forces, we must know the location of the interface. Thus, it is very important to track the interface between
two fluids efficiently and maintain as sharp an interface as possible. The accuracy of the interface advection scheme will directly affect the computation of the interfacial tension forces which influence the balance of momentum equation. Several methods exist to track or capture the interface between two fluids. In this section, we briefly introduce some of these methods and describe the method we have chosen to include in our numerical implementation, the level-set method. The details of two other methods are presented in the appendix: Flux-Corrected Transport and Finite Element Method with Defect Correction. Much of the broad overview of methods is taken from [33, 37]. Methods are basically divided into two categories, front tracking and front capturing. Front-tracking methods typically identify the interface at a particular time and proceed to move just the interface itself. Front-capturing methods usually define a characteristic variable (called the color function) throughout the computational domain and reconstruct the interface based on the color values.

The earliest, and perhaps most straightforward, of the methods is a front-tracking method called the **Marker and Cell Method** (MAC). The MAC method identifies several points (markers) along the interface and moves these markers according to the given velocity field. The ordering of the markers is constant. Thus connecting the markers at any given time in the correct order yields the approximation to the interface. The interface clearly remains sharp, but there are some obvious drawbacks to the method. Consecutive marker particles may eventually spread very far apart or narrow to points very close to each other. Thus, information about the interface is easily lost unless new particles are added or taken away. The markers will be connected by a linear interpolant (or higher order if desired) while the actual interface may have many changes in curvature between the two markers. Thus calculating curvature and other differential quantities becomes inaccurate. Keeping track of when and where to add and remove markers becomes quite cumbersome. This could, of course, be avoided by initially including a very large number of markers, but this is computationally expensive. In addition to these difficulties, the method also has trouble handling the separation and agglomeration of two fluid regions, and its extension to three dimensional space is quite difficult.
Front-capturing methods are more commonly used in the recent literature. One of the most popular front-capturing methods is known as the Volume of Fluid (VOF) method. In actuality, there are several VOF methods. VOF methods determine the volume percentage of each fluid in each cell of the computational domain. Then the volume fractions, typically called color, are advected with the continuity equation

$$\frac{\partial C}{\partial t} + \nabla \cdot (uC) = 0.$$ \hspace{1cm} (2.11)

Thus, the interface itself is imbedded in the color values. That is to say that the interface must be reconstructed using the color values. These methods are usually implemented with a finite-difference scheme. Because of the character of the advection equation non-physical, spurious oscillations may occur. To minimize these oscillations, it is common to add diffusion to the numerical implementation of (2.11). However, this numerical diffusion causes a loss of sharpness of the interface and may cause problems with mass conservation. Some VOF methods have been written which attempt to maintain the sharpness of the interface. However, these methods must apply “fixes” to do so. Rigorous mathematical analysis of these methods is an open question. However, the accuracy of these methods is quite good. Some of the VOF methods which attempt to maintain a sharp interface are the simplified line interface calculation (SLIC)[27] of Noh and Woodward, Young’s method [43], and flux-corrected transport (FCT)[20]. Summaries of these methods are given in [33]. SLIC allows the interface within a cell to be either parallel to the x-axis or parallel to the y-axis, then draws the appropriate line to provide the correct volume fraction for that cell. Young’s method expands to allow for continuous or nearly continuous interfaces by using a set of cell templates to determine the orientation of the interface. We now turn to a description of the interface-capturing technique which we have chosen to use. Two other techniques are described in detail in Appendix I.

2.2.1 The Level Set Method

In this section, we describe an algorithm called the Level Set Method. This method was first introduced and developed by Osher and Sethian in the late 1980’s (see [28]). This is a front-capturing technique with a slightly different interpretation of the “color” function.
Instead of the color function indicating in which phase each point resides, it indicates the nearest distance from the point to the interface. Distance is negative inside the minor phase and positive within the major phase. Thus, if the distance function is given by $\phi(x, t)$, then the interface is defined by $\phi(x, t) = 0$. That is to say, the interface is a level curve of the distance function. We call a function $f$ a distance function if $|\nabla f| = 1$. We give insight into this definition by considering a simple example in 2-D.

Consider a function $f$ which is defined to be the distance between any point $(x, y)$ and the reference point $(a, b)$. If we parameterize the arc (line) between $(x, y)$ and $(a, b)$ with $(x(s), y(s))$, then $df/ds = 1$ for points away from $(a, b)$ in the direction of positive arclength, and $df/ds = -1$ for points away from $(a, b)$ in the direction of negative arclength. So $|df/ds| = 1$. Also,

$$\frac{df}{ds} = \nabla f \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right).$$

Since $\nabla f$ and $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ are both tangent to the arc, we have

$$\nabla f \cdot \left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \pm |\nabla f|,$$

as $\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ is a unit vector. Thus, $|\nabla f| = 1$.

Using this framework, the level set method handles merger of two fluid volumes naturally. This is a distinct advantage over the other methods discussed herein. The primary reference for our discussion and implementation is Tornberg and Engquist [41]. Once an initial distance profile is defined, $\phi(x, t_1) = \phi_0(x)$, the interface is advected throughout the domain using equation (2.11). That is, we solve

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (u\phi) = 0, \quad \text{in } \Omega, \quad t_1 < t < t_2$$

$$\phi(x, t_1) = \phi_0(x). \quad (2.13)$$

This advection, however, destroys the “distance” property of the function $\phi$. That is, $|\nabla \phi|$ no longer equals 1. Thus, the advected distance function must be corrected so that the distance property is re-established. To achieve this, we solve the following [39]:

$$\frac{\partial \psi}{\partial \tau} = \text{sign}(\phi)(1 - |\nabla \psi|), \quad \text{in } \Omega, \quad \tau > 0 \quad (2.14)$$
\[ \psi(x, 0) = \phi(x, t_2) \quad (2.15) \]

to steady state. Note that the current level set \( \phi(x, t_2) \) is needed only to determine the location of the interface. Solving equations (2.14) and (2.15) to steady state then give a distance function with the same 0-level set as \( \phi \). Thus, we then assign \( \phi = \psi_{\text{steady state}} \).

In actuality, this property needs to hold only in a small region about the interface. This process also has the property that the distances are corrected near the interface first. This can be seen by writing (2.14) as

\[
\frac{\partial \psi}{\partial \tau} + \text{sign}(\phi) \frac{\nabla \psi}{|\nabla \psi|} \cdot \nabla \psi = \text{sign}(\phi).
\]

Thus, we see that this is a hyperbolic equation with the characteristic velocities pointing outward from the interface in the normal direction \[39\]. Our implementation is motivated by the method of \[41\]. We use a finite element discretization in space and finite differences in time. Upwinding is used to add stability to the method. One significant difference between the implementation used here and that of \[41\] is that we have chosen an implicit time-stepping scheme to increase the temporal stability of the method. The numerical implementation is given below.

\( (i) \) Discretization

\[
\left( \frac{\phi^n - \phi^{n-1}}{\Delta t}, v \right) + (u \cdot \nabla \phi^n, v + \nu_c u \cdot \nabla v) = 0, \quad \forall v \in W_h,
\]

where \( W_h \) is the discrete space of test functions, and \( \nu_c = \frac{1}{2\sqrt{(\Delta t)^{-2} + |u|^2 h^{-2}}} \).

\( (ii) \) Correct m times

\[
\left( \frac{\psi^m - \psi^{m-1}}{\Delta \tau}, v \right) + (\epsilon \nabla \psi^m, \nabla v) = - \left( w^{m-1} \cdot \nabla \psi^m, v + \nu_c w^{m-1} \cdot \nabla v \right)
+ \left( S_\alpha(\psi), v + \xi w^{m-1} \cdot \nabla v \right), \quad \forall v \in W_h,
\]

where

\[
\psi_0 = \phi_n,
\]

\[
S_\alpha = \frac{\psi}{\sqrt{\psi^2 + \alpha^2}}.
\]
$w = S_\alpha(\psi_0) \frac{\nabla \psi}{|\nabla \psi|}.$

(iii) The updated distance function is given by

$$\phi^n = \psi^m.$$

Note that $S_\alpha$ is a smoothed “sign” function (+1 in fluid 1, −1 in fluid 2). In our implementation, we take $W_h$ to denote the set of piecewise quadratic functions on a given triangulation of $\Omega$. We also point out that the level set function must be at least quadratic because of the second derivatives needed to calculate curvature. This calculation will be discussed further in the Section 2.3.

2.2.2 Numerical Results

To test the method, a notched cylinder is used as the initial profile. The left picture of Figure 2.5 shows the initial color profile. The right shows the profile after 10 time steps with $\Delta t = 0.5$. The velocity field is constant $u = (1, 0)$.

Figure 2.5  Color profiles at $t = 0$ and $t = 5.0$
2.3 The Continuum Surface Force Model

This section presents a brief discussion of the continuum method for modeling surface tension which was first developed by Brackbill et al.[9]. While Brackbill’s work assumed a constant coefficient of surface tension, no such assumption is made here. Thus, from section (2.1) of this work, there exists a surface “pressure” denoted as

\[ F_{sa}(x_s) = -\sigma(x_s)\kappa(x_s)n(x_s) - \nabla_s \sigma(x_s) \]

where \( x_s \) is a point on the interface, \( \mathcal{I} \). Once again, consider two fluids separated by an interface. The two fluids are identified by a characterizing measure (such as density). This measure is commonly referred to as a color function, \( c(x) \). The color function is a piecewise continuous function defined as

\[
c(x) = \begin{cases} 
  c_1 & ; x \in \text{fluid 1} \\
  c_2 & ; x \in \text{fluid 2} \\
  \frac{1}{2} (c_1 + c_2) & ; x \in \mathcal{I}
\end{cases}
\]

See Figure 2.6.

![Figure 2.6 True color values](image)

The discontinuity in the color function is a result of the continuum assumption. That is, on the continuum scale, the interface may be considered as a surface (i.e. no thickness).
However, the interface is actually a “region” of transition (on the molecular scale) through which the color changes from $c_1$ to $c_2$. This reality provides motivation for a different model of surface tension. The interfacial surface is replaced by an interfacial region (see Figure 2.7. The color is modified so as to change continuously from $c_1$ to $c_2$. Clearly, the width of the transition region is limited by the fineness allowed by the computational tools used. Using this modification, the two-component fluid system may be considered a single component system with continuous (possibly greatly varying within the transition region) characteristics. The interfacial force may be then represented by a local volume force, $F_{sv}$ (i.e. body forces). Thus, the goal becomes that of finding a “continuizing” scheme that retains the character of the original equation. To do so, it is required that a volume force be formulated such that

$$\lim_{h \to 0} \int_V F_{sv}(x) dV = \int_I F_{sa}(x_s) dA,$$

where $h$ is the width of the transition region. This is accomplished with the use of the $\delta$ distribution. The color function is defined in terms of the orthogonal distance from the interface. So, if $x$ is the point of interest and $x_s$ is a point on the interface, the normal distance is given by $d = n(x_s) \cdot (x - x_s)$. Following [9], the surface tension forces may be

![Figure 2.7 Color values vary continuously through and interfacial region](image)
represented by

\[ \int_{\mathcal{I}} F_{sa}(x_s) \, dA = \int_{V} F_{sa}(x) \delta(n(x) \cdot (x - x_s)) \, dV \]  

(2.16)

Thus, the interfacial tension forces are included in the momentum balance as

\[ \int_{V} -\sigma(x)\kappa(x)n(x)\delta(n(x) \cdot (x - x_s)) - \nabla_s \sigma(x)\delta(n(x) \cdot (x - x_s)) \, dV. \]

Finally, the balance of momentum given in (2.6) is given by

\[ \frac{d}{dt} \int_{V} \rho u \, dV = \int_{V} \rho \cdot b \, dV + \int_{V} \nabla \cdot T - \int_{V} \sigma \kappa \delta_{\mathcal{I}} + \nabla_s \sigma \delta_{\mathcal{I}} \, dV \]

which implies, since V is arbitrary, the pointwise equation

\[ \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = \rho \cdot b + \nabla \cdot T - (\sigma \kappa + \nabla_s \sigma) \delta_{\mathcal{I}}, \quad \text{in } \Omega. \]  

(2.17)

Thus, the goal now becomes that of obtaining a numerical implementation of (2.17). Brackbill, et. al. [9] have shown that

\[ \lim_{h \to 0} \int_{V} \sigma \kappa(x) \frac{\nabla \tilde{c}(x)}{|c|} \, dV = \int_{\mathcal{I}} \sigma \kappa n \, dS \]  

(2.18)

where \( \tilde{c} \) is a mollified version of c, and \( [c] = c_2 - c_1 \).

Since this result was set forth in 1991, several analogs have been developed. In our work we use a formulation derived by Chang in [12]. We have modified the result to allow for a spatially varying coefficient of interfacial tension, \( \sigma \). As in the previous section, we designate a level set function by \( \phi \). We then have:

**Lemma 2** The interfacial tension forces may be expressed as

\[ \int_{\mathcal{I}} (\sigma \kappa n + \nabla_s \sigma) \, dS = \int_{V} (\sigma \kappa \nabla \phi + \nabla_s \sigma) \delta(\phi) \, dV \]  

(2.19)

where \( \delta \) is the Dirac delta.

**Proof:**

Let \( \psi \) and \( \gamma \) be functions with the following properties.

\[ \nabla \phi \cdot \nabla \psi = 0, \]  

(2.20)
\[ \nabla \gamma = \nabla \phi \times \nabla \psi, \quad (2.21) \]
\[ |\nabla \psi| \neq 0. \quad (2.22) \]

Thus, \( \phi, \psi, \gamma \) represent a 3-D orthogonal, moving coordinate system. We then introduce the change of variable

\[ x^* = \psi(x, y, z) \]
\[ y^* = \phi(x, y, z) \]
\[ z^* = \gamma(x, y, z) \]

Note that the Jacobian of the transformation is given by

\[ |J| = |\gamma_x(\psi_y \phi_z - \phi_y \psi_z) - \gamma_y(\psi_x \phi_z - \phi_x \psi_z) + \gamma_z(\psi_x \phi_y - \phi_x \psi_y)| \]
\[ = |\nabla \gamma \cdot -\nabla \gamma|, \quad \text{(using (2.21))} \]
\[ = |\nabla \gamma| |\nabla \phi \times \nabla \psi| \]
\[ = |\nabla \gamma| |\nabla \phi| |\nabla \psi|, \quad \text{using (2.20).} \]

Hence, the change of variables is well-defined because \( J \neq 0 \). Then the right side of (2.19) may be written as

\[
\int_V (\sigma \kappa \nabla \phi + |\nabla \phi| \nabla_s \sigma) \delta(\phi) \, dV = \int_{V^*} (\sigma \kappa \nabla \phi + |\nabla \phi| \nabla_s \sigma) \delta(y^*) \frac{1}{|\nabla \phi| |\nabla \psi| |\nabla \gamma|} \, dx^* dy^* dz^* \\
= \int_{\phi=0} (n \sigma \kappa + \nabla_s \sigma) \frac{1}{|\nabla \psi| |\nabla \gamma|} \, dx^* dz^*. \quad (2.23)
\]

Now let \( (\tilde{x}(r, s), \tilde{y}(r, s), \tilde{z}(r, s)) \) be a parameterization of the interface where \( r \) denotes the arclength parameter in the direction of \( \nabla \psi \) and \( s \) the arclength parameter in the direction of \( \nabla \gamma \). Then when \( \phi = 0 \), we have

\[ dx^* = (\psi_x, \psi_y, \psi_z) \cdot (\tilde{x}_r, \tilde{y}_r, \tilde{z}_r) \, dr + (\psi_x, \psi_y, \psi_z) \cdot (\tilde{x}_s, \tilde{y}_s, \tilde{z}_s) \, ds, \]
\[ = (\psi_x, \psi_y, \psi_z) \cdot (\tilde{x}_r, \tilde{y}_r, \tilde{z}_r) \, dr, \quad \text{(as } (\psi_x, \psi_y, \psi_z) \cdot (\tilde{x}_s, \tilde{y}_s, \tilde{z}_s) = 0) \]
\[ = |\nabla \psi| \, dr. \]
Likewise,

\[ dz^* = |\nabla \gamma| \, ds. \]

Thus,

\[ dx^* dz^* = |\nabla \psi| |\nabla \gamma| \, dr \, ds, \]

\[ = |\nabla \psi| |\nabla \gamma| \, dS \]

Therefore

\[
\int_V (\sigma \kappa \nabla \phi + |\nabla \phi| \nabla_s \sigma) \delta(\phi) \, dV = \int_V (\sigma \kappa \nabla \phi + \nabla_s \sigma) \delta(\phi) \, dV \quad \text{(as } |\nabla \phi| = 1) \\
= \int_I (\sigma \kappa \mathbf{n} + \nabla_s \sigma) \, dS.
\]

In our numerical implementation, \( \delta \) is replaced by an approximation. We use [12]

\[
\delta_\epsilon(x) = \begin{cases} 
\frac{1}{\epsilon} \left( 1 + \cos \left( \frac{\pi x}{\epsilon} \right) \right) / \epsilon & \text{if } |x| < \epsilon, \\
0 & \text{otherwise}
\end{cases}
\]

In [41], it was remarked that in advecting \( \phi \) discontinuities in \( \nabla \phi \) can create some small (in magnitude) spurious oscillations. These errors will be amplified when taking further derivatives needed for the curvature. Thus, to avoid this complication, we use a diffused version of \( \phi \), denoted \( \tilde{\phi} \), when calculating the curvature. Thus, we calculate

\[
(\tilde{\phi}, v) + (\epsilon \nabla \tilde{\phi}, \nabla v) = (\phi, v), \quad \forall \, v \in W_h.
\]

Finally, we compute the curvature as

\[
\tilde{n} = \frac{\nabla \tilde{\phi}}{|\nabla \tilde{\phi}|}, \quad \kappa = -\nabla \cdot \tilde{n}.
\]

Thus, we now have all the pieces needed to advect the interface and compute the interfacial tension terms.
CHAPTER 3
APPROXIMATION OF TRANSIENT, VISCOELASTIC, FLUID FLOWS: SINGLE FLUID CASE

3.1 Introduction

Accurate numerical simulations of time dependent viscoelastic flows are important to the understanding of many phenomena in non-Newtonian fluid mechanics, particularly those associated with flow instabilities. Aside from [6], previous numerical analysis in this area has been for steady state flows.

In the case of Newtonian fluid flow the assumption that the extra stress tensor is proportional to the deformation tensor allows the stress to be eliminated from the modeling equations, giving the Navier-Stokes equations. In viscoelasticity, assuming an Oldroyd B type fluid, the stress is defined by a (hyperbolic) differential constitutive equation. Very different from computational fluid dynamics simulations, in viscoelasticity because of a “slow flow” assumption, the non-linearity in the momentum equation is often neglected. The difficulty in performing accurate numerical computations arises from the hyperbolic character of the constitutive equation, which does not contain a dissipative (stabilizing) term for the stress. Care must be used in discretizing the constitutive equation to avoid the introduction of spurious oscillations into the approximation.

The first error analysis for the steady-state finite element approximation of viscoelastic fluid was presented by Baranger and Sandri [5]. In [5] a discontinuous finite element formulation was used for the discretization of the constitutive equation, with the approximation for the stress being discontinuous. Motivated by implementation consideration, Najib and Sandri in [6] modified the discretization in [5] to obtain a decoupled system of two equations, showed the algorithm was convergent, and derived a priori error estimates. In [35], Sandri
presented an analysis of a finite element approximation to this problem wherein the constitutive equation was discretized using a Streamline Upwind Petrov Galerkin (SUPG) method. For the constitutive equation discretized using the method of characteristics, Baranger and Machmoum in [4] analysed this approach and gave error estimates for the approximations.

For the analysis of the time dependent problem, Baranger and Wardi [6] studied a DG approximation to inertialess flow in $\mathbb{R}^2$, using similar techniques as used for the steady state problem. With the Hood-Taylor finite element (FE) pair used to approximate the velocity and pressure, and a discontinuous linear approximation for the stress they showed, under the assumption $\Delta t \leq C_1 h^{3/2}$, that the discrete $H^1$ and $L^2$ errors for the velocity and stress, respectively, were bounded by $C(\Delta t + h^{3/2})$.

In this chapter we analyse the SUPG approximation to the time dependent equations in $\mathbb{R}^d$, $d = 2, 3$. For the fully discrete analysis we extend the approach used in [25] for compressible Navier-Stokes to non-Newtonian flow. For $\nu$ denoting the SUPG coefficient, and assuming Hood-Taylor FE pair approximation for the velocity, pressure, and a continuous FE approximation for the viscoelastic stress, under the assumption $\Delta t, \nu \leq C_1 h^{d/2}$, we obtain that the discrete $H^1$ and $L^2$ errors for the velocity and stress, respectively, are bounded by $C(\Delta t + \nu + h^2)$.

This chapter is organized as follows. A description of the modeling equations is presented in section 3.2. Section 3.3 describes the variational formulation. Section 3.4 contains a description of the mathematical notation, and several lemmas used in the analysis. The semi-discrete and fully discrete approximations are then presented and analysed in sections 3.5 and 3.6, respectively.

### 3.2 The Oldroyd B Model and the Approximating System

In this section we describe the modeling equations for viscoelastic fluid flow (see also [5]).
3.2.1 The Problem

Consider a fluid flowing in a bounded, connected domain $\Omega \subset \mathbb{R}^d$. The boundary of $\Omega$, $\partial \Omega$, is assumed to be Lipschitzian. The vector $\mathbf{n}$ represents the outward unit normal to $\partial \Omega$. The velocity vector is denoted by $\mathbf{u}$, pressure by $p$, total stress by $\mathbf{T}$, and extra stress by $\tau$.

For ease of notation, we use the convention of summation on repeated indices and denote differentiation with a comma. For example, $\frac{\partial \mathbf{u}}{\partial x_i}$ is written $u_i$. Then for a tensor $\tau$ and a vector $\mathbf{w}$, $\nabla \cdot \tau$ denotes $\tau_{ij,j}$, and $\mathbf{w} \cdot \nabla$ denotes the operator $w_i \frac{\partial}{\partial x_i}$. The deformation tensor, $D(\mathbf{u})$, and the vorticity tensor, $W(\mathbf{u})$, are given by

$$D(\mathbf{u}) = \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right), \quad W(\mathbf{u}) = \frac{1}{2} \left( \nabla \mathbf{u} - (\nabla \mathbf{u})^T \right).$$

The Oldroyd model can be described using an objective derivative\([5]\), denoted by $\dot{\sigma}/\partial t$, where

$$\dot{\sigma} := \frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma + g_a(\sigma, \nabla \mathbf{u}), \quad a \in [-1, 1]$$

and

$$g_a(\sigma, \nabla \mathbf{u}) := \sigma W(\mathbf{u}) - W(\mathbf{u})\sigma - a(D(\mathbf{u})\sigma + \sigma D(\mathbf{u}))$$

$$= \frac{1-a}{2} \left( \sigma \nabla \mathbf{u} + (\nabla \mathbf{u})^T \sigma \right) - \frac{1+a}{2} \left( (\nabla \mathbf{u})\sigma + \sigma (\nabla \mathbf{u})^T \right).$$

Oldroyd’s model for stress employs a decomposition of the extra stress into two parts: a Newtonian part and a viscoelastic part. So $\tau = \tau_N + \tau_V$. The Newtonian part is given by $\tau_N = 2(1-\alpha)D(\mathbf{u})$. The $(1-\alpha)$ represents that part of the total viscosity which is considered Newtonian. Hence $\alpha \in (0,1)$ represents the proportion of the total viscosity that is considered to be viscoelastic in nature. For example, if a polymer is immersed within a Newtonian carrier fluid, $\alpha$ is related to the percentage of polymer in the mix. The constitutive law is \([5]\)

$$\tau_N + \lambda \frac{\partial \tau_V}{\partial t} - 2\alpha D(\mathbf{u}) = 0, \quad (3.1)$$

where $\lambda$ is the Weissenberg number, which is a dimensionless constant defined as the product of the relaxation time and a characteristic strain rate \([8]\). For notational simplicity, the subscript, $V$, is dropped, and below $\tau$ will be used to denote the viscoelastic component of the extra stress.
The momentum balance for the fluid is given by

\[
Re \left( \frac{du}{dt} \right) = -\nabla p + \nabla \cdot (2(1-\alpha)D(u) + \tau) + f,
\]

where \( Re \) is the Reynolds number, \( f \) the body forces acting on the fluid, and \( du/dt \) is the material derivative. Recall that

\[
Re = \frac{L \sqrt{\rho}}{\mu}, \quad L = \text{characteristic length scale},
\]

\[
V = \text{characteristic velocity scale},
\]

\[
\rho = \text{fluid density},
\]

\[
\mu = \text{fluid viscosity}.
\]

In addition to (3.1) and (3.2) we also have the incompressibility condition:

\[
\nabla \cdot u = 0 \quad \text{in } \Omega.
\]

To fully specify the problem, appropriate boundary conditions must also be given. The simplest such condition is the homogeneous Dirichlet condition for velocity. In this case, there is no inflow boundary, and, thus, no boundary condition is required for stress. Summarizing, the modeling equations are:

\[
Re \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla p - 2(1-\alpha)\nabla \cdot D(u) - \nabla \cdot \tau = f \quad \text{in } \Omega,
\]

\[
\tau + \lambda \left( \frac{\partial \tau}{\partial t} + u \cdot \nabla \tau + ga(\tau, \nabla u) \right) - 2\alpha D(u) = 0 \quad \text{in } \Omega,
\]

\[
\nabla \cdot u = 0 \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega,
\]

\[
u(0, x) = u_0(x) \quad \text{in } \Omega,
\]

\[	au(0, x) = \tau_0(x) \quad \text{in } \Omega.
\]

In [18], Guillope and Saut proved the following for the “slow-flow” model of (3.3)-(3.8) (i.e. \( u \cdot \nabla u \) term in (3.3) is ignored):

1. local existence, in time, of a unique, regular solution, and
2. under a small data assumption on $f, f', u_0, \tau_0$, the global existence (in time) of a unique solution for $u$ and $\tau$.

In contrast to the Navier–Stokes equations, well-posedness for general models in viscoelasticity is still not well understood. Results which are known fall into one of three types [31]:

1. for initial value problems, solutions have been shown to exist locally in time,
2. global existence (in time) of solutions if the initial conditions are small perturbations of the rest state, and
3. for steady-state problems, existence of solutions which are small perturbations of the analogous Newtonian case.

### 3.3 The Variational Formulation

In this section, we develop the variational formulation of (3.3)-(3.6). The following notation will be used. The $L^2(\Omega)$ norm and inner product will be denoted by $\| \cdot \|$ and $(\cdot, \cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W_p^k}$, respectively. For the semi-norm in $W_p^k(\Omega)$ we use $\| \cdot \|_{W_p^k}$. $H^k$ is used to represent the Sobolev space $W^k_2$, and $\| \cdot \|_k$ denotes the norm in $H^k$. The following function spaces are used in the analysis:

- **Velocity Space**: $X := H_0^1(\Omega) := \{ u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega \}$,
- **Stress Space**: $S := \{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} ; \tau_{ij} \in L^2(\Omega) ; 1 \leq i, j \leq 3 \} \cap \{ \tau = (\tau_{ij}) : u \cdot \nabla \tau \in L^2(\Omega), \forall u \in X \}$,
- **Pressure Space**: $Q := L_0^2(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \}$,
- **Divergence–free Space**: $Z := \{ v \in X : \int_\Omega q (\nabla \cdot v) \, dx = 0, \forall q \in Q \}$.

The variational formulation of (3.3)-(3.6) proceeds in the usual manner. Taking the inner product of (3.3), (3.4), and (3.5) with a velocity test function, a stress test function, and a pressure test function respectively, we obtain

\[
\text{Re} \left( \frac{\partial u}{\partial t} + u \cdot \nabla u, v \right) - (p, \nabla \cdot v) + (2(1-\alpha)D(u) + \tau, D(v)) = (f, v), \forall v \in X \tag{3.9}
\]

\[
\tau + \lambda \left( \frac{\partial \tau}{\partial t} + u \cdot \nabla \tau + g_a(\tau, \nabla u) \right) - 2\alpha D(u), \psi \right) = 0, \forall \psi \in S \tag{3.10}
\]

\[
(\nabla \cdot u, q) = 0, \forall q \in Q \tag{3.11}
\]
The space $Z$ is the space of weakly divergence free functions. Note that the condition

$$(\nabla \cdot \mathbf{u}, q) = 0, \quad \forall \, q \in Q, \, \mathbf{u} \in X,$$

is equivalent in a “distributional” sense to

$$(\mathbf{u}, \nabla q) = 0, \quad \forall \, q \in Q, \, \mathbf{u} \in X,$$  \hspace{1cm} (3.12)

where in (3.12), $(\cdot, \cdot)$ denotes the duality pairing between $H^{-1}$ and $H^1_0$ functions. In addition, note that the velocity and pressure spaces, $X$ and $Q$, satisfy the inf-sup condition

$$\inf_{q \in Q} \sup_{v \in X} \frac{(q, \nabla \cdot v)}{\|q\| \|v\|_1} \geq \beta > 0.$$  \hspace{1cm} (3.13)

Since the inf-sup condition (3.13) holds, an equivalent variational formulation to (3.9)-(3.11) is: Find $(\mathbf{u}, \tau): [0, T] \rightarrow X \times S$ such that

$$\text{Re} \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v} \right) + (2(1 - \alpha)D(\mathbf{u}) + \tau, D\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \, \mathbf{v} \in Z,$$  \hspace{1cm} (3.14)

$$\left( \tau + \lambda \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + g_\alpha(\tau, \nabla \mathbf{u}) \right) - 2\alpha D(\mathbf{u}), \psi \right) = 0, \quad \forall \, \psi \in S.$$  \hspace{1cm} (3.15)

Before discussion the numerical approximation of (3.14),(3.15), we summarize the mathematical notation and interpolation properties used in the analysis.

### 3.4 Mathematical Notation

In this section the mathematical framework and approximation properties are summarized. Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a polygonal domain and let $T_h$ be a triangulation of $\Omega$ made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\Omega = \bigcup K; \quad K \in T_h.$$  

We assume that there exist constants $c_1, c_2$ such that

$$c_1 h \leq h_K \leq c_2 \rho_K$$

where $h_K$ is the diameter of triangle (tetrahedral) $K$, $\rho_K$ is the diameter of the greatest ball (sphere) included in $K$, and $h = \max_{K \in T_h} h_K$. Let $P_k(A)$ denote the space of polynomials
on $A$ of degree no greater than $k$. Then we define the finite element spaces as follows.

$$
X_h := \{ v \in X \cap C(\Omega)^2 : v|_K \in P_k(K), \forall K \in T_h \},
$$

$$
S_h := \{ \sigma \in S \cap C(\Omega)^4 : \sigma|_K \in P_m(K), \forall K \in T_h \},
$$

$$
Q_h := \{ q \in Q \cap C(\Omega) : q|_K \in P_q(K), \forall K \in T_h \},
$$

$$
Z_h := \{ v \in X_h : (q, \nabla \cdot v) = 0, \forall q \in Q_h \}.
$$

Analogous to the continuous spaces, we assume that $X_h$ and $Q_h$ satisfy the discrete inf-sup condition

$$
\inf_{q \in Q_h} \sup_{v \in X_h} \frac{(q, \nabla \cdot v)}{\|q\| \|v\|_1} \geq \beta > 0 . \tag{3.1}
$$

We summarize several properties of finite element spaces and Sobolev’s spaces which we will use in our subsequent analysis. For $(u, p) \in H^{k+1}(\Omega)^d \times H^{q+1}(\Omega)$ we have (see [16]) that there exists $(\mathcal{U}, \mathcal{P}) \in Z_h \times Q_h$ such that

$$
\|u - \mathcal{U}\| \leq C_I h^{k+1} \|u\|_{W_2^{k+1}} , \tag{3.2}
$$

$$
\|u - \mathcal{U}\|_{W_2^1} \leq C_I h^k \|u\|_{W_2^{k+1}} , \tag{3.3}
$$

$$
\|p - \mathcal{P}\| \leq C_I h^{q+1} \|p\|_{W_2^{q+1}} . \tag{3.4}
$$

Let $T \in S_h$ be a $P_1$ continuous interpolant of $\tau$. For $\tau \in H^{m+1}(\Omega)^{d \times d}$ we have that

$$
\|\tau - T\| + h \|\tau - T\|_{W_2^1} \leq C_I h^{m+1} \|\tau\|_{W_2^{m+1}} , \tag{3.5}
$$

$$
\|\tau - T\|_{L^1} + h \|\tau - T\|_{W_2^1} \leq C_I h^{m+1-d/4} \|\tau\|_{W_2^{m+1}} . \tag{3.6}
$$

From [10], we have the following results.

**Lemma 3**: Let \( \{T_h\}, 0 < h \leq 1 \), denote a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subset \mathbb{R}^d$. Let $(\hat{K}, P, N)$ be a reference finite element such that $P \subset W_p^1(\hat{K}) \cap W_q^m(\hat{K})$ where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $0 \leq m \leq l$. For $K \in T_h$, let $(K, P_K, N_K)$ be the affine equivalent element, and $V_h = \{ v : v$ is measurable and $v|_K \in P_K, \forall K \in T_h \}$. Then there exists $C = C(l, p, q)$ such that

$$
\left[ \sum_{K \in T_h} \|v\|_{W_p^1(K)}^p \right]^{1/p} \leq C h^{m-l+\min(0, \frac{d}{p} - \frac{d}{q})} \left[ \sum_{K \in T_h} \|v\|_{W_q^m(K)}^q \right]^{1/q} , \tag{3.7}
$$
for all $v \in V_h$.

**Lemma 4**: Let $I_h$ denote the interpolant of $v$. Then for all $v \in W^m_p(\Omega) \cap C^r(\Omega)$ and $0 \leq s \leq \min\{m, r + 1\}$,

$$
\|v - I_h\|_{s,\infty} \leq C h^{m-s-d/p} \|v\|_{W^m_p} .
$$  \hfill (3.8)

When $v(x,t)$ is defined on the entire time interval $(0, T)$, we define

$$
\|v\|_{\infty,k} := \sup_{0 < t < T} \|v(\cdot, t)\|_k ,
$$

$$
\|v\|_{0,k} := \left( \int_0^T \|v(\cdot, t)\|_k^2 \, dt \right)^{1/2} .
$$

For the analysis of the fully discrete approximation we use $\Delta t$ to denote the step size for $t$ so that $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$, and define

$$
d_t f := \frac{f(t_n) - f(t_{n-1})}{\Delta t} .
$$  \hfill (3.9)

We also use the following additional norms:

$$
\|v\|_{\infty,k} := \max_{1 \leq n \leq N} \|v^n\|_k ,
$$

$$
\|v\|_{0,k} := \left[ \sum_{n=1}^N \Delta t \|v^n\|_k^2 \right]^{1/2} .
$$

### 3.5 Semi-Discrete Approximation

In this section we present the analysis of a semi–discrete approximation to (3.14),(3.15). We begin by introducing some notation specific to the semi–discrete approximation and cite some lemmas used in the analysis.

For $\sigma = \sigma + \nu h \mathbf{u} \cdot \nabla \sigma$ we define

$$
A(w, (\mathbf{u}, \tau), (v, \psi)) := (\tau, \psi_w) - 2\alpha(D(\mathbf{u}), \psi_w) + 2\alpha(\tau, D(v)) + \alpha(1 - \alpha)(\nabla \mathbf{u}, \nabla v) ,
$$  \hfill (3.1)
\[ B(u, v, \tau, \sigma) := (u \cdot \nabla \tau, \sigma) + \frac{1}{2}(\nabla \cdot u \tau, \sigma), \quad (3.2) \]
\[ c(w, u, v) := (w \cdot \nabla u, v), \quad (3.3) \]
\[ \tilde{c}(w, u, v) := \frac{1}{2}(c(w, u, v) - c(w, v, u)). \quad (3.4) \]

**Lemma 5:** [23] For \( u, v, w \in X \), there exists a constant \( C_1 \) such that
\[ |\tilde{c}(u, v, w)| \leq C_1 \|u\|^{1/2} \|\nabla u\|^{1/2} \|\nabla v\|^{1/2} \|\nabla w\|^{1/2}. \quad (3.5) \]

**Note:**
(i) \( \tilde{c}(u, v, w) = c(u, v, w) \) when \( \nabla \cdot u = 0 \) in \( \Omega \), and \( u = 0 \) on \( \partial \Omega \). \quad (3.6)
(ii) \( \tilde{c}(u, v, v) = 0 \), even when \( \nabla \cdot u \neq 0 \). \quad (3.7)
(iii) For \( u \in X \), from the Poincare–Friedrich’s inequality we have that there exists a constant \( C_{PF} = C(\Omega) \) such that
\[ \|u\|^2 \leq C_{PF}^2 \|\nabla u\|^2. \quad (3.8) \]

The operators \( A(\cdot, (\cdot, (\cdot, (\cdot))) : X \times (X \times H^1(\Omega)^{n \times n}) \times (X \times H^1(\Omega)^{n \times n}) \rightarrow \mathbb{R} \), and \( B(\cdot, (\cdot, (\cdot, (\cdot)) : X \times X \times H^1(\Omega)^{n \times n} \times H^1(\Omega)^{n \times n} \rightarrow \mathbb{R} \) are the same as that used in [5],[35]. When \( u = v \) we omit the second variable in \( B(\cdot, (\cdot, (\cdot, (\cdot)) \).

**Lemma 6:** We have that
\[ B(u, \tau, \tau) = \nu h(u \cdot \nabla \tau, u \cdot \nabla \tau). \quad (3.9) \]

**Proof:**

On integrating \((u \cdot \nabla \tau, \sigma)\) by parts we have:
\[ B(u, v, \tau, \sigma) := -(u \cdot \nabla \sigma, \tau) + \nu h(u \cdot \nabla \tau, v \cdot \nabla \sigma) - \frac{1}{2}(\nabla \cdot u \sigma, \tau). \quad (3.10) \]

Setting \( v = u \), \( \sigma = \tau \), and combining (3.2) and (3.10) the stated result follows.
Lemma 7 : For \( w \in X, (u, \tau) \in X \times S \), and \( h \) sufficiently small, we have

\[
A(w, (u, \tau), (u, \tau)) + \lambda B(w, \tau, \tau) \geq C_A \left( \|\tau\|^2 + \|u\|^2 \right).
\]

Proof:

Using the definitions of \( A \) and \( B \) we obtain

\[
A(w, (u, \tau), (u, \tau)) + \lambda B(w, \tau, \tau) =
\]

\[
\|\tau\|^2 + (\tau, \nu h w \cdot \nabla \tau) - 2\alpha(D(u), \tau) - 2\alpha(D(u), \nu h w \cdot \nabla \tau)
\]

\[
+ 2\alpha(\tau, D(u)) + \alpha(1 - \alpha)\|\nabla u\| + \lambda \nu h \|w \cdot \nabla \tau\|^2
\]

\[
\geq \|\tau\|^2 + \alpha(1 - \alpha)\|\nabla u\|^2 + \lambda \nu h \|w \cdot \nabla \tau\|^2 - \frac{1}{2}\|\tau\|^2
\]

\[
- \frac{1}{2} \nu^2 h^2 \|w \cdot \nabla \tau\|^2 - \frac{1}{2} \alpha(1 - \alpha)\|\nabla u\|^2
\]

\[
- \frac{\alpha \nu^2 h^2}{2(1 - \alpha)} \|w \cdot \nabla \tau\|^2
\]

\[
\geq \frac{1}{2} \|\tau\|^2 + \frac{\alpha(1 - \alpha)}{2}\|\nabla u\|^2
\]

\[
+ \left( \lambda \nu h - \frac{\nu^2 h^2}{2} - \frac{\alpha \nu^2 h^2}{2(1 - \alpha)} \right) \|w \cdot \nabla \tau\|^2
\]

(3.11)

\[
\geq C_A \left( \|\tau\|^2 + \|u\|^2 \right), \text{for } h \text{ sufficiently small, using (3.8)}.
\]

Now, we define the semi–discrete approximation of (3.14),(3.15) as:

Find \((u_h, \tau_h) : [0, T] \rightarrow X_h \times S_h\) such that

\[
Re(\tau_{ht}, \nu) + Re \tilde{c}(u_h, u_h, \nu) + (1 - \alpha)(\nabla u_h, \nabla \nu) +
\]

\[
\lambda(\tau_{ht}, \sigma) + \lambda B(u_h, \tau_h, \sigma) + \lambda(g_a(\tau_h, \nabla u_h), \sigma_{u_h}) +
\]

\[
(\tau_h, \sigma_{u_h}) - 2\alpha(D(u_h), \sigma_{u_h}) = 0, \forall \sigma \in S_h.
\]

(3.12)

3.5.1 Analysis of the semi–discrete approximation

In this section, we show that, under suitable conditions, a unique solution to the discretized system exists. Fixed point theory is used to establish the desired result. The proof is established using the following four steps.
1. Define an iterative map in such a way that a fixed point of the map is a solution to (3.12),(3.13).

2. Show the map is well-defined and bounded on bounded sets.

3. Show there exists an invariant ball on which the map is a contraction.

4. Apply Schauder’s fixed point theorem to establish the existence and uniqueness of the discrete approximation.

**Theorem 2**: Assume that the system (3.3)-(3.8) (and thus, (3.14)-(3.15)) has a solution \((u, \tau, p) \in L^2(0, T; H^{k+1}) \times L^\infty(0, T; H^{m+1}) \times L^2(0, T; H^{q+1})\). In addition assume that \(k, m \geq \frac{d}{2}\), and

\[
\|\nabla u\|_\infty, \|\tau\|_\infty, \|\nabla \tau\|_\infty, \|u\|_{k+1}, \|\tau\|_{m+1}, \|p\|_{q+1} \leq D_0 \quad \text{for all } t \in [0, T]. \tag{3.14}
\]

Then, for \(D_0\) and \(h\) sufficiently small, there exists a unique solution to (3.12)-(3.13) satisfying

\[
\int_0^T \left(\|u - u_h\|^2 + \|\nabla (u - u_h)\|^2\right) dt \leq Ch^{\min\{k,m,q+1\}}, \tag{3.15}
\]

\[
\sup_{0 \leq t \leq T} \|\tau - \tau_h\| \leq Ch^{\min\{k,m,q+1\}}. \tag{3.16}
\]

**Proof:**

**Step 1: The Iterative Map**

A mapping \(\xi : L^2(0, T; Z_h) \times L^\infty(0, T; S_h) \to L^2(0, T; Z_h) \times L^\infty(0, T; S_h)\) is defined via:

\[
(u_2, \tau_2) = \xi(u_1, \tau_1) \text{ where } (u_2, \tau_2) \text{ satisfies}
\]

\[
\text{Re}(u_2, v) + \text{Re} c(u_1, u_2, v) + (1 - \alpha)(\nabla u_2, \nabla v) + (\tau_2, D(v)) = (f, v), \quad \forall \ v \in Z_h, \tag{3.17}
\]

\[
\lambda(\tau_2, \sigma) + \lambda B(u_1, \tau_2, \sigma) + (\tau_2, \sigma_{u_1}) - 2\alpha(D(u_h), \sigma_{u_1}) = -\lambda(g(\tau_1, \nabla u_1), \sigma_{u_1}), \quad \forall \ \sigma \in S_h. \tag{3.18}
\]

Thus, given an initial guess \((u_h, \tau_h) \approx (u_1, \tau_1)\), solving (3.17),(3.18) for \((u_2, \tau_2)\) gives a new approximation to the solution. Also, it is clear that a fixed point of (3.17),(3.18) is
a solution to the approximating system (3.12),(3.13) (i.e. $\xi(u_1, \tau_1) = (u_1, \tau_1)$ implies that $(u_1, \tau_1)$ is a solution to (3.12),(3.13)).

Step 2: Show $\xi$ is well-defined and bounded on bounded sets

Note that (3.17)(3.18) corresponds to a first order system of ODEs for the FEM coefficients $c_{u_2}$ and $c_{\tau_2}$ of $u_2$ and $\tau_2$, respectively. That is, (3.17)(3.18) is equivalent to

$$
\begin{bmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{bmatrix}
\begin{bmatrix}
c_{u_2} \\
c_{\tau_2}
\end{bmatrix} = F(t, c_{u_2}, c_{\tau_2}),
$$

where

$$
F(t, c_{u_2}, c_{\tau_2}) = 
\begin{bmatrix}
(f, v) - Re \tilde{c}(u_1, u_2, v) - (1 - \alpha)(\nabla u_2, \nabla v) - (\tau_2, v) \\
-\lambda(g_a(\tau_h, \nabla u_h), \sigma_{u_1}) - \lambda B(u_1, \tau_2, \sigma) - (\tau_2, \sigma_{u_1}) + 2\alpha(D(u_h), \sigma_{u_1})
\end{bmatrix},
$$

and $A_{11}$ and $A_{22}$ are “mass” (invertible) matrices.

Note that $F : [0, T] \times \mathbb{R}^{dim(c_{u_2})} \times \mathbb{R}^{dim(c_{\tau_2})} \rightarrow \mathbb{R}^{dim(c_{u_2})} \times \mathbb{R}^{dim(c_{\tau_2})}$ is a linear function with respect to the FEM coefficients $c_{u_2}, c_{\tau_2}$. Thus, for $f(t)$ a continuous function of $t$, we have that $F$ is Lipschitz continuous. Then, from ODE theory (see [15]), we are guaranteed that there exists a unique local solution for $(c_{u_2}, c_{\tau_2})$, and hence for $(u_2, \tau_2)$.

Next, to establish the existence of $(u_2, \tau_2)$ on $[0, T]$, we show that it remains bounded in the appropriate norms on that interval.

Multiplying (3.17) through by $2\alpha$ and adding the result to (3.18), $(u_2, \tau_2)$ is equivalently determined via

$$
2\alpha Re(u_2, v) + 2\alpha Re \tilde{c}(u_1, u_2, v) + A(u_1, (u_2, \tau_2), (v, \sigma)) + \lambda(\tau_2, \sigma) + \lambda B(u_1, \tau_2, \sigma) = 2\alpha(f, v) - \lambda(g_a(\tau_1, \nabla u_1), \sigma_{u_1}), \quad \forall (v, \sigma) \in Z_h \times S_h.
$$

Choosing $v = u_2$, $\sigma = \tau_2$ in (3.19), and using (3.7),(3.11), implies

$$
\alpha Re \|u_2\|_t^2 + \frac{\lambda}{2}\|\tau_2\|_t^2 + \frac{1}{2}\|\tau_2\|_t^2 + \frac{\alpha(1 - \alpha)}{2}\|\nabla u_2\|_t^2 + \left(\lambda h - \nu^2 h^2\left(\frac{1}{2} + \frac{\alpha}{2(1 - \alpha)}\right)\right)\|u_1 \cdot \nabla \tau_2\|_t^2 
\leq 2\alpha \|f\|_{-1}\|u_2\|_1
$$
Subtracting (3.19) from (3.23) implies that

We begin by defining an invariant ball.

Thus for \( c_1 = \min\{\alpha Re, \lambda/2\} \), and the restriction \( \nu h \leq 2\lambda(1 - \alpha)/(2 - \alpha) \),

Hence for \( 0 \leq t \leq T \),

By the equivalence of norm in finite dimensional spaces, (and \( u_2(0) = u_1(0), \tau_2(0) = \tau_1(0) \)), we therefore have that \( (u_2, \tau_2) \in L^2(0, T; Z_h) \times L^\infty(0, T; S_h) \).

Note that (3.21) also establishes that the mapping \( \xi \) is bounded on bounded sets.

**Step 3: Existence of an invariant ball for \( \xi \).**

We begin by defining an invariant ball.

Let \( R = \epsilon^* h^{\min\{k, m, q + 1\}} \) for \( 0 < \epsilon^* < 1 \), and define the ball \( B_h \) as

The exact solution \( (u, p, \tau) \) of (3.9)-(3.11) satisfies

Subtracting (3.19) from (3.23) implies that

\[
2\alpha Re \left( (u - u_2)_t, v \right) + 2\alpha Re \tilde{c}(u, u, v) - 2\alpha Re \tilde{c}(u_1, u_2, v) + A(u_1, ((u - u_2), (\tau - \tau_2), (v, \sigma))\lambda((\tau - \tau_2)_t, \sigma)
\]
\[ + \lambda B(u_1, (\tau - \tau_2), \sigma) \]
\[ = 2\alpha(p, \nabla \cdot v) - \lambda ((g_a(\tau, \nabla u), \sigma_{u_1}) - (g_a(\tau_1, \nabla u_1), \sigma_{u_1})) \]
\[ - \lambda B(u, u_1, \tau, \sigma) + \lambda B(u_1, \tau, \sigma) \forall (v, \sigma) \in Z_h \times S_h. \]  
\[ (3.24) \]

Let
\[ \Lambda := u - U, \quad E := U - u_2 \]
\[ \Gamma := \tau - T, \quad F := T - \tau_2 \]
\[ \text{and } \epsilon_u := \Lambda + E = u - u_2, \quad \epsilon_\tau := \Gamma + F = \tau - \tau_2. \]  
\[ (3.25) \]
\[ (3.26) \]
\[ (3.27) \]

Rewriting (3.24) using these definitions, along with the choice \( \sigma = F, v = E \), we obtain
\[ 2\alpha Re(E_t, E) + 2\alpha Re(\tilde{c}(u, u, E) - 2\alpha Re(\tilde{c}(u_1, u_2, E) + A(u_1, (E, F),(E, F)) \]
\[ + \lambda (F_t, F) + \lambda B(u_1, F, F) \]
\[ = -2\alpha Re(\Lambda_t, E) - A(u_1, (\Lambda, \Gamma), (E, F)) - \lambda (\Gamma_t, F) - \lambda B(u_1, \Gamma, F) \]
\[ + 2\alpha(p, \nabla \cdot E) - \lambda ((g_a(\tau, \nabla u), F_{u_1}) - (g_a(\tau_1, \nabla u_1), F_{u_1})) \]
\[ - \lambda B(u, u_1, \tau, F) + \lambda B(u_1, \tau, F). \]  
\[ (3.28) \]

We now proceed to bound \( E \) in terms of \( F, u, \) and \( u_1 \).

For the \( \tilde{c} \) terms we have:
\[ \tilde{c}(u, u, E) - \tilde{c}(u_1, u_2, E) = \tilde{c}(u - u_1, u, E) + \tilde{c}(u_1, u - u_2, E) \]
\[ = \tilde{c}(u - u_1, u, E) + \tilde{c}(u_1, E + \Lambda, E) \]
\[ = \tilde{c}(u - u_1, u, E) + \tilde{c}(u_1, E, E) \]  
\[ \text{ (using (3.7))}. \]  
\[ (3.29) \]

We estimate the first term on the rhs of (3.29) by
\[ |\tilde{c}(u - u_1, u, E)| \leq C_1 \|u - u_1\|^{1/2} \|\nabla(u - u_1)\|^{1/2} \||\nabla u|||\nabla E|| \]  
\[ \leq \epsilon_1 ||\nabla E||^2 + \frac{C_1^2}{4\epsilon_1} ||u - u_1|| \|\nabla(u - u_1)|| \||\nabla u||^2. \]  
\[ (3.30) \]

For the second term on the rhs of (3.29)
\[ |\tilde{c}(u_1, E)| \leq | - \tilde{c}((u - u_1), \Lambda, E)| + |\tilde{c}(u, \Lambda, E)| \]

\[ |\tilde{c}(u_1, E)| \leq | - \tilde{c}((u - u_1), \Lambda, E)| + |\tilde{c}(u, \Lambda, E)| \]
\[ C_1 \| \mathbf{u} - \mathbf{u}_1 \|^2 \| \nabla (\mathbf{u} - \mathbf{u}_1) \|^2 \| \nabla \mathbf{A} \| \| \nabla E \| + C_2 \| \mathbf{u} \|_\infty \| \nabla \mathbf{A} \| \| \nabla E \| \leq \epsilon_3 \| \nabla E \|^2 + \frac{C_2^2}{4\epsilon_3} \| \mathbf{u} - \mathbf{u}_1 \|^2 \| \nabla \mathbf{A} \|^2 + \epsilon_4 \| \nabla E \|^2 + \frac{C_2^2}{4\epsilon_4} \| \mathbf{u} \|_\infty^2 \| \nabla \mathbf{A} \|^2. \quad (3.31) \]

In view of the estimates (3.11) and (3.9) we proceed next to consider the terms on the rhs of equation (3.28).

\[(\Lambda_t, E) \leq \| \Lambda_t \| \| E \| \leq \epsilon_5 \| \nabla E \|^2 + \frac{C_{PE}^2}{4\epsilon_5} \| \Lambda_t \|^2, \quad (3.32)\]

\[(\Gamma_t, F) \leq \| \Gamma_t \| \| F \| \leq \epsilon_6 \| F \|^2 + \frac{1}{4\epsilon_6} \| \Gamma_t \|^2. \quad (3.33)\]

For the pressure term we have

\[ 2\alpha |(p, \nabla \cdot E)| = 2\alpha |((p - \overline{P}), \nabla \cdot E)| \leq 2\alpha \| p - \overline{P} \| \| \nabla \cdot E \| \leq 2\alpha \| p \| \| E \| \| \nabla \| \leq 2\alpha d \| p - \overline{P} \| \| \nabla E \| \leq \frac{\alpha^2 d}{\epsilon_7} \| p - \overline{P} \|^2 + \epsilon_7 \| \nabla E \|^2. \quad (3.34)\]

Writing out the \( A \) term on the rhs of (3.28) we have the terms

\[ A(\mathbf{u}_1, (\Lambda, \Gamma, (E, F))) = (\Gamma, F_{u_1}) - 2\alpha (D(\Lambda), F_{u_1}) + 2\alpha (\Gamma, D(E)) + \alpha (1 - \alpha)(\nabla \Lambda, \nabla E). \]

For the first term in \( A \):

\[(\Gamma, F_{u_1}) = (\Gamma, F) + (\Gamma, \nu h \mathbf{u}_1 \cdot \nabla F) = \| \Gamma \| \| F \| + \| \Gamma \| \nu h \| \mathbf{u}_1 \cdot \nabla F \| = \epsilon_8 \| F \|^2 + \frac{1}{4\epsilon_8} \| \Gamma \|^2 + \nu^2 h^2 \| \mathbf{u}_1 \cdot \nabla F \|^2 + \frac{1}{4} \| \Gamma \|^2. \quad (3.36)\]

Similarly,

\[2\alpha (D(\Lambda), F_{u_1}) \leq \epsilon_9 \| F \|^2 + \frac{\alpha^2}{\epsilon_9} \| D(\Lambda) \|^2 + \nu^2 h^2 \| \mathbf{u}_1 \cdot \nabla F \|^2 + \alpha^2 \| D(\Lambda) \|^2, \quad (3.37)\]

\[2\alpha (\Gamma, D(E)) \leq \epsilon_{10} \| \nabla E \|^2 + \frac{\alpha^2}{4\epsilon_{10}} \| \Gamma \|^2, \quad (3.38)\]
Bounding the $g_a(\cdot, \cdot)$ terms on the rhs of (3.28) is more involved. We rewrite the difference as the sum of three terms and then bound each of the terms individually.

We have that

\[
(g_a(\tau, \nabla u) - g_a(\tau_1, \nabla u_1), F_{u_1}) = (g_a(\tau - \tau_1, \nabla u), F_{u_1}) + (g_a(\tau_1, \nabla (u - u_1), F_{u_1}) \\
= (g_a(\tau - \tau_1, \nabla u), F_{u_1}) + (g_a(\tau_1 - \tau, \nabla (u - u_1), F_{u_1}) \\
+ (g_a(\tau, \nabla (u - u_1), F_{u_1}).
\]

For the first term on the rhs of (3.40)

\[
(g_a(\tau - \tau_1, \nabla u), F_{u_1}) \leq 4\|\nabla (\tau - \tau_1)\| \|\nabla u\| \|F\| + 4\|\tau - \tau_1\| \|\nabla u\| \|\nu h u_1 \cdot \nabla F\| \\
\leq 4\|\nabla u\|_{\infty\|(\tau - \tau_1)\| \|F\| + 4\|\tau - \tau_1\| \|\nu h u_1 \cdot \nabla F\| \\
\leq \epsilon_{12}\|F\|^2 + \frac{4\epsilon_{12}^2}{\epsilon_1} \|\nabla u\|_{\infty\|(\tau - \tau_1)\| \|\nu h u_1 \cdot \nabla F\|^2}
\]

For the second term

\[
(g_a(\tau - \tau_1, \nabla (u - u_1)), F_{u_1}) \leq 4\|\tau - \tau_1\| \|\nabla (u - u_1)\| \|\nabla F\| \\
+ 4\|\tau - \tau_1\| \|\nabla (u - u_1)\| \|\nu h u_1 \cdot \nabla F\| \\
\leq \epsilon_{13}\|F\|^2 + \frac{4\epsilon_{13}^2}{\epsilon_1} \|\nabla (u - u_1)\|^2 \\
+ \nu^2 h^2 \|\nu h u_1 \cdot \nabla F\|^2 + 4\|\tau - \tau_1\| \|\nabla (u - u_1)\|^2.
\]

Note that

\[
\|\tau - \tau_1\| \nabla (u - u_1)\| \leq \|\tau - \tau_1\|_L^4 \|\nabla (u - u_1)\|_L^4,
\]

and, using (3.7),

\[
\|\tau_1 - T\|_L^4 \leq C_T h^{-d/4} \|\tau_1 - T\| \\
\leq C_T h^{-d/4} \|\tau_1 - \tau\| + C_T h^{-d/4} \|\tau - T\|.
\]
Thus,
\[
\| \tau - \tau_1 \|_{L^4} \leq \| \tau - T \|_{L^4} + \| T - \tau_1 \|_{L^4} \\
\leq \| \tau - T \|_{L^4} + Ch^{-d/4} \| \tau_1 - \tau \| + Ch^{-d/4} \| \tau - T \| \\
\leq 2Ch^{m+1-d/4} \| \tau \|_{m+1} + C_ch^{-d/4} \| \tau_1 - \tau \|. \tag{3.43}
\]

Similarly,
\[
\| \nabla(u - u_1) \|_{L^4} \leq \| \nabla(u - U) \|_{L^4} + Ch^{-d/4} \| u - u_1 \|_1 + Ch^{-d/4} \| u - U \|_1 \\
\leq 2Ch^{k-\theta/2} \| u \|_{k+1} + C_ch^{-d/4} \| u - u_1 \|_1. \tag{3.44}
\]

Combining (3.43),(3.44) with (3.42) yields
\[
| (g_a(\tau - \tau_1, \nabla(u - u_1)), F_{u_1}) | \leq \epsilon_{13} \| F \|^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \\
+ \left( \frac{4}{\epsilon_{13}} + 4 \right) \left( 2Ch^{m+1-d/4} \| \tau \|_{m+1} + C_ch^{-d/4} \| \tau_1 - \tau \| \right)^2. \\
\left( 2Ch^{k-\theta/4} \| u \|_{k+1} + C_ch^{-d/4} \| u - u_1 \|_1 \right)^2. \tag{3.45}
\]

For the third \( g_a(\cdot, \cdot) \) terms on the rhs of (3.40) we have
\[
| (g_a(\tau, \nabla(u - u_1)), F_{u_1}) | \leq 4\| \tau \nabla(u - u_1) \| \| F \| + 4\| \tau \nabla(u - u_1) \| \| \nu h u_1 \cdot \nabla F \|
\leq 4d\| \tau \|_\infty \| \nabla(u - u_1) \| \| F \| + 4d\| \tau \|_\infty \| \nabla(u - u_1) \| \| \nu h u_1 \cdot \nabla F \|
\leq \epsilon_{14} \| F \| + \frac{4d^2}{\epsilon_{14}} \| \tau \|_\infty^2 \| \nabla(u - u_1) \|^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \\
+ 4d^2 \| \tau \|_\infty^2 \| \nabla(u - u_1) \|^2. \tag{3.46}
\]

What remains is to estimate the three \( B \) terms on the rhs of (3.28). We begin by rewriting the terms in a more convenient form.
\[
-B(u_1, \Gamma, F) - B(u, u_1, \tau, F) + B(u_1, \tau, F) = B(u_1, T, F) - B(u, u_1, \tau, F) \\
= -B(u - u_1, u_1, T, F) - B(u, u_1, \Gamma, F) \\
= B(u - u_1, u_1, \Gamma, F) - B(u - u_1, u_1, \tau, F) \\
- B(u, u_1, \Gamma, F). \tag{3.47}
\]
For the first $B$ term in (3.47) we have

\[
B(u - u_1, u_1, \Gamma, F) = ((u - u_1) \cdot \nabla \Gamma, F) + ((u - u_1) \cdot \nabla \Gamma, \nu h u_1 \cdot \nabla F) \\
+ \frac{1}{2} (\nabla \cdot (u - u_1) \Gamma, F) \\
\leq \|u - u_1\| \|\nabla \Gamma\| + \|u - u_1\| \|\nabla \Gamma\| \|\nu h u_1 \cdot \nabla F\| \\
+ \frac{1}{2} \|\nabla \cdot (u - u_1) \Gamma\| \|F\| \\
\leq \epsilon_15 \|F\|^2 + \left(\frac{1}{4\epsilon_15} + \frac{1}{4}\right) \|u - u_1\| \|\nabla \Gamma\|^2 + \nu^2 h^2 \|u_1 \cdot \nabla F\|^2 \\
+ \epsilon_16 \|F\|^2 + \frac{1}{16\epsilon_16} \|\nabla \cdot (u - u_1) \Gamma\| \|F\|^2 .
\] (3.48)

For $I_u$ the interpolant of $u$ we have, using (3.7),(3.8),

\[
\|u - u_1\|_\infty \leq \|u - I_u\|_\infty + \|I_u - u_1\|_\infty \\
\leq C_n h^{k+1 - \frac{d}{2}} \|u\|_{k+1} + C_v h^{-\frac{d}{2}} \|I_u - u_1\| \\
\leq C_n h^{k+1 - \frac{d}{2}} \|u\|_{k+1} + C_v h^{-\frac{d}{2}} \|I_u - u\| + C_v h^{-\frac{d}{2}} \|u - u_1\| \\
\leq C_{nv} h^{k+1 - \frac{d}{2}} \|u\|_{k+1} + C_v h^{-\frac{d}{2}} \|u - u_1\| .
\] (3.49)

Using this estimate we obtain that

\[
\|(u - u_1) \cdot \nabla \Gamma\| \leq \hat{d} \|u - u_1\|_\infty \|\nabla \Gamma\| \\
\leq \hat{d} \left( C_n h^{k+1 - \frac{d}{2}} \|u\|_{k+1} + C_v h^{-\frac{d}{2}} \|u - u_1\| \right) \|\nabla \Gamma\| .
\] (3.50)

Also,

\[
\|\nabla \cdot (u - u_1) \Gamma\| \leq \hat{d}^2 \|\nabla (u - u_1)\|_\infty \|\Gamma\|_\infty \\
\leq C_v \hat{d}^{3/2} h^{m+1 - \frac{d}{2}} \|u - u_1\|_1 \|\tau\|_{m+1} .
\] (3.51)

Combining (3.48),(3.50), and (3.51) we have

\[
B(u - u_1, u_1, \Gamma, F) \leq (\epsilon_15 + \epsilon_16) \|F\|^2 + \nu^2 h^2 \|u_1 \cdot \nabla F\|^2 \\
+ (\frac{1}{4\epsilon_15} + \frac{1}{4}) \hat{d}^2 \left( C_n h^{k+1 - \frac{d}{2}} \|u\|_{k+1} + C_v h^{-\frac{d}{2}} \|u - u_1\| \right)^2 \|\nabla \Gamma\|^2 \\
+ \frac{1}{16\epsilon_16} \left( C_v \hat{d}^{3/2} h^{m+1 - \frac{d}{2}} \|u - u_1\|_1 \|\tau\|_{m+1} \right)^2 .
\] (3.52)
For the third $B$ term on the rhs of (3.47)

$$B(u - u_1, u_1, \tau, F) = ((u - u_1) \cdot \nabla \tau, F) + ((u - u_1) \cdot \nabla \tau, v h u_1 \cdot \nabla F)$$

$$+ \frac{1}{2} (\nabla \cdot (u - u_1) \tau, F)$$

$$\leq \| (u - u_1) \cdot \nabla \tau \| F \| + \| (u - u_1) \cdot \nabla \tau \| v h u_1 \cdot \nabla F \|$$

$$+ \frac{1}{2} \| \nabla \cdot (u - u_1) \| F \|$$

$$\leq \epsilon_{17} \| F \| ^2 + \frac{1}{4 \epsilon_{17}} \| (u - u_1) \cdot \nabla \tau \| ^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2$$

$$+ \frac{1}{4} \| (u - u_1) \cdot \nabla \tau \| ^2 + \epsilon_{18} \| F \| ^2 + \frac{1}{16 \epsilon_{18}} \| \nabla \cdot (u - u_1) \| ^2$$

$$\leq (\epsilon_{17} + \epsilon_{18}) \| F \| ^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2$$

$$+ d^3 \left( \frac{1}{4 \epsilon_{17}} + \frac{1}{4} \right) \| \nabla \tau \| ^2 \| u - u_1 \| ^2$$

$$+ \frac{d^3}{16 \epsilon_{18}} \| \tau \| ^2 \| \nabla (u - u_1) \| ^2. \quad (3.53)$$

For the third $B$ term on the rhs of (3.47)

$$B(u, u_1, \Gamma, F) = (u \cdot \nabla \Gamma, F) + (u \cdot \nabla \Gamma, v h u_1 \cdot \nabla F) + \frac{1}{2} (\nabla \cdot u \Gamma, F)$$

$$\leq \| u \cdot \nabla \Gamma \| F \| + \| u \cdot \nabla \Gamma \| v h \| u_1 \cdot \nabla F \| + \frac{1}{2} \| \nabla \cdot u \Gamma \| F \|$$

$$\leq (\epsilon_{19} + \epsilon_{20}) \| F \| ^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2$$

$$+ d \left( \frac{1}{4 \epsilon_{19}} + \frac{1}{4} \right) \| u \| ^2 \| \nabla \Gamma \| ^2 + \frac{d^2}{16 \epsilon_{20}} \| \nabla u \| ^2 \| \Gamma \| ^2. \quad (3.54)$$

Returning to (3.28) and putting everything back together:

$$\alpha Re \frac{d}{dt} \| E \| ^2 + \frac{\lambda}{2} \frac{d}{dt} \| F \| ^2 + \frac{1}{2} \| F \| ^2 + \frac{\alpha (1 - \alpha)}{2} \| \nabla E \| ^2$$

$$\left( \nu h - \frac{\nu^2 h^2}{2 \alpha} - \frac{\alpha C^2}{2 (1 - \alpha)} \right) \| u_1 \cdot \nabla F \| ^2 - 2 \alpha (\epsilon_1 + \epsilon_3 + \epsilon_4) \| \nabla E \| ^2$$

$$- 2 \alpha \frac{C^2}{4 \epsilon_1} \| \nabla u \| ^2 \| u - u_1 \| ^2 - 2 \alpha \frac{C^2}{4 \epsilon_3} \| u - u_1 \| ^2 \| \nabla \Lambda \| ^2 - 2 \alpha \frac{C^2}{4 \epsilon_4} \| u \| ^2 \| \nabla \Lambda \| ^2$$

$$\leq 2 \alpha Re \frac{C^2}{4 \epsilon_5} \| \Lambda t \| ^2 + 2 \alpha \epsilon_5 \| \nabla E \| ^2 + \frac{1}{4 \epsilon_6} \| \Gamma \| ^2 + \epsilon_6 \| F \| ^2$$

$$+ \epsilon_8 \| F \| ^2 + \frac{1}{4 \epsilon_8} \| \Gamma \| ^2 + \frac{1}{4} \| \Gamma \| ^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2$$

$$+ \frac{1}{4} \| \Gamma \| ^2 + \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2 + \frac{\alpha^2}{2 \epsilon_9} \| \nabla \Lambda \| ^2 + \epsilon_9 \| F \| ^2 + \frac{\alpha^2}{2 \epsilon_9} \| \nabla \Lambda \| ^2$$

$$+ \nu^2 h^2 \| u_1 \cdot \nabla F \| ^2 + \frac{\alpha^2}{4 \epsilon_{10}} \| \Gamma \| ^2 + \epsilon_{10} \| \nabla E \| ^2 + \frac{\alpha^2 (1 - \alpha)^2}{4 \epsilon_{11}} \| \nabla \Gamma \| ^2.$$
\[ + \epsilon_{11} \| \nabla E \|^2 + \frac{\lambda}{4e_6} \| \Gamma_t \|^2 + \lambda \epsilon_6 \| F \|^2 + \frac{\alpha^2}{\epsilon_7} d \| p - P \|^2 + \epsilon_7 \| \nabla E \|^2 \]
\[ + \lambda \frac{d^2}{\epsilon_{12}} \| \nabla u \|^2_{\infty} \| \tau - \tau_1 \|^2 + \lambda 4d^2 \| \nabla u \|^2_{\infty} \| \tau - \tau_1 \|^2 + \lambda \epsilon_{12} \| F \|^2 \]
\[ + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \]
\[ + \lambda \left( \frac{4}{\epsilon_{13}} + 4 \right) \left( 2C_I h^{m+1-d/4} \| \tau \|_{m+1} + C_I h^{-d/4} \| \tau - \tau_1 \| \right)^2 \cdot \]
\[ \left( 2C_I h^{k-d/4} \| u \|_{k+1} + C_I h^{-d/4} \| u - u_1 \| \right)^2 \]
\[ + \lambda \epsilon_{13} \| F \|^2 + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 + \lambda 4d^2 \left( \frac{1}{\epsilon_{14}} + 1 \right) \| \tau \|^2_{\infty} \| u - u_1 \|^2 \]
\[ + \lambda \epsilon_{14} \| F \|^2 + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \]
\[ + \lambda \left( \frac{1}{4\epsilon_{15}} + \frac{1}{4} \right) d^2 \left( C_{nv} h^{k+1-d/2} \| u \|_{k+1} + C_{v} h^{-d/2} \| u - u_1 \| \right)^2 \| \nabla \Gamma \|^2 \]
\[ + \lambda \frac{1}{16\epsilon_{16}} \left( C_{v} d^3/2 h^{m+1-d/2} \| u - u_1 \|_{m+1} \right)^2 + \lambda \left( \epsilon_{15} + \epsilon_{16} \right) \| F \|^2 \]
\[ + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \]
\[ + \lambda \left( \epsilon_{17} + \epsilon_{18} \right) \| F \|^2 + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \]
\[ + \lambda \frac{1}{4\epsilon_{17}} \| \nabla \tau \|^2_{\infty} \| u - u_1 \| + \lambda \frac{d^3}{16\epsilon_{18}} \| \tau \|^2_{\infty} \| \nabla (u - u_1) \|^2 \]
\[ + \lambda \left( \epsilon_{19} + \epsilon_{20} \right) \| F \|^2 + \lambda \nu^2 h^2 \| u_1 \cdot \nabla F \|^2 \]
\[ + \lambda \frac{1}{4\epsilon_{19}} \| u \|^2_{\infty} \| \nabla \Gamma \|^2 + \lambda \frac{d^2}{16\epsilon_{20}} \| \nabla u \|^2_{\infty} \| \Gamma \|^2 . \] (3.55)

Now, rewriting (3.55) with all the \( E \) and \( F \) terms on the LHS and the RHS terms written in terms “controlled” by the \emph{ball}, terms controlled by interpolation approximation, and terms controlled by both the \emph{ball} and interpolation approximation, we have:

\[ \alpha \text{Re} \frac{d}{dt} \| E \|^2 + \frac{\lambda}{2} \frac{d}{dt} \| F \|^2 \]
\[ + \left( \frac{\alpha(1-\alpha)}{2} - 2\alpha(\epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5) - (\epsilon_7 + \epsilon_{10} + \epsilon_{11}) \right) \| \nabla E \|^2 \]
\[ + \left( \frac{1}{2} - (\epsilon_6 + \epsilon_8 + \epsilon_9) \right) \| \nabla \tau \|^2_{\infty} \| u - u_1 \|^2 \]
\[ + \lambda(\epsilon_6 + \epsilon_{12} + \epsilon_{13} + \epsilon_{14} + \epsilon_{15} + \epsilon_{16} + \epsilon_{17} + \epsilon_{18} + \epsilon_{19} + \epsilon_{20}) \| F \|^2 \]
\[ + \left( \nu h - \nu^2 h^2 \left( \frac{7}{2} - \frac{\alpha}{2(1-\alpha)} + 6\lambda \right) \right) \| u_1 \cdot \nabla F \|^2 \]
\[ \leq \| u - u_1 \|^2 \left\{ \lambda 4d^2 \left( \frac{1}{\epsilon_{14}} + 1 \right) \| \tau \|^2_{\infty} + \lambda d^3 \left( \frac{1}{4\epsilon_{17}} + \frac{1}{4} \right) \| \nabla \tau \|^2_{\infty} \right\} \]
\[ + \| \tau - \tau_1 \|^2 \left\{ \lambda \frac{4d^2}{\epsilon_{12}} \| \nabla u \|^2_{\infty} + 4\lambda d^2 \| \nabla u \|^2_{\infty} \right\} \]
+ \|u - u_1\|^2 \{ 2\alpha \frac{C_2^2}{4 \epsilon_1} \|\nabla u\|^2 + \lambda \frac{d^3}{16 \epsilon_{18}} \|\tau\|^2 \}
+ \|u - u_1\|^2 \left\{ \frac{1}{4} + \frac{1}{4e_6} + \frac{\alpha^2}{4e_10} + \lambda \frac{d^2}{16 \epsilon_{20}} \|\nabla u\|^2 \right\} + \|\Gamma\|^2 \left\{ \frac{\lambda}{4e_6} \right\} + \|\nabla \Gamma\|^2 \left\{ \frac{\alpha^2(1 - \alpha)^2}{4e_{11}} + 2C_{m}^2 h^{2k+2-d} \|u\|^2_{k+1} + \lambda \frac{d}{2} \left( \frac{1}{4e_{19}} + \frac{1}{4} \right) \|u\|^2_{2}\right\}
+ \|\Lambda_t\|^2 \left\{ 2\alpha \text{Re} \frac{C_{2}^2}{4 \epsilon_5} \right\} + \|\nabla \Lambda\|^2 \left\{ 2\alpha \frac{C_2^2}{4 \epsilon_4} \|\nabla u\|^2 \right\} + \|p - \mathcal{P}\|^2 \left\{ \frac{d}{2} \frac{\alpha^2}{\epsilon_7} \right\} + \lambda \left( \frac{4}{\epsilon_{13}} + 4 \right) C_j^2 h^{2m+2-d} \|\tau\|^2_{m+1} \|u\|^2_{k+1}
+ \|u - u_1\|^2 \left\{ \frac{1}{4} + \frac{1}{4} \frac{d}{2} \frac{\alpha^2}{\epsilon_7} \right\} + \|\tau - \tau_1\|^2 \left\{ \lambda \left( \frac{4}{\epsilon_{13}} + 4 \right) C_j^2 h^{2k-d} \|u\|^2_{k+1} \right\}
+ \|u - u_1\|^2 \left\{ \lambda \left( \frac{4}{\epsilon_{13}} + 4 \right) C_j^2 h^{2m+2-d} \|\tau\|^2_{m+1} \right\}
+ \frac{1}{16 \epsilon_{16}} C_j^2 \alpha \frac{d}{2} h^{2m+2-d} \|\tau\|^2_{m+1} \right\} . \quad (3.56)

With our assumptions that \(0 < \alpha < 1\), and \(\lambda > 0\), we can choose values for the \(\epsilon_i\)'s, and \(\nu h\) sufficiently small such that the left hand side of (3.56) is bounded below by

\[
\alpha \text{Re} \frac{d}{dt} \|E\|^2 + \frac{\lambda}{2} \frac{d}{dt} \|F\|^2 + \frac{1}{4} \|F\|^2 + \frac{\alpha(1 - \alpha)}{4} \|\nabla E\|^2 + \frac{\nu h}{2} \|u_1 \cdot \nabla F\|^2 . \quad (3.57)
\]

Let \(D_i, i = 1, \ldots, 6\) denote constants dependent upon \(u, p, \tau, \) their derivatives and \(T.\) (Recall the definition of \(c^*, R\) in (3.22), and \(D_0\) in theorem 2.) As usual \(C_j, j = 4, \ldots, 7,\) denote constants independent of the solution \(u, p, \tau\) and the mesh parameter \(h.\)

Using (3.57) and integrating (3.56) we obtain

\[
\|E\|^2(t) + \|F\|^2(t) + \int_0^t \|\nabla E\|^2(s) ds \leq R^2 C_4 D_0
+ R^4 C_5 h^{-d}
+ D_1 h^{2m+2} + D_2 h^{2m+2}
+ C_6 D_0 h^{2m} + D_3 h^{2k+2m+2-d}
\]
Now, in view of (3.27), we have that for $h, D_0$, and $c^*$ sufficiently small

$$\| \tau - \tau_2 \|^2(t) \leq 2\|F\|^2(t) + 2\|\Gamma\|^2(t) \leq cR^2 + 2D_0h^{2m+2} \leq \tilde{c}R^2,$$  \hfill (3.59)

where $0 < \tilde{c} < 1$. Similarly, for $h$ sufficiently small

$$\| u - u_2 \|^2(t) \leq 2\|E\|^2(t) + 2\|\Lambda\|^2(t) \leq cR^2 + 2D_0h^{2k+2} \leq \tilde{c}R^2,$$  \hfill (3.60)

hence

$$\int_0^T \| u - u_2 \|^2(t) \, dt \leq \frac{\tilde{c}}{2} R^2.$$  \hfill (3.61)

Also, for $h$ sufficiently small

$$\int_0^T \| \nabla (u - u_2) \|^2(t) \, dt \leq 2\int_0^T \| \nabla E \|^2(t) \, dt + 2\int_0^T \| \nabla \Lambda \|^2(t) \, dt \leq c1R^2 + 2D_0Th^{2k} \leq \frac{\tilde{c}}{2} R^2.$$  \hfill (3.62)

Combining (3.59)–(3.62) we have that for $h$ sufficiently small that $\xi$ is a strict contraction on the ball defined in (3.22).

**Step 4:** A direct application of Schauder’s fixed point theorem now establishes the uniqueness of the approximation and the stated error estimates.
3.6 Fully-Discrete Approximation

In this section we analyse we a fully discrete approximation to (3.14), (3.15).

We assume that the fluid flow satisfies the following properties:

\[
\|u\|_\infty, \|\tau\|_\infty, \|\nabla u\|_\infty, \|\nabla \tau\|_\infty \leq M, \tag{3.1}
\]

for all \(t \in [0, T]\).

Note that it follows from (3.1) and inverse estimates that

\[
\|U^n\|_\infty, \|\nabla U^n\|_\infty \leq \tilde{M} \approx M. \tag{3.2}
\]

Below, for simplicity, we take \(\tilde{M} = M\).

To simplify the notation, the following definition is used in the analysis.

**Definitions:**

\[
b(u, \tau, \psi) := (u \cdot \nabla \tau, \psi). \tag{3.3}
\]

To obtain the fully discretized approximation, the time derivatives are replaced by backward differences and the nonlinear terms are lagged. As we are assuming "slow flow", i.e. \(Re \equiv O(1)\), we use a conforming finite element method to discretize the momentum equation. For the constitutive equation for stress, we use a streamline upwind Petrov-Galerkin (SUPG) discretization to control the production of spurious oscillations in the approximation. The discrete approximating system of equations is then:

**Approximating System**

**For** \(n = 1, 2, \ldots, N\), **find** \(u^n_h \in Z_h, \tau^n_h \in S_h\) **such that**

\[
Re \left( d_t u^n_h, v \right) + Re c \left( u^{n-1}_h, u^n_h, v \right) + (1 - \alpha) (\nabla u^n_h, \nabla v) + (\tau^n_h, D(v)) = (f^n, v), \quad v \in Z_h
\]

\[
\frac{1}{\lambda} (\tau^n_h, \tilde{\sigma}) + (d_t \tau^n_h, \sigma) + b \left( u^{n-1}_h, \tau^n_h, \tilde{\sigma} \right) - \bar{\lambda} (D(u^n_h), \tilde{\sigma}) = - \left( g_0 (\tau^{n-1}_h, \nabla u^{n-1}_h, \tilde{\sigma}) \right), \quad \sigma \in S_h
\]

where \(\tilde{\sigma} := \sigma + \nu \sigma^{n}_u, \sigma^{n}_u := u^{n-1}_h \cdot \nabla \sigma\), \(\nu\) is a small positive constant, and \(\bar{\lambda} = \lambda/(2\alpha)\).

The parameter \(\nu > 0\) is used to supress the production of spurious oscillations in the approximation. Note that for \(\nu = 0\) the discretization of the constitutive equation is a conforming Galerkin method. The goal in choosing \(\nu\) is to keep it as small as possible, but large enough to control the generation of catastrophic spurious oscillations in the approximate stress.
Choosing $\nu = u^n_h, \sigma = \tau^n_h$, multiplying (3.4) by $\bar{\lambda}$ and adding to (3.5) we obtain

$$a(u^n, \tau^n; u^n, \tau^n) = \bar{\lambda} (f^n, u^n) + \frac{\lambda}{\Delta t} (u^{n-1}, u^n)$$

$$- (g_a (\tau^{n-1}, \nabla u^{n-1}), \tau^n) + \frac{1}{\Delta t} (\tau^{n-1}, \tau^n), \quad (3.6)$$

where the bilinear form $a(u, \tau; v, \sigma)$ is defined as:

$$a(u, \tau; v, \sigma) := \frac{\lambda}{\Delta t} (u, v) + \bar{\lambda} Re c(u^{n-1}, u, v) + \bar{\lambda} (1 - \alpha) (\nabla u, \nabla v) + \frac{1}{\lambda} (\tau, \bar{\sigma})$$

$$+ \frac{1}{\Delta t} (\tau, \bar{\sigma}) + b(u^{n-1}, \tau, \sigma) + b(u^{n-1}, \tau, \nu u^{n-1} \cdot \nabla \sigma)$$

$$- \bar{\lambda} (D(u), \nu u^{n-1} \cdot \nabla \sigma).$$

We now estimate the terms in $a(u^n, \tau^n; u^n, \tau^n)$. We have

$$|c(u^{n-1}, u, u)| = \left| (u^{n-1} \cdot \nabla u, u) \right| \leq d_2^2 \|u^{n-1}\|_\infty \|\nabla u\| \|u\|$$

$$\leq \epsilon_1 \|\nabla u\|^2 + \frac{\lambda K^2}{4 \epsilon_1} \|u\|^2,$$

$$|b(u^{n-1}, \tau, \tau)| = \left| (u^{n-1} \cdot \nabla \tau, \tau) \right| \leq \|u^{n-1} \cdot \nabla \tau\| \|\tau\|$$

$$\leq \epsilon_2 \|u^{n-1} \cdot \nabla \tau\|^2 + \frac{1}{4 \epsilon_2} \|\tau\|^2.$$

$$b(u^{n-1}, \tau, \nu u^{n-1} \cdot \nabla \tau) = \nu \|u^{n-1} \cdot \nabla \tau\|^2,$$

$$\left| (D(u), \nu u^{n-1} \cdot \nabla \tau) \right| \leq \|D(u)\| \|\nu u^{n-1} \cdot \nabla \tau\|$$

$$\leq \epsilon_3 \|D(u)\|^2 + \frac{\nu^2}{4 \epsilon_3} \|u^{n-1} \cdot \nabla \tau\|^2$$

$$\leq \epsilon_3 \|\nabla u\|^2 + \frac{\nu^2}{4 \epsilon_3} \|u^{n-1} \cdot \nabla \tau\|^2.$$
Applying these inequalities to the bilinear form \( a(\cdot, \cdot; \cdot, \cdot) \) yields

\[
a(u^n, \tau^n; u^n, \tau^n) \geq \bar{\lambda} Re \left( \frac{1}{\Delta t} - \frac{dK^2}{4\epsilon_1} \right) \|u^n\|^2 + \bar{\lambda}((1 - \alpha) - Re \epsilon_1 - \epsilon_3) \|\nabla u\|^2 + \left( \frac{1}{\lambda} + \frac{1}{\Delta t} - \frac{1}{4\epsilon_2} \right) \|\tau^n\|^2 + \left( \nu - \epsilon_2 - \frac{\nu^2}{4\epsilon_3} \right) \|u^{n-1} \cdot \nabla \tau^n\|^2.
\]

Choosing \( \epsilon_1 = \frac{(1-\alpha)}{4 Re}, \epsilon_2 = \frac{\nu}{3}, \epsilon_3 = \frac{(1-\alpha)}{4}, \nu = \frac{2(1-\alpha)}{3} \), and \( \Delta t \leq \min \left\{ \frac{1-\alpha}{Re dK^2}, \nu \right\} \), it follows that the bilinear form \( a(\cdot, \cdot; \cdot, \cdot) \) is positive. Hence, (3.6) has at most one solution. Since (3.6) is a finite dimensional linear system, the uniqueness of the solution implies the existence of the solution.

The discrete Gronwall’s lemma plays an important role in the following analysis.

**Lemma 9 (Discrete Gronwall’s Lemma)** [19] Let \( \Delta t, H, \) and \( a_n, b_n, c_n, \gamma_n \), (for integers \( n \geq 0 \)), be nonnegative numbers such that

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} \gamma_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \quad \text{for } l \geq 0.
\]

Suppose that \( \Delta t \gamma_n < 1 \), for all \( n \), and set \( \sigma_n = (1 - \Delta t \gamma_n)^{-1} \). Then,

\[
a_l + \Delta t \sum_{n=0}^{l} b_n \leq \exp \left( \Delta t \sum_{n=0}^{l} \sigma_n \gamma_n \right) \left\{ \Delta t \sum_{n=0}^{l} c_n + H \right\} \quad \text{for } l \geq 0. \quad (3.7)
\]

### 3.6.1 Analysis of the fully–discrete approximation

In this section we analyze the error between the finite element approximation given by (3.4),(3.5) and the true solution. A priori error estimates for the approximation are in theorem 3.

**Theorem 3** Assume that the system (3.3)-(3.8) (and thus, (3.14)-(3.15)) has a solution \((u, \tau, p) \in C^2(0, T; H^{k+1}) \times C^2(0, T; H^{m+1}) \times C(0, T; H^{q+1})\). In addition assume that \( \Delta t, \nu \leq ch^{d/2} \), and

\[
\|u\|_\infty, \|\nabla u\|_\infty, \|\tau\|_\infty, \|\nabla \tau\|_\infty \leq M \quad \text{for all } t \in [0, T]. \quad (3.8)
\]
Then, the finite element approximation (3.4)-(3.5) is convergent to the solution of (3.14)-(3.15) on the interval $(0, T)$ as $\Delta t, h \to 0$. In addition, the approximation $(u_h, \tau_h)$ satisfies the following error estimates:

\[
\||u_h - u||_{\infty,0} + ||\tau_h - \tau||_{\infty,0} \leq F(\Delta t, \nu, h) \tag{3.9}
\]
\[
\||u_h - u||_{0,1} + ||\tau_h - \tau||_{0,0} \leq F(\Delta t, \nu, h) \tag{3.10}
\]

where

\[
F(\Delta t, \nu, h) = C \left( h^k \||u||_{0,k+1} + h^{k+1} \||u_t||_{0,k+1} \right) + C \left( h^m \||\tau||_{0,m+1} + h^{m+1} \||\tau_t||_{0,m+1} \right) + C \left( h^{k+1} \||u||_{\infty,k+1} + h^{m+1} \||\tau||_{\infty,m+1} \right) + C |\Delta t| \left( ||u_t||_{0,1} + ||u_{tt}||_{0,0} + ||\tau_t||_{0,1} + ||\tau_{tt}||_{0,0} \right) + C \nu \left( ||\tau_t||_{0,1} + ||\tau||_{\infty,0} \right).
\]

In order to establish the estimates (3.9)-(3.10), we begin by introducing the following notation. Let $u^n = u(t_n), \tau^n = \tau(t_n)$ represent the solution of (3.14)-(3.15), and $u^n_h, \tau^n_h$ denote the solution of (3.4)-(3.5).

Define $\Lambda^n, E^n, F^n, \epsilon_u, \epsilon_\tau$ as

\[
\Lambda^n = u^n - U^n, \quad E^n = U^n - u^n_h, \\
F^n = T^n - \tau^n, \quad \epsilon_u = u - u^n_h, \\
\epsilon_\tau = \tau - \tau^n_h.
\]

The proof of theorem 3 is established in three steps.

1. Prove a lemma, assuming two induction hypotheses.
2. Show that the induction hypotheses are true.
3. Prove the error estimates given in (3.9),(3.10).

**Step 1.** We prove the following lemma.

**Lemma 10** Under the induction hypothesis (IH1) and the additional assumption

\[
(IH2) \quad \sum_{n=1}^{l-1} \Delta t \||E^n||_{\infty} \leq 1,
\]
th we have that
\[ \|E^t\|^2 + \|r^t\|^2 \leq G(\Delta t, h, \nu), \quad (3.11) \]

where
\[
G(\Delta t, h, \nu) = C \left( h^{2k} \|u\|^2_{0,k+1} + h^{2k+2} \|u_t\|^2_{0,k+1} \right) + C \left( h^{2m} \|\tau\|^2_{0,m+1} + h^{2m+2} \|\tau_t\|^2_{0,m+1} \right) + C h^{2q+2} \|p\|^2_{0,q+1} + C |\Delta t|^2 \left( \|u\|^2_{0,1} + \|u_t\|^2_{0,0} + \|\tau\|^2_{0,1} + \|\tau_t\|^2_{0,0} \right) + C \nu^2 \left( \|\tau\|^2_{\infty,1} + \|\tau_t\|^2_{\infty,0} \right).
\]

**Proof of lemma 10:** From (3.14)-(3.15), it is clear that the true solution \((u, \tau)\) satisfies
\[
Re (dtu^n, v) + Re c(u^n_{h^{-1}}, u^n, v) + (1 - \alpha) (\nabla u^n, \nabla v) + (\tau^n, D(v)) = (f^n, v) + (p^n, \nabla \cdot v) + R_1(v), \quad \forall v \in Z_h, \quad (3.12)
\]
\[
(dt\tau^n, \sigma) + b(u^n_{h^{-1}}, \tau^n, \bar{\sigma}) - \lambda (D(u^n), \bar{\sigma}) + \frac{1}{\lambda} (\tau^n, \bar{\sigma}) = - \left( g_a \left( \tau^n_{h^{-1}}, \nabla u^n_{h^{-1}} \right), \bar{\sigma} \right) + R_2(\sigma), \quad \forall \sigma \in S_h, \quad (3.13)
\]
where
\[
R_1(v) := Re (dtu^n, v) - Re (u^n_t, v) + Re c(u^n_{h^{-1}}, u^n, v) - Re c(u^n, u^n, v),
\]
and
\[
R_2(\sigma) := (dt\tau^n, \sigma) - (\tau^n_t, \sigma) - \nu \left( \tau^n_t, u^n_{h^{-1}} \cdot \nabla \sigma \right) + b(u^n_{h^{-1}}, \tau^n, \bar{\sigma}) + b(u^n, \tau^n, \bar{\sigma}) - \left( g_a \left( \tau^n_{h^{-1}}, \nabla u^n_{h^{-1}} \right), \bar{\sigma} \right) - \left( g_a \left( \tau^n, \nabla u^n \right), \bar{\sigma} \right).
\]

Subtracting (3.4)-(3.5) from (3.12)-(3.13) we obtain the following equations for \(\epsilon_u\) and \(\epsilon_\tau\):
\[
Re (dt\epsilon_u, v) + Re c(u^n_{h^{-1}}, \epsilon_u, v) + (1 - \alpha) (\nabla \epsilon_u, \nabla v) + (\epsilon_\tau, D(v)) = (p^n, \nabla \cdot v) + R_1(v), \quad \forall v \in Z_h, \quad (3.14)
\]
\[
(dt\epsilon_\tau, \sigma) + b(u^n_{h^{-1}}, \epsilon_\tau, \bar{\sigma}) - \lambda (D(\epsilon_u), \bar{\sigma}) + \frac{1}{\lambda} (\epsilon_\tau, \bar{\sigma}) = R_2(\sigma), \quad \forall \sigma \in S_h. \quad (3.15)
\]
Substituting $\epsilon_u = E^n + \Lambda^n$, $\epsilon_F = F^n + \Gamma^n$, $v = E^n$, $\sigma = F^n$ into (3.14)-(3.15), we obtain

\[
Re \ (d_t E^n, E^n) + Re \ c(u_h^{n-1}, E^n, E^n) + (1 - \alpha) (\nabla E^n, \nabla E^n) + (F^n, D(E^n)) = \mathcal{F}_1(E^n), \tag{3.16}
\]

\[
(d_t F^n, F^n) + b(u_h^{n-1}, F^n, \tilde{F}^n) - \hat{\lambda} \left( D(E^n), \tilde{F}^n \right) + \frac{1}{\lambda} \left( F^n, \tilde{F}^n \right) = \mathcal{F}_2(F^n), \tag{3.17}
\]

where,

\[
\mathcal{F}_1(E^n) = (p^n, \nabla \cdot E^n) + R_1(E^n) - Re \ (d_t \Lambda^n, E^n) - Re \ c(u_h^{n-1}, \Lambda^n, E^n) - (1 - \alpha) (\nabla \Lambda^n, \nabla E^n) - (\Gamma^n, D(E^n)),
\]

\[
\mathcal{F}_2(F^n) = R_2(F^n) - (d_t \Gamma^n, F^n) - b(u_h^{n-1}, \Gamma^n, \tilde{F}^n) + \hat{\lambda} \left( D(\Lambda^n), \tilde{F}^n \right) - \frac{1}{\lambda} \left( \Gamma^n, \tilde{F}^n \right).
\]

Multiplying (3.16) by $\hat{\lambda}$ and adding to (3.17) we obtain the single equation

\[
Re \ \hat{\lambda} \ (d_t E^n, E^n) + Re \ \hat{\lambda} \ c(u_h^{n-1}, E^n, E^n) + (1 - \alpha) \hat{\lambda} (\nabla E^n, \nabla E^n) + (d_t F^n, F^n) + b(u_h^{n-1}, F^n, \tilde{F}^n) - \hat{\lambda} \left( D(E^n), \tilde{F}^n \right) + \frac{1}{\lambda} \left( F^n, \tilde{F}^n \right) = \hat{\lambda} \mathcal{F}_1(E^n) + \mathcal{F}_2(F^n). \tag{3.18}
\]

Note that

\[
(d_t E^n, E^n) = \frac{1}{\Delta t} \left[ (E^n, E^n) - (E^{n-1}, E^n) \right] \geq \frac{1}{\Delta t} \left[ \|E^n\|^2 - \|E^n\| \|E^{n-1}\| \right] \geq \frac{1}{2\Delta t} \left[ \|E^n\|^2 - \|E^{n-1}\|^2 \right],
\]

and similarly, $(d_t F^n, F^n) \geq \frac{1}{2\Delta t} \left[ \|F^n\|^2 - \|F^{n-1}\|^2 \right]$. Thus, we have

\[
\frac{Re \ \hat{\lambda}}{2\Delta t} \left[ \|E^n\|^2 - \|E^{n-1}\|^2 \right] + \frac{1}{2\Delta t} \left[ \|F^n\|^2 - \|F^{n-1}\|^2 \right] + (1 - \alpha) \hat{\lambda} \|\nabla E^n\|^2 + \nu \|u_h^{n-1} \cdot \nabla F^n\|^2 + \frac{1}{\lambda} \|F^n\|^2 \leq -Re \ \hat{\lambda} c(u_h^{n-1}, E^n, E^n) - b(u_h^{n-1}, F^n, \tilde{F}^n) + \hat{\lambda} \left( D(E^n), \nu u_h^{n-1} \cdot \nabla F^n \right) - \frac{1}{\lambda} \left( F^n, \nu u_h^{n-1} \cdot \nabla F^n \right) + \hat{\lambda} \mathcal{F}_1(E^n) + \mathcal{F}_2(F^n). \tag{3.19}
\]

Multiplying (3.19) by $\Delta t$ and summing from $n = 1$ to $l$ yields:

\[
\frac{Re \ \hat{\lambda}}{2} \left[ \|E^l\|^2 - \|E^0\|^2 \right] + \frac{1}{2} \left[ \|F^l\|^2 - \|F^0\|^2 \right] + (1 - \alpha) \hat{\lambda} \sum_{n=1}^{l} \Delta t \|\nabla E^n\|^2
\]
\[ + \nu \sum_{n=1}^{l} \Delta t \left\| u_n^{n-1} \cdot \nabla F^n \right\|^2 + \frac{1}{\lambda} \sum_{n=1}^{l} \Delta t \| F^n \|^2 \] (3.20)

\[ \leq \Delta t \sum_{n=1}^{l} \left[ -Rc \bar{\lambda} c(u_h^{n-1}, E^n, E^n) - b(u_h^{n-1}, F^n, F^n) + \bar{\lambda} \left( D(E^n), \nu u_h^{n-1} \cdot \nabla F^n \right) - \frac{1}{\lambda} \left( F^n, \nu u_h^{n-1} \cdot \nabla F^n \right) \right] \]

\[ + \bar{\lambda} \Delta t \sum_{n=1}^{l} F_1(E^n) + \Delta t \sum_{n=1}^{l} F_2(F^n). \] (3.21)

We now estimate each term on the right hand side of (3.21). For \( c(u_h^{n-1}, E^n, E^n) \) we have that

\[ \left| c(u_h^{n-1}, E^n, E^n) \right| \leq \left| \left( u_h^{n-1} \cdot \nabla E^n, E^n \right) \right| \]

\[ \leq \left| u_h^{n-1} \cdot \nabla E^n \right| \| E^n \| \]

\[ \leq \left| u_h^{n-1} \right| \| \nabla E^n \| \| E^n \| \]

\[ \leq \epsilon_1 \| \nabla E^n \|^2 + \frac{dK^2}{4\epsilon_1} \| E^n \|^2, \text{ using (IH1)}. \] (3.22)

Note that for \( v = 0 \) on \( \partial \Omega \), applying Green’s theorem we have

\[ b(v, \tau, \sigma) = -b(v, \sigma, \tau) - (\nabla \cdot v, \tau, \sigma), \] (3.23)

\[ \Rightarrow b(v, \tau, \tau) = -\frac{1}{2} (\nabla \cdot v, \tau, \tau). \] (3.24)

Using (3.24),

\[ \left| b(u_h^{n-1}, F^n, F^n) \right| = \frac{1}{2} \left| \left( \nabla \cdot u_h^{n-1} F^n, F^n \right) \right| \]

\[ = \frac{1}{2} \left| \left( \nabla \cdot u_h^{n-1} U^{n-1} F^n, F^n \right) + \left( \nabla U^{n-1} F^n, F^n \right) \right| \]

\[ \leq \frac{1}{2} \left\| \nabla \cdot E^{n-1} \right\|_\infty \| F^n \|^2 + \frac{1}{2} \left\| \nabla U^{n-1} \right\|_\infty \| F^n \|^2 \]

\[ \leq \frac{1}{2} \left\| \nabla \cdot E^{n-1} \right\|_\infty \| F^n \|^2 + \frac{1}{2} M \| F^n \|^2, \text{ using (3.2)}. \]

Next,

\[ \left| \left( D(E^n), \nu u_h^{n-1} \cdot \nabla F^n \right) \right| \leq \left| D(E^n) \right| \left| \nu u_h^{n-1} \cdot \nabla F^n \right| \]

\[ \leq \| \nabla E^n \| \| \nu u_h^{n-1} \cdot \nabla F^n \| \]

\[ \leq \epsilon_2 \| \nabla E^n \|^2 + \frac{\nu^2}{4\epsilon_2} \| u_h^{n-1} \cdot \nabla F^n \|^2. \]
Also,
\[
\left| \left( F^n, \nu u_h^{n-1} \cdot \nabla F^n \right) \right| = \nu \left| \left( F^n, u_h^{n-1} \cdot \nabla F^n \right) \right|
\]
\[
\leq \nu \| F^n \| \| u_h^{n-1} \cdot \nabla F^n \|
\]
\[
\leq \| F^n \|^2 + \frac{\nu^2}{4} \| u_h^{n-1} \cdot \nabla F^n \|^2.
\]

Thus, for the first summation on the right hand side of (3.21), we have
\[
\Delta t \sum_{n=1}^{t} \left[ - \text{Re} \lambda c(u_h^{n-1}, E^n, E^n) - b(u_h^{n-1}, F^n, F^n) \right.
\]
\[
\quad + \lambda \left( D(E^n), \nu u_h^{n-1} \cdot \nabla F^n \right) - \frac{1}{\lambda} \left( F^n, \nu u_h^{n-1} \cdot \nabla F^n \right) \right]
\]
\[
\leq \Delta t \sum_{n=1}^{t} (Re \lambda \epsilon_1 + \dot{\lambda} \epsilon_2) \| \nabla E^n \|^2 + \Delta t \sum_{n=1}^{t} \frac{Re \lambda \dot{\lambda} K^2}{4 \epsilon_1} \| E^n \|^2
\]
\[
+ \Delta t \sum_{n=1}^{t} \left( \frac{\dot{\lambda} \nu^2}{4 \epsilon_2} + \frac{\nu^2}{\lambda \epsilon_3} \right) \| u_h^{n-1} \cdot \nabla F^n \|^2
\]
\[
+ \Delta t \sum_{n=1}^{t} \left( \frac{1}{2} M + \frac{1}{2} \| \nabla F^{n-1} \|_{\infty} + \frac{\epsilon_3}{\lambda} \right) \| F^n \|^2. \quad (3.25)
\]

Next we consider \( \mathcal{F}_1(E^n) \).
\[
\left| (p^n, \nabla \cdot E^n) \right| = \left| (p^n - p^n, \nabla \cdot E^n) \right|
\]
\[
\leq \| p^n - p^n \| d \frac{1}{2} \| \nabla E^n \|
\]
\[
\leq \epsilon_4 \| \nabla E^n \|^2 + \frac{d}{4 \epsilon_4} \| p^n - p^n \|^2. \quad (3.26)
\]
\[
\left| (d_t A^n, E^n) \right| \leq \| E^n \| \| d_t A^n \|
\]
\[
\leq \| E^n \|^2 + \frac{1}{4} \| d_t A^n \|^2.
\]
\[
\left| c(u_h^{n-1}, A^n, E^n) \right| \leq \| E^n \| \| u_h^{n-1} \cdot \nabla A^n \|
\]
\[
\leq \| E^n \| \| u_h^{n-1} \|_{\infty} d \frac{1}{2} \| \nabla A^n \|
\]
\[
\leq \| E^n \|^2 + \frac{K^2 d}{4} \| \nabla A^n \|^2, \quad \text{using (IH1).} \quad (3.27)
\]
\[
\left| (\nabla A^n, \nabla E^n) \right| \leq \| \nabla E^n \| \| \nabla A^n \|
\]
\[
\leq \epsilon_5 \| \nabla E^n \|^2 + \frac{1}{4 \epsilon_5} \| \nabla A^n \|^2. \quad (3.28)
\]
\[
\left| (\Gamma^n, D(E^n)) \right| \leq \| D(E^n) \| \| \Gamma^n \|
\]
\[
\leq \| \nabla E^n \| \| \Gamma^n \|
\]
\[ \leq \epsilon_6 \| \nabla E^n \|^2 + \frac{1}{4\epsilon_6} \| \Gamma^n \|^2. \] (3.29)

For the \( R_1(E^n) \) terms we have:

\[
| (d_t u^n, E^n) - (u^n_t, E^n) | \leq \| E^n \|^2 + \frac{1}{4} \| d_t u^n - u^n_t \|^2 \] (3.30)

\[
| c(u^n_{h-1}, u^n, E^n) - c(u^n, E^n) | = | c(u^n_{h-1} - U^n - 1, u^n, E^n) + c(U^n - 1, u^n, E^n) + c(u^n - 1, u^n, E^n) | \leq \| E^{n-1} \cdot \nabla u^n \| \| E^n \| + \| \Lambda^{n-1} \cdot \nabla u^n \| \| E^n \|
+ \| (u^n - u^n_{h-1}) \cdot \nabla u^n \| \| E^n \|
\leq \hat{d} M \| E^{n-1} \| \| E^n \| + \hat{d} M \| \Lambda^{n-1} \| \| E^n \|
+ \hat{d} M \| (u^n - u^n_{h-1}) \| \| E^n \|
\leq \hat{d} M \left( \frac{E^{n-1}}{2} + \left( \frac{\hat{d} M}{2} + 2 \right) \| E^n \|^2 + \frac{\hat{d}^2 M^2}{4} \| \Lambda^{n-1} \|^2 \right)
+ \frac{\hat{d}^2 M^2}{4} \Delta t \int_{t_{n-1}}^{t_n} \| u_t \|^2 \, dt. \] (3.31)

Combining (3.26)-(3.31) we have the following estimate for \( F_1(E^n) \):

\[
| \hat{\lambda} F_1(E^n) | \leq \hat{\lambda} (\epsilon_4 + \epsilon_5 + \epsilon_6) \| \nabla E^n \|^2 + \hat{\lambda} Re \left( \frac{\hat{d} M}{2} + 5 \right) \| E^n \|^2
+ \hat{\lambda} Re \frac{\hat{d} M}{2} \| E^{n-1} \|^2 + \hat{\lambda} Re \frac{\hat{d}^2 M}{4} \| (p^n - P^n) \|^2 + \hat{\lambda} Re \frac{\hat{d}^2 M^2}{4} \| \Lambda^{n-1} \|^2
+ \hat{\lambda} Re \frac{\hat{d} M}{4} \| (u^n - u^n_{h-1}) \| \| E^n \|
+ \hat{\lambda} Re \frac{\hat{d}^2 M^2}{4} \Delta t \int_{t_{n-1}}^{t_n} \| u_t \|^2 \, dt. \] (3.32)

Next we consider the terms in \( F_2(F^n) \).

\[
| (d_t \Gamma^n, F^n) | \leq \| F^n \|^2 + \frac{1}{4} \| d_t \Gamma^n \|^2. \] (3.33)

\[
| b(u^n_{h-1}, \Gamma^n, F^n) | = | b(u^n_{h-1}, \Gamma^n, F^n) + b(u^n_{h-1}, \Gamma^n, \nu F^n) |
\leq \| u^n_{h-1} \cdot \nabla \Gamma^n \| \| F^n \| + \| u^n_{h-1} \cdot \nabla \Gamma^n \| \| \nu F^n \|
\leq \frac{\hat{d}^2}{4} \| u^n_{h-1} \|_{\infty} \| \nabla \Gamma^n \| \| F^n \| + \frac{\hat{d}^2}{4} \| u^n_{h-1} \|_{\infty} \| \nabla \Gamma^n \| \| \nu F^n \|
\leq \| F^n \|^2 + \nu^2 \| F^n \|^2 + \frac{\hat{d}^2 K^2}{2} \| \nabla \Gamma^n \|^2. \] (3.34)
\[
\left| D(\Lambda^n), \bar{F}^n \right| = \left| (D(\Lambda^n), F^n) + (D(\Lambda^n), \nu F^n) \right|.
\]
\[
\leq \| F^n \|^2 + \nu^2 \| F^n \|^2 + \frac{1}{2} \| \nabla \Lambda^n \|^2.
\] (3.35)

\[
\left| (\Gamma^n, \bar{F}^n) \right| = \left| (\Gamma^n, F^n) + \nu (\Gamma^n, \nu F^n) \right|
\]
\[
\leq \| F^n \|^2 + \nu^2 \| F^n \|^2 + \frac{1}{2} \| \Gamma^n \|^2.
\] (3.36)

For the terms making up \( R_2(F^n) \) we have:

\[
\left| (d_t \tau^n, F^n) - (\tau^n, F^n) \right| \leq \| F^n \|^2 + \frac{1}{4} \| d_t \tau^n - \tau^n \|^2.
\] (3.37)

\[
\left| (\tau^n, \nu F^n) \right| = \left| \left( \tau^n, \nu u_h^{n-1} \cdot \nabla F^n \right) \right|
\]
\[
= b(\nu u_h^{n-1}, F^n, \tau^n)
\]
\[
\leq b(\nu u_h^{n-1}, \tau^n, F^n)
\]
\[
+ \left| \left( \nabla \cdot u_h^{n-1} \nu F^n, \tau^n \right) \right| \quad (\text{using (3.23)})
\]
\[
\leq \nu \| u_h^{n-1} \cdot \nabla \tau^n \| \| F^n \|
\]
\[
+ \left| \left( \nabla \cdot (u_h^{n-1} - \mathcal{U}^{n-1}) \nu F^n, \tau^n \right) \right|
\]
\[
+ \left| \left( \nabla \mathcal{U}^{n-1} \nu F^n, \tau^n \right) \right|
\]
\[
\leq \nu \| u_h^{n-1} \| \| \nabla \tau^n \| \| F^n \|
\]
\[
+ \nu \left| \nabla \cdot (u_h^{n-1} - \mathcal{U}^{n-1}) \| F^n \| \| \tau^n \|
\]
\[
+ \| \nabla \mathcal{U}^{n-1} \| \nu \| F^n \| \| \tau^n \|
\]
\[
\leq \left( 2 + \| \nabla E^{n-1} \|_\infty \right) \| F^n \|^2
\]
\[
+ \frac{\nu^2}{4} \left( M^2 + \| \nabla E^{n-1} \|_\infty \right) \| \tau^n \|^2
\]
\[
+ \frac{\nu^2}{4} K^2 d \| \nabla \tau^n \|, \quad (\text{using (3.2) and (IH)}).
\] (3.38)

\[
| b(u_h^{n-1}, \tau^n, \bar{F}^n) - b(u^n, \tau^n, \bar{F}^n) | = \left| \left( u_h^{n-1} - u^n \right) \cdot \nabla \tau^n, \bar{F}^n \right|
\]
\[
\leq \left| \left( u_h^{n-1} - u^n \right) \cdot \nabla \tau^n \right| \| \bar{F}^n \|
\]
\[
\leq \frac{1}{2} \| \bar{F}^n \|^2 + \frac{1}{2} \delta^2 \| \nabla \tau^n \|_\infty \| u_h^{n-1} - u^n \|^2
\]
\[
\leq \| F^n \|^2 + \nu^2 \| F^n \|^2
\]
\[
+ \frac{1}{2} \delta^2 M^2 \| E^{n-1} + \mathcal{U}^{n-1} + u^{n-1} - u^n \|^2
\]
\[ \begin{align*}
\leq & \; \|F^n\|^2 + \nu^2 \|F_u^n\|^2 + \frac{3}{2} d^3 M^2 \|E^{n-1}\|^2 \\
& + \frac{3}{2} d^3 M^2 \|\Lambda^{n-1}\|^2 + \frac{3}{2} d^3 M^2 \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|^2 \, dt \tag{3.39}
\end{align*} \]

In order to estimate the \( g_a \) terms in \( F_2(\cdot) \) note that

\[ g_a \left( \tau^{n-1}_h, \nabla u^{n-1}_h \right) - g_a (\tau^n, \nabla u^n) = g_a \left( \tau^{n-1}_h, \nabla (u^{n-1}_h - U^{n-1}) \right) \]

\[ + g_a \left( \tau^{n-1}_h, \nabla (U^{n-1} - u^{n-1}) \right) \]

\[ + g_a \left( \tau^{n-1}_h, \nabla (u^{n-1} - u^n) \right) \]

\[ + g_a \left( \tau^{n-1}_h - T^{n-1}, \nabla u^n \right) \]

\[ + g_a \left( T^{n-1} - \tau^{n-1}, \nabla u^n \right) \]

\[ + g_a \left( \tau^{n-1} - \tau^n, \nabla u^n \right) \]

\[ = -g_a \left( \tau^{n-1}_h, \nabla E^{n-1} \right) \]

\[ - g_a \left( \tau^{n-1}_h, \nabla \Lambda^{n-1} \right) \]

\[ - g_a \left( \tau^{n-1}_h, \nabla (u^n - u^{n-1}) \right) \]

\[ - g_a (F^{n-1}, \nabla u^n) \]

\[ - g_a (\Gamma^{n-1}, \nabla u^n) \]

\[ - g_a (\tau^n - \tau^{n-1}, \nabla u^n) \] \tag{3.40}

Bounding each of the terms on the right hand side of (3.40) we obtain

\[ \left| g_a \left( \tau^{n-1}_h, \nabla E^{n-1} \right), \tilde{F}_n \right| \leq \left| g_a \left( \tau^{n-1}_h, \nabla E^{n-1} \right) \right| \| \tilde{F}_n \| \]

\[ \leq 4 \delta \left\| \tau^{n-1}_h \right\|_{\infty} \left\| \nabla E^{n-1} \right\| \| \tilde{F}_n \| \]

\[ \leq \epsilon_7 \left\| \nabla E^{n-1} \right\|^2 + \frac{8 \delta^2 K^2}{\epsilon_7} \| F^n \|^2 \]

\[ + \frac{8 \delta^2 K^2}{\epsilon_7} \nu^2 \| F_u^n \|^2 \], \tag{3.41}

\[ \left| g_a \left( \tau^{n-1}_h, \nabla \Lambda^{n-1} \right), \tilde{F}_n \right| \leq 8 \delta^2 K^2 \left\| \nabla \Lambda^{n-1} \right\|^2 + \left| \nabla F^n \right|^2 + \nu^2 \| F_u^n \|^2 \], \tag{3.42}

\[ \left| g_a \left( \tau^{n-1}_h, \nabla (u^n - u^{n-1}) \right), \tilde{F}_n \right| \leq 8 \delta^2 K^2 \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|^2 \, dt + \left| \nabla F^n \right|^2 \]

\[ + \nu^2 \| F_u^n \|^2 \], \tag{3.43}

\[ \left| g_a (F^{n-1}, \nabla u^n), \tilde{F}_n \right| \leq 8 \delta^2 M^2 \left| F^{n-1} \right|^2 + \left| \nabla F^n \right|^2 + \nu^2 \| F_u^n \|^2 \], \tag{3.44}
\[
\left\| \left( g_a \left( \Gamma^{n-1}, \nabla u^n \right), \vec{F}^n \right) \right\| \leq 8d^2 M^2 \left\| \Gamma^{n-1} \right\|^2 + \| \mathbf{F}^n \|^2 + \nu^2 \| \mathbf{u}_t^n \|^2,
\]
\[
\left\| \left( g_a \left( \tau^n - \tau^{n-1}, \nabla u^n \right), \vec{F}^n \right) \right\| \leq 8d^2 M^2 \Delta t \int_{t_{n-1}}^{t_n} \| \tau_t \|^2 dt + \| \mathbf{F}^n \|^2 + \nu^2 \| \mathbf{u}_t^n \|^2.
\]

(3.45)

(3.46)

Combining the estimates in (3.33)-(3.39), (3.41)-(3.46), we obtain the following estimate for \( \mathcal{F}_2(\mathbf{F}^n) \):

\[
| \mathcal{F}_2(\mathbf{F}^n) | \leq \epsilon_7 \left\| \nabla \mathbf{E}^{n-1} \right\|^2 + \nu^2 \| \mathbf{F}_u^n \|^2 \left( 7 + \frac{8d^2 K^2}{\epsilon_7} + \lambda + \frac{1}{\lambda} \right) + \| \mathbf{F}^n \|^2 \left( 11 + \frac{8d^2 K^2}{\epsilon_7} + \| \nabla \mathbf{E}^{n-1} \|_\infty + \lambda + \frac{1}{\lambda} \right) + 8d^2 M^2 \| \mathbf{F}^{n-1} \|^2 (8d^2 M^2) + \| \nabla \mathbf{A}^n \|^2 \left( \frac{\lambda}{2} \right) + \| \nabla \mathbf{G}^n \|^2 \left( \frac{dK^2}{2} \right) + \| \mathbf{G}^{n-1} \|^2 \left( \frac{1}{2} \right) + \| d_t \mathbf{G}^n \|^2 \left( \frac{1}{4} \right)
\]

\[
+ \| \nabla \mathbf{A}^{n-1} \|^2 (8d^2 K^2) + \| \mathbf{A}^{n-1} \|^2 \left( \frac{3}{2} d^3 M^2 \right) + \| \mathbf{G}^{n-1} \|^2 (8d^2 M^2) + \frac{1}{4} \| d_t \tau^n - \tau^n \|^2 + \frac{\nu^2}{4} \left( M^2 + \| \nabla \mathbf{E}^{n-1} \|_\infty \right) \| \tau_t^n \|^2 + \frac{\nu^2}{4} K^2 d \| \nabla \tau_t^n \|^2 + \frac{3}{2} d^3 M^2 \Delta t \int_{t_{n-1}}^{t_n} \| \mathbf{u}_t \|^2 dt + 8d^2 M^2 \Delta t \int_{t_{n-1}}^{t_n} \| \tau_t \|^2 dt
\]

(3.47)

With the following choices: \( \epsilon_1 = \frac{(1-\alpha)}{12 \Re \lambda} \), \( \epsilon_2 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = -\frac{(1-\alpha)}{12 \lambda} \), \( \mathbf{u}_h^0 = \mathbf{E}^0 = 0 \), \( \tau_h^0 = \mathbf{T}^0 \Rightarrow \mathbf{F}^0 = 0 \), substituting (3.25), (3.32), (3.47) into (3.21) yields

\[
\frac{\Re \dot{\lambda}}{2} \left\| \mathbf{E}^l \right\|^2 + \frac{1}{2} \left\| \mathbf{F}^l \right\|^2 + \frac{(1-\alpha)}{2} \lambda \sum_{n=1}^{l} \Delta t \| \nabla \mathbf{E}^n \|^2 + \left[ \nu - \nu^2 \left( \frac{3\lambda^2 + 96d^2 K^2 \lambda}{\Re \lambda} \right) + 7 + \lambda + \frac{5}{4\lambda} \right] \sum_{n=1}^{l} \Delta t \| \mathbf{F}_u^n \|^2
\]

\[
\leq C_1 \sum_{n=1}^{l} \Delta t \| \mathbf{E}^n \|^2 + C_2 \sum_{n=1}^{l} \Delta t \| \mathbf{F}^n \|^2 + C_3 \sum_{n=1}^{l} \Delta t \| \nabla \mathbf{E}^{n-1} \|_\infty \| \mathbf{F}^n \|^2
\]

\[
+ C_4 \sum_{n=1}^{l} \Delta t \| \mathbf{A}^n \|^2 + C_5 \sum_{n=1}^{l} \Delta t \| \nabla \mathbf{A}^n \|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \| d_t \mathbf{A}^n \|^2
\]

\[
+ C_6 \sum_{n=1}^{l} \Delta t \| \mathbf{G}^n \|^2 + Re \dot{\lambda} \sum_{n=1}^{l} \Delta t \| d_t \mathbf{u}^n - \mathbf{u}_t^n \|^2
\]

\[
+ \left( \frac{dK^2}{2} \right) \sum_{n=1}^{l} \Delta t \| \nabla \mathbf{G}^n \|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \| d_t \mathbf{G}^n \|^2 + \frac{1}{4} \sum_{n=1}^{l} \Delta t \| d_t \tau^n - \tau_t^n \|^2
\]
\[ + \frac{\nu^2}{4} \sum_{n=1}^{l} \Delta t \left( M^2 + \| \nabla E^{n-1} \|_\infty \right) \| \tau^n \|^2 + \sum_{n=1}^{l} \Delta t \| p^n - \mathcal{P}^n \|^2 \]
\[ + |\Delta t|^2 \hat{d} \left( \text{Re} \hat{d} M^2 \lambda \| u_{0,0} \|^2 + \frac{3}{2} \hat{d}^2 M^2 \| u_{0,0} \|^2 + 8 \hat{d} M^2 \| \tau_{0,0} \|^2 + 8 \hat{d}^2 K^2 \| u_{0,0} \|^2 \right) \]
\[ + \frac{\nu^2}{4} K^2 \hat{d} \| \nabla \tau \|_{0,0}^2. \] (3.48)

We now apply the interpolation properties of the approximating spaces to estimate the terms on the right hand side of (3.48). Using elements of order \( k \) for velocity, elements of order \( m \) for stress, and elements of order \( q \) for pressure, we have

\[ \sum_{n=1}^{l} \Delta t \| \nabla \Lambda^n \|^2 + \sum_{n=1}^{l} \Delta t \| \nabla \Gamma^n \|^2 \leq C \left( h^{2k} \sum_{n=1}^{l} \Delta t \| u^n \|_{k+1}^2 + h^{2m} \sum_{n=1}^{l} \Delta t \| \tau^n \|_{m+1}^2 \right) \]
\[ \leq C \left( h^{2k} \| u \|_{0,k+1}^2 + h^{2m} \| \tau \|_{0,m+1}^2 \right), \] (3.49)

\[ \sum_{n=1}^{l} \Delta t \| \Lambda^n \|^2 + \sum_{n=1}^{l} \Delta t \| \Gamma^n \|^2 + \sum_{n=1}^{l} \Delta t \| p - \mathcal{P}^n \|^2 \]
\[ \leq C \left( h^{2k+2} \sum_{n=1}^{l} \Delta t \| u^n \|_{k+1}^2 + h^{2m+2} \sum_{n=1}^{l} \Delta t \| \tau^n \|_{m+1}^2 + h^{2q+2} \sum_{n=1}^{l} \Delta t \| p^n \|_{q+1}^2 \right) \]
\[ \leq C \left( h^{2k+2} \| u \|_{0,k+1}^2 + h^{2m+2} \| \tau \|_{0,m+1}^2 + h^{2q+2} \| p \|_{0,q+1}^2 \right), \] (3.50)

\[ \sum_{n=1}^{l} \Delta t \| d_t \Lambda^n \|^2 = \sum_{n=1}^{l} \Delta t \left( \left| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \frac{\partial \Lambda}{\partial t} \, dt \right|^2 \right) \]
\[ \leq \sum_{n=1}^{l} \Delta t \left( \frac{1}{\Delta t} \right)^2 \int_{\Omega} \left( \int_{t_{n-1}}^{t_n} \left( \frac{1}{\Delta t} \right)^2 \, dt \right) \, dx \]
\[ \leq Ch^{2k+2} \| u \|_{0,k+1}^2, \] (3.51)

and similarly,

\[ \sum_{n=1}^{l} \Delta t \| d_t \Gamma^n \|^2 \leq Ch^{2m+2} \| \tau \|^2_{0,m+1}. \] (3.52)

Note that \( d_t u^n - u_t^n \) may be expressed as

\[ d_t u^n - u_t^n = \frac{1}{2\Delta t} \int_{t_{n-1}}^{t_n} u_{tt}(\cdot,t)(t_{n-1} - t) \, dt. \]
Also,
\[
\left( \frac{1}{2 \Delta t} \int_{t_{n-1}}^{t_n} u_t(\cdot,t)(t_n - t) \, dt \right)^2 \leq \frac{1}{4 |\Delta t|^2} \int_{t_{n-1}}^{t_n} u_t^2(\cdot,t) \, dt \int_{t_{n-1}}^{t_n} (t_n - t)^2 \, dt \\
= \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} u_t^2(\cdot,t) \, dt.
\]
Therefore it follows that
\[
\sum_{n=1}^{l} \Delta t \| d_t u^n - u^n_t \|^2 \leq \sum_{n=1}^{l} \Delta t \int_{\Omega} \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} u_{tt}^2(\cdot,t) \, dt \, dx \\
= \frac{1}{12} |\Delta t|^2 \| u_{tt} \|^2_{0,0}.
\] (3.53)

Similarly, for \( d_t \tau^n - \tau^n_t \) we have
\[
\sum_{n=1}^{l} \Delta t \| d_t \tau^n - \tau^n_t \|^2 \leq \frac{1}{12} |\Delta t|^2 \| \tau_{tt} \|^2_{0,0}.
\] (3.54)

In view of (3.49)-(3.54), our induction hypotheses (IH1),(IH2), and with \( \nu \) chosen such that
\[
\nu \leq \frac{1}{2} \left( \frac{3 \lambda^2 + 96 \delta^2 K^2 \lambda}{(1 - \alpha)} + 7 + \frac{5}{4\lambda} \right)^{-1},
\] (3.55)
from (3.48) we obtain
\[
\frac{\Re \lambda}{2} \| E' \|^2 + \frac{1}{2} |F'|^2 + \frac{(1 - \alpha)}{2} \lambda \sum_{n=1}^{l} \Delta t \| \nabla E^n \|^2 + \nu \sum_{n=1}^{l} \Delta t \| F^n_t \|^2 \\
\leq C \sum_{n=1}^{l} \Delta t \left( \| E^n \|^2 + \| F^n \|^2 \right) + C \sum_{n=1}^{l} \Delta t \| \nabla E^{n-1} \|_{\infty} \| F^n \|^2 \\
+ C \nu^2 \left( \| \tau_t \|_{0,1}^2 + \| \tau_t \|^2_{\infty,0} \right) \\
+ C |\Delta t|^2 \left( \| u_t \|_{0,1}^2 + \| u_{tt} \|_{0,0}^2 + \| \tau_t \|_{0,0}^2 + \| \tau_{tt} \|_{0,0}^2 \right) + Ch^{2k+2} \| u \|^2_{0,k+1} \\
+ Ch^{2m+2} \| \tau \|^2_{0,m+1} + Ch^{2q+2} \| p \|^2_{0,q+1} + Ch^{2k} \| u \|^2_{0,k+1} \\
+ Ch^{2k+2} \| u_t \|_{0,k+1} \\
+ Ch^{2m} \| \tau \|^2_{0,m+1} + Ch^{2m+2} \| \tau_t \|^2_{0,m+1},
\] (3.56)
where the \( C' \)'s denote constants independent of \( l, \Delta t, h, \nu \). Applying Gronwall’s lemma and (IH2) to (3.56), the estimate given in (3.11) follows.
Step 2. We show that the induction hypotheses, \((IH_1)\) and \((IH_2)\), are true.

**Verification of \((IH_1)\)**

Assume that \((IH_1)\) holds true for \(n = 1, 2, \ldots, l - 1\). By interpolation properties, inverse estimates and (3.11), we have that

\[
\left\| \mathbf{u}_h^l \right\|_\infty \leq \left\| \mathbf{u}_h^l - \mathbf{u}^l \right\|_\infty + \left\| \mathbf{u}^l \right\|_\infty
\]

\[
\leq \left\| E^l \right\|_\infty + \left\| \Lambda^l \right\|_\infty + M
\]

\[
\leq Ch^{-\frac{d}{2}} \left\| E^l \right\|_0 + Ch^{-\frac{d}{2}} \left\| \Lambda^l \right\|_0 + M
\]

\[
\leq C \left( |\Delta t| h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+\frac{d}{2}} + h^{q+1+\frac{d}{2}} \right) + M. \quad (3.57)
\]

Note that the expression \(C \left( |\Delta t| h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+\frac{d}{2}} + h^{q+1+\frac{d}{2}} \right)\) is independent of \(l\). Hence, if we set \(k, m \geq \frac{d}{2}, q \geq \frac{d}{2} - 1\), and choose \(h, \Delta t, \nu\) such that

\[
h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1+\frac{d}{2}} \leq \frac{1}{C}, \quad \Delta t, \nu \leq \frac{h^{\frac{d}{2}}}{C}, \quad (3.58)
\]

then from (3.57)

\[
\left\| \mathbf{u}_h^l \right\|_\infty \leq M + 6.
\]

Similarly it follows that \(\left\| \tau_h^l \right\|_\infty \leq M + 6\).

\[\blacksquare\]

**Verification of \((IH_2)\)**

Assume that \((IH_2)\) is true for \(n = 1, 2, \ldots, l - 1\). Equations (3.11) and (3.56) imply

\[
\sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^n \right\|_2^2 \leq C \left( h^{2k} + h^{2m} + h^{2q+2} + |\Delta t|^2 + \nu^2 \right). \quad (3.59)
\]

Applying the inverse estimate and using the inequality

\[
\sum_{n=1}^{l} a_n \leq \sqrt{l} \left( \sum_{n=1}^{l} a_n^2 \right)^{\frac{1}{2}}
\]

from (3.59) we obtain

\[
\sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^n \right\|_\infty \leq Ch^{-\frac{d}{2}} \sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^n \right\|
\]

\[
\leq Ch^{-\frac{d}{2}} \sqrt{\Delta t} \sqrt{l} \left( \sum_{n=1}^{l} \Delta t \left\| \nabla \mathbf{E}^n \right\|_2^2 \right)^{\frac{1}{2}}
\]
\[ \leq \tilde{C} \left( \Delta t h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} \right), \]

where \( \tilde{C} = C \sqrt{T} \) is a constant independent of \( l, h, \Delta t, \) and \( \nu \). Hence when

\[ \nu, \Delta t \leq \frac{h^{\frac{d}{2}}}{5C}, \quad (3.60) \]

and

\[ h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \leq \frac{1}{5C}, \]

\((IH2)\) holds. \blacksquare

**Step 3.** We derive the error estimates in (3.9) and (3.10).

**Proof of the Theorem 3.**

Using estimates (3.11) and (approximation properties), we have

\[ \| u - u_h \|^2_{\infty,0} + \| \tau - \tau_h \|^2_{\infty,0} \leq \| E \|^2_{\infty,0} + \| A \|^2_{\infty,0} + \| F \|^2_{\infty,0} \]

\[ \leq C(T + 1) G(\Delta t, h, \nu) \]

Note the restrictions on \( \nu \) from (3.55), (3.58), (3.60), and on \( \Delta t \) from (3.1), (3.58), (3.60).

Hence, we obtain the stated estimate (3.9).

To establish (3.10), from (3.11), (3.56) we have

\[ \| \nabla E \|^2_{0,0} + \Delta t \| F \|^2_{0,0} \leq C(T + 1) G(\Delta t, h, \nu) \quad (3.61) \]

and

\[ \| E \|^2_{0,0} + \| F \|^2_{0,0} \leq T G(\Delta t, h, \nu). \quad (3.62) \]

Hence

\[ \| E \|^2_{1,0} + \| F \|^2_{1,0} \leq \tilde{C} G(\Delta t, h, \nu). \quad (3.63) \]

\( \blacksquare \)

We conclude this analysis with some comments on the sensitivity of the error bounds to the physical parameters in the modeling equations. From (3.48) we note that the constants
$C_1, C_2, C_3$, involve the terms $K^2, M^2, Re, \tilde{\lambda}(=\lambda/2\alpha), \lambda^{-1}$. Thus, in view of the exponential multiplicative factor in the discrete Gronwall’s lemma, we have that the generic constants $C$ in (3.9),(3.10),(3.11), depend exponentially on these terms.
CHAPTER 4
APPROXIMATION OF TRANSIENT, VISCOELASTIC, FLUID FLOWS:
MULTI-FLUID CASE

4.1 Introduction

In this chapter, we present an error analysis for a fully discrete approximation to a time dependent, multicomponent, viscoelastic fluid flow problem. This extends the result of section (3.6) to flows which involve more than one fluid.

For the governing equations of multicomponent viscoelastic fluid flow, we have that within each fluid component the viscoelastic equations must hold. In addition, along the interfaces separating components, a free-surface boundary condition must be satisfied. This boundary condition accounts for the discontinuity in the stress tensor across the interface between components and, in part, determines the shape of the interface. Using a continuum surface force model (CSF), we replace the interfacial surface with an interfacial region in which we use a continuous interpolate to describe the fluid characteristics. The CSF approach enables us to model and analyse the multicomponent fluid problem as a single fluid with varying material parameters. See Chapter 2 for a further description of the boundary condition and the CSF model.

This chapter is organized as follows. In section 4.2, we give a brief summary of the general equations which govern multicomponent, viscoelastic fluid flow. This is an overview of topics which were presented earlier in this work. Section 4.3 gives the variational formulation of the approximating system. The main approximation result is then given in theorem 4 in section 4.4, followed by its proof.
4.2 The Modeling Equations of Multicomponent Fluid Flow

In this section, we briefly present the modeling equations we use for multicomponent, viscoelastic fluid flow. We use $u$, $\rho$, and $T$ to denote the velocity, density, and total stress (tensor) of the fluid.

Let $\Omega$ denote a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$), with boundary $\partial \Omega$. For ease of exposition, we will present the formulation for two viscoelastic fluids in $\Omega$. Let $\Omega_1, \Omega_2$ denote the region in $\Omega$ occupied by fluids 1 and 2, respectively, and $I$, the interface between the two fluids. Note that $\Omega_1, \Omega_2$ and $I$ are functions of time, and $\Omega = \Omega_1 \cup \Omega_2 \cup I$.

Within each $\Omega_i$:

For $V$ a fixed region in $\Omega_i$, with boundary $\partial V$, the conservation of momentum and mass equations imply

$$\frac{d}{dt} \int_V \rho u \, dx = \int_V b \, dx + \int_{\partial V} T \cdot n \, dS - \int_{\partial V} \rho u(u \cdot n) \, dS,$$  \hspace{1cm} (4.1)

$$\frac{d}{dt} \int_V \rho \, dx = -\int_{\partial V} \rho u \cdot n \, dS,$$ \hspace{1cm} (4.2)

where $n$ denotes the unit outward normal on $\partial V$ and $b$ the body forces acting on $V$.

Along the Interface $I$:

The boundary condition which holds along, and determines the interface $I$ is (see section (2.1) and [7])

$$|T \cdot n| = -\sigma \kappa n - \nabla s \sigma,$$ \hspace{1cm} (4.3)

where $\kappa$ denotes the mean curvature of $I$, $\sigma$ the coefficient of interfacial tension, $\nabla s \sigma$ the surface gradient of $\sigma$, $n$ the unit normal on $I$ pointing into fluid 2, and $|T \cdot n|$ the jump of the normal component of stress across $I$ defined by

$$|T \cdot n|_x = \lim_{\epsilon \to 0^+} T|_{x+\epsilon n} - T|_{x-\epsilon n}.$$

Using the continuum surface force model of Brackbill et. al. [9], the force along the interface is rewritten as a volume force using a delta distribution, i.e.

$$\int_{V \cap I} [t \cdot n] \, dS = \int_V [T \cdot n] \delta(x - x_s).$$
where $x_s$ denotes a nearest point to $x$ on $I$.

Using the divergence theorem to replace the surface integrals in (4.1), (4.2) with volume integrals, the fact that $V$ is an arbitrary volume, and the incompressibility of the fluid, we obtain the following pointwise equations for the conservation of momentum and mass:

\[
\frac{\partial u}{\partial t} + \rho u \cdot \nabla u = b + \nabla \cdot T - (\sigma \kappa \mathbf{n} + \nabla_s \sigma) \delta_z, \quad \text{in } \Omega, \quad (4.4)
\]

\[
\nabla \cdot u = 0, \quad \text{in } \Omega. \quad (4.5)
\]

**Modeling Equation for the Stress Tensor $T$:**

The stress tensor $T$ is usually written in the form

\[
T = -pI + \tau \quad (4.6)
\]

where $p$ denotes the internal fluid pressure, $I$ the identity tensor, and $\tau$ the extra stress tensor. For a Newtonian fluid $\tau$ is usually modelled as

\[
\tau = 2\eta D(u) \quad (4.7)
\]

where $D(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ is the deformation tensor and $\eta$ is the fluid viscosity. For viscoelastic fluids, because of the internal elasticity of the fluid, the modeling equation for the extra stress is in general considerably more complicated than (4.7), (see [8] for a description of various models).

In this paper we assume that extra stress is governed by an Oldroyd B model. For this model $\tau$ is expressed as

\[
\tau = \tau_n + \tau_v \quad (4.8)
\]

where the Newtonian contribution to the extra stress, $\tau_n$ satisfies

\[
\tau_n = 2(1 - \alpha)D(u) \quad (4.9)
\]
and the viscoelastic contribution $\tau_v$ is given by

$$\tau_v + \lambda \frac{\partial \tau_v}{\partial t} - 2\alpha \mathcal{D}(u) = 0,$$

(4.10)

where

$$\frac{\partial \tau_v}{\partial t} := \frac{\partial \tau_v}{\partial t} + u \cdot \nabla \tau_v + g_a(\tau_v, \nabla u), \quad a \in [-1, 1]$$

(4.11)

and

$$g_a(\sigma, \nabla u) := \frac{1-a}{2} (\tau_v \nabla u + (\nabla u)^T \tau_v) - \frac{1+a}{2} ((\nabla u)\tau_v + \tau_v (\nabla u)^T).$$

(4.12)

In (4.9), $\alpha \in (0, 1)$ may be interpreted as the proportion of the viscosity which is considered to be viscoelastic in nature. The *Weissenberg number*, $\lambda$, is a dimensionless constant which is defined as the product of the relaxation time and a characteristic strain rate [8]. In (4.11) the choices $a = 1, -1, 0$ correspond to the upper, lower, and corotational convected derivatives of $\tau_v$, respectively.

In what follows, for ease of notation, we use $\tau$ to denote $\tau_v$. Using (4.6), (4.8)-(4.11) and (4.4), (4.5) we obtain, on nondimensionalization of the problem, the modeling system of equations:

$$Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p - 2(1-\alpha)\nabla \cdot \mathbf{D}(u) - \nabla \cdot \tau = f \quad \text{in } \Omega,$$

(4.13)

$$\tau + \lambda \left( \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla \tau + g_a(\tau, \nabla u) \right) - 2\alpha \mathcal{D}(u) = 0 \quad \text{in } \Omega,$$

(4.14)

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega,$$

(4.15)

where

$$f = \mathbf{b} - (\sigma \kappa \mathbf{n} + \nabla \sigma) I,$$

(4.16)

$$Re = \frac{L V \bar{\rho}}{\bar{n}}$$

(4.17)

In (4.17), $L, V, \bar{\rho}, \bar{n}$ denote a characteristic length scale, velocity scale, density, and viscosity. To fully specify the problem, together with (4.13)-(4.15), we require initial conditions for the velocity and stress, boundary conditions for the velocity, and the stress specified on the
inflow boundary of $\Omega, \Omega_{in}$,

\[ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in} \; \Omega, \]  

\[ \tau(\mathbf{x}, 0) = \tau_0(\mathbf{x}) \quad \text{in} \; \Omega. \]  

\[ \mathbf{u} = \mathbf{u}_{bdy} \quad \text{on} \; \partial\Omega, \]  

\[ \tau = \tau_{bdy} \quad \text{on} \; \partial\Omega_{in}. \]  

Note: Equations (4.13)-(4.15), (4.18)-(4.21) only specify the pressure $p$ up to an arbitrary constant.

The existence and uniqueness of $(\mathbf{u}, \tau, p)$ satisfying (4.13)-(4.15), (4.18)-(4.21) is still largely an open research question. The local existence (in time), and under a “small data” assumption on $\mathbf{f}, \mathbf{f}', \mathbf{u}_0, \tau_0$, global existence (in time) of solutions to (4.13)-(4.15), (4.18)-(4.21) have been established [18]. For a more complete discussion of existence and uniqueness issues, see [31].

In order to simplify the numerical analysis of the approximation scheme to (4.13)-(4.15), (4.18)-(4.21), we will assume homogeneous boundary conditions for the velocity (i.e. $\mathbf{u}_{bdy} = 0$). Consequently, as there is no inflow boundary, below we study the specific system of equations (4.13)-(4.15), (4.18)-(4.20) with $\mathbf{u}_{bdy} = 0$).

### 4.3 The Variational Formulation

In this section, we develop the variational formulation of (4.13)-(4.15), (4.18)-(4.20). The following notation will be used. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot, \cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W^{k,p}(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{k,p}$, respectively. $H^k$ is used to represent the Sobolev space $W^{k,2}$, and $\|\cdot\|_k$ denotes the norm in $H^k$. The following function spaces are used in the analysis:

Velocity Space : \[ X := H^1_0(\Omega) := \left\{ \mathbf{u} \in H^1(\Omega) : \mathbf{u} = 0 \; \text{on} \; \partial\Omega \right\}, \]

Stress Space : \[ S := \left\{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} ; \tau_{ij} \in L^2(\Omega) ; 1 \leq i, j \leq 3 \right\} \]
\[ \cap \left\{ \tau = (\tau_{ij}) : \mathbf{u} \cdot \nabla \tau \in L^2(\Omega), \forall \mathbf{u} \in X \right\}, \]
Pressure Space : \( Q := L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \} \),

Divergence – free Space : \( Z := \{ v \in X : \int_{\Omega} q(\nabla \cdot v) \, dx = 0, \, \forall \, q \in Q \} \).

The variational formulation of (4.13)-(4.15), (4.18)-(4.20) proceeds in the usual manner. Taking the inner product of (4.13), (4.14), and (4.15) with a velocity test function, a stress test function, and a pressure test function respectively, we obtain

\[
\left( Re \frac{\partial u}{\partial t} + Re \, u \cdot \nabla u, v \right) - (p, \nabla \cdot v) + (2(1 - \alpha)D(u) + \tau, D(v)) = (f, v), \, \forall \, v \in X \quad (4.22)
\]

\[
\left( \tau + \left( \lambda \frac{\partial \tau}{\partial t} + \lambda u \cdot \nabla \tau + \lambda g_a(\tau_V, \nabla u) \right) - 2\alpha D(u), \psi \right) = 0, \quad \forall \, \psi \in S \quad (4.23)
\]

\[
(\nabla \cdot u, q) = 0, \quad \forall \, q \in Q \quad (4.24)
\]

Note that \( Re \) and \( \lambda \) are functions of time and space, determined by which fluid is occupying the point \( x \) at time \( t \). We use

\[
0 < Re_m := \min Re \\
0 < Re_M := \max Re \\
0 < \lambda_m := \min \lambda \\
0 < \lambda_M := \max \lambda.
\]

The space \( Z \) is the space of weakly divergence free functions. The condition

\[
(\nabla \cdot u, q) = 0, \quad \forall \, q \in Q, \, u \in X,
\]

is equivalent in a “distributional” sense to

\[
(u, \nabla q) = 0, \quad \forall \, q \in Q, \, u \in X, \quad (4.25)
\]

where in (4.25), \((\cdot, \cdot)\) denotes the duality pairing between \( H^{-1} \) and \( H^1_0 \) functions. In addition, note that the velocity and pressure spaces, \( X \) and \( Q \), satisfy the inf-sup condition

\[
\inf_{\varphi \in Q} \sup_{v \in X} \frac{(q, \nabla \cdot v)}{||q|| \, ||v||_1} \geq \beta > 0. \quad (4.26)
\]
Since the inf-sup condition (4.26) holds, an equivalent variational formulation to (4.22)-(4.24) is: find \( u \in Z, \tau \in S \) satisfying

\[
\begin{align*}
Re \frac{\partial u}{\partial t} + Re \ u \cdot \nabla u, v \right) + (2(1 - \alpha)D(u) + \tau, D(v)) &= (f, v), \quad \forall \ v \in Z, \quad (4.27) \\
\left( \tau + \lambda \left( \frac{\partial \tau}{\partial t} + u \cdot \nabla \tau + g_a(\tau, \nabla u) \right) - 2\alpha D(u), \psi \right) &= 0, \quad \forall \ \psi \in S. \quad (4.28)
\end{align*}
\]

We assume that the fluid flow satisfies the following properties:

\[
\|u\|_\infty, \|\tau\|_\infty, \|\nabla u\|_\infty, \|\nabla \tau\|_\infty \leq M, \quad (4.29)
\]

for all \( t \in [0,T] \).

The following definitions are used in the analysis below:

\[
\begin{align*}
b(u, \tau, \psi) &:= (u \cdot \nabla \tau, \psi), \quad (4.30) \\
c(w, u, v) &:= (w \cdot \nabla u, v). \quad (4.31)
\end{align*}
\]

4.3.1 Finite Element Approximation

In this section we formulate a fully discrete finite element method for solving the viscoelastic fluid flow equations, and prove the solvability of the approximation at each step (for sufficiently small \( \Delta t \) and \( h \)). We begin by describing the finite element approximation framework and listing the approximating properties and inverse estimates used in the analysis.

Let \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) be a polygonal domain and let \( T_h \) be a triangulation of \( \Omega \) made of triangles (in \( \mathbb{R}^2 \)) or tetrahedrals (in \( \mathbb{R}^3 \)). Thus, the computational domain is defined by

\[
\Omega = \bigcup K; \ K \in T_h.
\]

We assume that there exist constants \( c_1, c_2 \) such that

\[
c_1 h \leq h_K \leq c_2 \rho_K
\]
where \( h_K \) is the diameter of triangle (tetrahedral) \( K \), \( \rho_K \) is the diameter of the greatest ball (sphere) included in \( K \), and \( h = \max_{K \in T_h} h_K \). Let \( P_k(K) \) denote the space of polynomials on \( A \) of degree no greater than \( k \). Then we define the finite element spaces as follows.

\[
X_h := \left\{ \mathbf{v} \in X \cap C(\Omega)^2 : \mathbf{v}|_K \in P_k(K), \forall K \in T_h \right\},
\]

\[
S_h := \left\{ \sigma \in S \cap C(\Omega)^4 : \sigma|_K \in P_m(K), \forall K \in T_h \right\},
\]

\[
Q_h := \left\{ q \in Q \cap C(\Omega) : q|_K \in P_q(K), \forall K \in T_h \right\},
\]

\[
Z_h := \{ \mathbf{v} \in X_h : (q, \nabla \cdot \mathbf{v}) = 0, \forall q \in Q_h \}.
\]

We assume that the velocity and pressure spaces are chosen so as to satisfy the discrete \( \inf-sup \) condition:

\[
\inf_{Q_h} \sup_{\mathbf{v} \in X_h} \frac{(q, \nabla \cdot \mathbf{v})}{\|q\| \|v\|_1} \geq \beta > 0. \tag{4.32}
\]

Let \( \Delta t \) denote the step size for \( t \), \( t_n = n\Delta t, n = 0, 1, 2, \ldots, N \), and let

\[
d_t f = \frac{f(t_n) - f(t_{n-1})}{\Delta t}.
\]

We also define the following additional norms:

\[
\|\|v\|\|_{\infty,k} := \max_{1 \leq n \leq N} \|v^n\|_k,
\]

\[
\|\|v\|\|_{0,k} := \left\{ \sum_{n=1}^{N} \|v^n\|_k^2 \Delta t \right\}^{\frac{1}{2}}.
\]

When \( v(x, t) \) is defined on the entire time interval \( (0, T) \), we use

\[
\|v\|_{\infty,k} := \sup_{0 < t < T} \|v(\cdot, t)\|_k,
\]

\[
\|v\|_{0,k} := \left( \int_0^T \|v(\cdot, t)\|_k^2 \, dt \right)^{1/2}.
\]

In addition, we make use of the following approximation properties,[13]:

\[
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\| \leq Ch^{k+1} \|\mathbf{u}\|_{k+1}, \; \mathbf{u} \in H^{k+1}(\Omega)^d,
\]

\[
\inf_{\mathbf{v} \in X_h} \|\mathbf{u} - \mathbf{v}\|_1 \leq Ch^k \|\mathbf{u}\|_{k+1}, \; \mathbf{u} \in H^{k+1}(\Omega)^d,
\]

\[
\inf_{\tau \in S_h} \|\tau - \sigma\| \leq Ch^{m+1} \|\tau\|_{m+1}, \; \tau \in H^{m+1}(\Omega)^{d \times d}, \tag{4.33}
\]

\[
\inf_{\sigma \in S_h} \|\tau - \sigma\|_1 \leq Ch^m \|\tau\|_{m+1}, \; \tau \in H^{m+1}(\Omega)^{d \times d},
\]

\[
\inf_{\mathbf{v} \in X_h} \|\mathbf{w} - \mathbf{v}\| \leq Ch^{k+1} \|\mathbf{w}\|_{k+1}, \; \mathbf{w} \in H^{k+1}(\Omega)^d,
\]

\[
\inf_{\mathbf{v} \in X_h} \|\mathbf{w} - \mathbf{v}\|_1 \leq Ch^k \|\mathbf{w}\|_{k+1}, \; \mathbf{w} \in H^{k+1}(\Omega)^d,
\]

\[
\inf_{\tau \in S_h} \|\sigma - \tau\| \leq Ch^{m+1} \|\tau\|_{m+1}, \; \sigma \in H^{m+1}(\Omega)^{d \times d},
\]

\[
\inf_{\sigma \in S_h} \|\tau - \sigma\|_1 \leq Ch^m \|\tau\|_{m+1}, \; \tau \in H^{m+1}(\Omega)^{d \times d},
\]
\[
\inf_{\sigma \in S_h} \| \tau - \sigma \|_1 \leq Ch^n \| \tau \|_{m+1}, \quad \tau \in H^{m+1}(\Omega)^d \times d, \\
\inf_{r \in Q_h} \| p - r \| \leq Ch^{q+1} \| p \|_{q+1}, \quad p \in H^{q+1}(\Omega).
\]

The following inverse estimates, \[13\], are also used:

\[
\begin{align*}
\| u_h \|_{\infty} & \leq ch^{-\frac{d}{2}} \| u_h \| \quad \forall u_h \in X_h, \\
\| q_h \|_{\infty} & \leq ch^{-\frac{d}{2}} \| q_h \| \quad \forall q_h \in Q_h.
\end{align*}
\]

To solve the time-dependent flow equations numerically, time derivatives are replaced by backward differences, and nonlinear terms are lagged. As we are assuming “slow flow”, i.e. \( Re \equiv O(1) \), we use a conforming finite element method to discretize the momentum equation. For the constitutive equation for stress, which is hyperbolic, we use a streamline upwind Petrov-Galerkin (SUPG) discretization to control the production of spurious oscillations in the approximation. The discrete approximating system of equations is then:

**Approximating System**

For \( n = 1, 2, \ldots, N \), find \( u_h^n \in Z_h, \tau_h^n \in S_h \) such that

\[
(Re \, d_t u_h^n, v) + c \left( Re \, u_h^{n-1}, u_h^n, v \right) + 2(1 - \alpha) (D(u_h^n), D(v)) \\
+ (\tau_h^n, D(v)) = (f^n, v), \quad \forall v \in Z_h,
\]

\[
(\tau_h^n, \tilde{\sigma}) + (\lambda \, d_t \tau_h^n, \sigma) + b \left( \lambda \, u_h^{n-1}, \tau_h^n, \tilde{\sigma} \right) - 2\alpha (D(u_h^n), \tilde{\sigma}) \\
= - \left( \lambda \, g_d(\tau_h^{n-1}, \nabla u_h^{n-1}), \tilde{\sigma} \right), \quad \forall \sigma \in S_h,
\]

where \( \tilde{\sigma} := \sigma + \nu \sigma_h^n, \sigma_h^n := u_h^{n-1} \cdot \nabla \sigma \), and \( \nu \) is a small positive constant.

The parameter \( \nu > 0 \) is used to suppress the production of spurious oscillations in the approximation. Note that for \( \nu = 0 \) the discretization of the constitutive equation is the usual Galerkin method. The goal in choosing \( \nu \) is to keep it as small as possible, but large enough to control the generation of catastrophic spurious oscillations in the approximate stress.
Choosing Proof:

We now estimate the terms in solution (4.36)-(4.37) is uniquely solvable for $u_h$ and $\tau_h$ at each time step $n$. We use the following induction hypothesis.

\[(IH1) \quad \|u_h^{n-1}\|_\infty \cdot \|\tau_h^{n-1}\|_\infty \leq K.\]

**Lemma 11** Assume (IH1) is true. For sufficiently small step size $\Delta t$, there exists a unique solution $(u_h^n, \tau_h^n) \in Z_h \times S_h$ satisfying (4.36)-(4.37).

**Proof:** For notational simplicity, in this proof we drop the subscript $h$ from the variables. Choosing $v = u_h^n, \sigma = \tau_h^n$, multiplying (4.36) by $2\alpha$ and adding to (4.37) we obtain

\[
a(u^n, \tau^n; u^n, \tau^n) = 2\alpha (f^n, u^n) + 2\alpha \frac{1}{\Delta t} \Bigl( Re \ u^{n-1}, u^n \Bigr) + \frac{1}{\Delta t} \Bigl( \lambda \tau^{n-1}, \tau^n \Bigr) \quad \text{(4.38)}
\]

where the bilinear form $a(u, \tau; v, \sigma)$ is defined as:

\[
a(u, \tau; v, \sigma) := \frac{2\alpha}{\Delta t} (Re \ u, u) + 2\alpha c (Re \ u^{n-1}, u, v) + 4\alpha(1-\alpha) (D(u), D(v)) + 2\alpha (\tau, D(v)) + (\tau, \tilde{\sigma}) + \frac{1}{\Delta t} (\lambda \tau, \sigma) + B \Bigl( \lambda u^{n-1}, \tau, \sigma \Bigr) + B \bigl( \lambda u^{n-1}, \nu u^{n-1} \cdot \nabla \sigma \bigr) + 2\alpha (D(u), \sigma) - 2\alpha \bigl( D(u), \nu u^{n-1} \cdot \nabla \sigma \bigr).
\]

Thus,

\[
a(u, \tau; u, \tau) = \frac{2\alpha}{\Delta t} (Re \ u, u) + 2\alpha c (Re \ u^{n-1}, u, u) + 4\alpha(1-\alpha) (D(u), D(u)) + (\tau, \tau) + \nu(\tau, \tau_u) + \frac{1}{\Delta t} (\lambda \tau, \tau) + b \bigl( \lambda u^{n-1}, \tau, \tau \bigr) + b \bigl( \lambda u^{n-1}, \tau, \nu \tau_u \bigr) - 2\alpha \nu (D(u), \tau_u).
\]

We now estimate the terms in $a(u^n, \tau^n; u^n, \tau^n)$. We have

\[
\frac{2\alpha}{\Delta t} (Re \ u^n, u^n) \geq \frac{2\alpha \text{Re}_m}{\Delta t} \|u^n\|^2
\]

\[
|2\alpha c(Re \ u^{n-1}, u^n, u^n)| = 2\alpha \left| \left( Re \ u^{n-1} \cdot \nabla u^n, u^n \right) \right| \leq 2\alpha d^\frac{3}{2} \|Re \ u^{n-1}\|_\infty \|\nabla u^n\| \|u^n\|
\]

\[
\leq 2\alpha d^\frac{3}{2} C_K \text{Re}_M \left\|u^{n-1}\right\|_\infty \|D(u^n)\|
\]

(Using Korn’s lemma)
Applying these inequalities to the bilinear form \( a(\cdot, \cdot; \cdot, \cdot) \) yields

\[
a(u^n, \tau^n; u^n, \tau^n) \geq \left( \frac{2\alpha \Re m}{\Delta t} - \frac{d K^2 C_k^2}{\Delta t} \frac{\alpha^2 \Re M}{\epsilon_1} \right) \| u^n \|^2 + \left( \frac{4\alpha (1 - \alpha) - \epsilon_1 - \epsilon_3}{\epsilon_1} \right) \| D(u^n) \|^2
+ \left( 1 + \frac{\lambda_m}{\Delta t} - \frac{\lambda_M^2}{4\epsilon_2} \right) \| \tau^n \|^2 + \left( \lambda_m \nu - \frac{\nu^2}{4} - \frac{\alpha^2 \nu^2}{\epsilon_2} \right) \| \tau^n \|^2.
\]

For

\[
\nu \leq \frac{2\lambda_m (1 - \alpha)}{1 + 3\alpha},
\]

and choosing \( \epsilon_1 = \epsilon_3 = \alpha (1 - \alpha) \), \( \epsilon_2 = \frac{\lambda_M \nu}{2} \), we have that for

\[
\Delta t < \min \left\{ \frac{4\epsilon_2 \lambda_m}{\frac{\lambda_m}{\lambda_M^2}}, \frac{2\epsilon_1 \alpha \Re m}{d K^2 C_k^2 \alpha^2 \Re M} \right\},
\]

the bilinear form \( a(\cdot, \cdot; \cdot, \cdot) \) is positive. Hence, (4.38) has at most one solution. Since (4.38) is a finite dimensional linear system, the uniqueness of the solution implies the existence of the solution.
4.4 A Priori Error Estimate

In this section we analyze the error between the finite element approximation given by (4.36)-(4.37) and the true solution. A priori error estimates for the approximation are in theorem 4.

**Theorem 4** There exists constants $c_1, c_2 > 0$ such that if $\Delta t < c_1 h^{d/2}, \nu < c_2 h^{d/2}$, the finite element approximation (4.36)-(4.37) is convergent to the solution of (4.27)-(4.28) on the interval $(0, T)$ as $h \to 0$. In addition, the approximation $(u_h, \tau_h)$ and the true solution $(u, \tau)$ satisfy the following error estimates:

\[
\|u_h - u\|_{\infty,0} + \|\tau_h - \tau\|_{\infty,0} \leq \mathbf{F}(\Delta t, h) \tag{4.41}
\]
\[
\|u_h - u\|_{0,1} + \|\tau_h - \tau\|_{0,0} \leq \mathbf{F}(\Delta t, h) \tag{4.42}
\]

where

\[
\mathbf{F}(\Delta t, h) = Ch^k \|u\|_{0,k+1} + Ch^{k+1} \|u_t\|_{0,k+1} + Ch^m \|\tau\|_{0,m+1} + Ch^{m+1} \|\tau_t\|_{0,m+1}
\]
\[
+ C h^{q+1} \|p\|_{0,q+1} + C \left( h^{k+1} \|u\|_{\infty,k+1} + h^{m+1} \|\tau\|_{\infty,m+1} \right)
\]
\[
+ C |\Delta t| \left( \|u_t\|_{0,1} + \|u_{tt}\|_{0,0} + \|\tau_t\|_{0,1} + \|\tau_{tt}\|_{0,0} \right)
\]
\[
+ C \nu \left( \|\tau_{tt}\|_{0,1} + \|\tau_t\|_{\infty,0} \right).
\]

In order to establish the estimates (3.9)-(3.10), we begin by introducing the following notation. Let $u^n = u(t_n), \tau^n = \tau(t_n)$ represent the solution of (4.27)-(4.28) at time $t_n$, and $u^n_h, \tau^n_h$ denote the solution of (4.36)-(4.37). Let $(\mathcal{U}^n, \mathcal{P}^n)$ denote the Stokes projection of $(u^n, p^n)$ into $(Z_h, Q_h)$, and $T^n$ a Clément interpolant of $\tau^n$, [14]. We have the approximating properties:

\[
\|u^n - \mathcal{U}^n\| \leq Ch^{k+1} \|u^n\|_{k+1},
\]
\[
\|\tau^n - T^n\| \leq Ch^{m+1} \|\tau^n\|_{m+1},
\]
\[
\|p^n - \mathcal{P}^n\| \leq Ch^{q+1} \|p^n\|_{q+1},
\]
\[
\|\nabla(u^n - \mathcal{U}^n)\| \leq Ch^k \|u^n\|_{k+1},
\]
\[
\|\nabla(\tau^n - T^n)\| \leq Ch^m \|\tau^n\|_{m+1}.
\]
We will also use Lemma 3 from Section 3.4. Note that it follows from (3.1) and inverse estimates, [10], that
\[ \| \mathbf{U}^n \|_{\infty} , \| \nabla \mathbf{U}^n \|_{\infty} \leq \tilde{M} \approx M . \] (4.44)

Below, for simplicity, we take \( \tilde{M} = M \).

Define \( \mathbf{A}^n, \mathbf{E}^n, \mathbf{F}^n, \epsilon_u, \epsilon_\tau \) as
\[
\mathbf{A}^n = \mathbf{u}^n - \mathbf{U}^n, \quad \mathbf{E}^n = \mathbf{U}^n - \mathbf{u}_h^n, \\
\mathbf{F}^n = \tau^n - T^n, \quad \mathbf{F}^n = T^n - \tau_h^n, \\
\epsilon_u = \mathbf{u} - \mathbf{u}_h^n, \quad \epsilon_\tau = \tau - \tau_h^n.
\]

The proof of theorem 4 is established in three steps.

1. Prove a lemma, assuming two induction hypotheses.
2. Show that the induction hypotheses are true.
3. Prove the error estimates given in (3.9), (3.10).

**Step 1.** We prove the following lemma.

**Lemma 12** Under the induction hypothesis \((IH1)\) and the additional assumption
\[
(IH2) \quad \sum_{n=1}^{l-1} \Delta t \| \nabla \mathbf{E}^n \|_{\infty} \leq 1 ,
\]
we have that
\[
\| \mathbf{E}^l \|^2 + \| \mathbf{F}^l \|^2 \leq G(\Delta t, h, \nu),
\] (4.45)

where
\[
G(\Delta t, h, \nu) = C \left( h^{2k} \| \mathbf{u} \|^2_{0,k+1} + h^{2k+2} \| \mathbf{u}_t \|^2_{0,k+1} \right) \\
+ C \left( h^{2m} \| \tau \|^2_{0,m+1} + h^{2m+2} \| \tau_t \|^2_{0,m+1} \right) \\
+ C h^{2q+2} \| p \|^2_{0,q+1} + C |\Delta t|^2 \left( \| \mathbf{u}_t \|^2_{0,1} + \| \mathbf{u}_{tt} \|^2_{0,0} + \| \tau_t \|^2_{0,1} + \| \tau_{tt} \|^2_{0,0} \right) \\
+ C \nu^2 \left( \| \tau_t \|^2_{0,1} + \| \tau_{tt} \|^2_{\infty,0} \right).
\]

**Proof of lemma 12:** From (4.27)-(4.28), we have that the true solution \((\mathbf{u}, \tau)\) satisfies
\[
(Re d_t \mathbf{u}^n, \mathbf{v}) + c \left( Re \mathbf{u}_h^{n-1}, \mathbf{u}^n, \mathbf{v} \right) + 2(1 - \alpha) (D(\mathbf{u}^n), D(\mathbf{v})) + (\tau^n, D(\mathbf{v}))
\]
\[
\begin{align*}
\frac{\partial}{\partial t} \tau^n - \sigma &= b \left( \lambda \left( \mu u_h^{n-1}, \tau^n \right), \tilde{\sigma} \right) - 2\alpha \left( D(u^n), \tilde{\sigma} \right) + (\tau^n, \tilde{\sigma}) \\
&= - \left( \lambda g_a \left( \tau_h^{n-1}, \nabla u_h^{n-1} \right), \tilde{\sigma} \right) + R_2(\sigma), \quad \forall \sigma \in S_h, \\
\end{align*}
\]

where

\[
R_1(\mathbf{v}) := (Re \, d_t u^n, \mathbf{v}) - (Re \, u_t^n, \mathbf{v}) + c(Re \, u_h^{n-1}, u^n, \mathbf{v}) - c(Re \, u^n, u^n, \mathbf{v}),
\]

and

\[
R_2(\sigma) := (\lambda d_t \tau^n, \sigma) - (\lambda \tau_t^n, \sigma) - (\lambda \tau_t^n, \nu \sigma_u) + b(\lambda u_h^{n-1}, \tau^n, \tilde{\sigma}) - b(\lambda u^n, \tau^n, \tilde{\sigma}) + \left( \lambda g_a \left( \tau_h^{n-1}, \nabla u_h^{n-1} \right), \tilde{\sigma} \right) - (\lambda g_a (\tau^n, \nabla u^n), \tilde{\sigma}).
\]

Subtracting (4.36)-(4.37) from (4.46)-(4.47) we obtain the following equations for $\epsilon_u$ and $\epsilon_\tau$:

\[
(Re \, d_t \epsilon_u, \mathbf{v}) + c(Re \, u_h^{n-1}, \epsilon_u, \mathbf{v}) + 2(1-\alpha) \left( D(\epsilon_u), D(\mathbf{v}) \right) + (\epsilon_\tau, D(\mathbf{v})) = (p^n, \nabla \cdot \mathbf{v}) + R_1(\mathbf{v}), \quad \forall \mathbf{v} \in Z_h
\]

Substituting $\epsilon_u = E^n + A^n$, $\epsilon_\tau = F^n + G^n$, $\mathbf{v} = E^n$, $\sigma = F^n$ into (4.48)-(4.49), we obtain

\[
(Re \, d_t E^n, E^n) + c(Re \, u_h^{n-1}, E^n, E^n) + 2(1-\alpha) \left( D(E^n), D(E^n) \right) + (F^n, D(E^n)) = F_1(E^n),
\]

\[
(\lambda d_t F^n, F^n) + B(\lambda u_h^{n-1}, F^n, \tilde{F}^n) - 2\alpha \left( D(F^n), \tilde{F}^n \right) + \left( F^n, \tilde{F}^n \right) = F_2(F^n),
\]

where,

\[
F_1(E^n) = (p^n, \nabla \cdot E^n) + R_1(E^n) - (Re \, d_t A^n, E^n) - c(Re \, u_h^{n-1}, A^n, E^n) - 2(1-\alpha) \left( D(A^n), D(E^n) \right) - (G^n, D(E^n)),
\]

\[
F_2(F^n) = R_2(F^n) - (\lambda d_t G^n, F^n) - b(\lambda u_h^{n-1}, G^n, \tilde{F}^n) + 2\alpha \left( D(A^n), \tilde{F}^n \right) - (G^n, \tilde{F}^n).
\]
where $Re^*_n = Re_m$ or $Re_M$ depending on the sign of $[\|E^n\|^2 - \|E^{n-1}\|^2]$. Similarly, 

$(\lambda d_t F^n, F^n) \geq \frac{\lambda^*_n}{2\Delta t} \left(\|F^n\|^2 - \|F^{n-1}\|^2\right)$. Then, from (4.50), we have that 

$$
\frac{1}{2\Delta t} \left(\|E^n\|^2 - \|E^{n-1}\|^2\right) + \frac{1}{Re^*_n} c(Re u_h^{-1}, E^n, E^n) + \frac{1}{Re^*_n} 2(1 - \alpha) (D(E^n), D(E^n)) + \frac{1}{Re^*_n} R^*_n (F^n, D(E^n)) \leq \frac{1}{Re^*_n} F_1(E^n). (4.52)
$$

Multiplying (4.52) by $Re_m \Delta t$ and summing from $n = 1$ to $l$ gives 

$$
\frac{Re_m}{2} \left(\|E^l\|^2 - \|E^0\|^2\right) + \sum_{n=1}^l \left\{ \frac{Re_m \Delta t}{Re^*_n} c(Re u_h^{-1}, E^n, E^n) + \frac{Re_m \Delta t}{Re^*_n} 2(1 - \alpha) \|D(E^n)\|^2 + \frac{Re_m \Delta t}{Re^*_n} (F^n, D(E^n)) \right\} \leq \sum_{n=1}^l \frac{Re_m \Delta t}{Re^*_n} F_1(E^n). (4.53)
$$

Similarly, from (4.51) we have that 

$$
\frac{\lambda_m}{2} \left(\|F^l\|^2 - \|F^0\|^2\right) + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda^*_n} \left\{ \left(\lambda F^n, \lambda F^n\right) + b(\lambda u_h^{-1}, F^n, \lambda F^n) \right\} - 2\alpha \left(D(E^n), \lambda F^n\right) \leq \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda^*_n} F_2(F^n). (4.54)
$$

Multiplying (4.53) by $2\alpha$ and adding to (4.54) yields 

$$
\alpha Re_m \left(\|E^l\|^2 - \|E^0\|^2\right) + \frac{\lambda_m}{2} \left(\|F^l\|^2 - \|F^0\|^2\right) + 4\alpha(1 - \alpha) \sum_{n=1}^l \frac{Re_m \Delta t}{Re^*_n} \|D(E^n)\|^2 + \nu \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda^*_n} \|u^n\|^2 + \sum_{n=1}^l \frac{\lambda_m \Delta t}{\lambda^*_n} \|F^n\|^2 \leq \Delta t \sum_{n=1}^l \frac{2\alpha Re_m}{Re^*_n} \left(F_1(E^n) - c(Re u_h^{-1}, E^n, E^n) - (F^n, D(E^n))\right) + \Delta t \sum_{n=1}^l \frac{\lambda_m}{\lambda^*_n} \left(F_2(F^n) - b(\lambda u_h^{-1}, F^n, F^n) + 2\alpha \left(D(E^n), \lambda F^n\right) - \nu (F^n, F^n)\right). (4.55)
$$
Noting that \( Re_m \leq Re_n^* \leq Re_M, \lambda_m \leq \lambda_n^* \leq \lambda_M \), applying the triangle inequality to the right hand side of (4.55), we have that

\[
\alpha Re_m \left( \| F^t \|^2 - \| E^0 \|^2 \right) + \frac{\lambda_m}{2} \left( \| F^t \|^2 - \| F^0 \|^2 \right) + 4\alpha(1 - \alpha) \sum_{n=1}^{l} \frac{Re_m R}{Re_M} \| D(E^n) \|^2 \\
+ \nu \sum_{n=1}^{l} \frac{\lambda_n}{\lambda_M} \| F^n_u \|^2 + \sum_{n=1}^{l} \frac{\lambda_n}{\lambda_M} \| F^n \|^2 \\
\leq \Delta t \sum_{n=1}^{l} \left\{ 2\alpha \left[ c(Re u_h^{-1}, E^n, E^n) \right] + \left[ b(\lambda u_h^{-1}, F^n, F^n) \right] \\
+ 2\alpha \nu \left( |D(E^n), F^n_u| \right) + 2\alpha \left( 1 - \frac{Re_m}{Re_M} \right) \left( |D(E^n), F^n| \right) + \nu \left( |F^n, F^n_u| \right) \right\} \\
+ \Delta t \sum_{n=1}^{l} \left\{ 2\alpha |F_1(E^n)| + |F_2(F^n)| \right\}.
\]

We now estimate each term on the right hand side of (4.56). For \( c(u_h^{-1}, E^n, E^n) \) we have that

\[
2\alpha \ c(Re u_h^{-1}, E^n, E^n) \leq 2\alpha \ Re_M \left( u_h^{-1} \cdot \nabla E^n, E^n \right) \\
\leq 2\alpha \ Re_M \left\| u_h^{-1} \cdot \nabla E^n \right\| \left\| E^n \right\| \\
\leq 2\alpha \ Re_M \left\| u_h^{-1} \right\| \left\| \nabla E^n \right\| \left\| E^n \right\| \\
\leq 4\alpha^2 Re_M^2 \epsilon_1 \left\| \nabla E^n \right\|^2 + \frac{\hat{d} K^2}{4\epsilon_1} \left\| E^n \right\|^2, \text{ using (IH1).}
\]

\[
\leq 4\alpha^2 Re_M^2 \ C_1^2 \epsilon_1 \left\| D(E^n) \right\|^2 + \frac{\hat{d} K^2}{4\epsilon_1} \left\| E^n \right\|^2,
\]

(using Korn’s lemma)

Note that for \( v = 0 \) on \( \partial \Omega \), applying Green’s theorem we have

\[
b(v, \tau, \sigma) = -b(v, \sigma, \tau) - (\nabla \cdot v, \tau, \sigma),
\]

\[
\Rightarrow b(v, \tau, \tau) = -\frac{1}{2} (\nabla \cdot v, \tau, \tau).
\]

Using (4.59),

\[
\left| b(\lambda u_h^{-1}, F^n, F^n) \right| \leq \frac{\lambda_M}{2} \left| (\nabla \cdot u_h^{-1} F^n, F^n) \right| \\
= \frac{\lambda_M}{2} \left| (\nabla \cdot (u_h^{-1} - U^{n-1}) F^n, F^n) + (\nabla \cdot U^{n-1} F^n, F^n) \right| \\
\leq \frac{\lambda_M}{2} \left\| \nabla \cdot E^n \right\|_\infty \left\| F^n \right\|^2 + \frac{\lambda_M}{2} \left\| \nabla \cdot U^{n-1} \right\|_\infty \left\| F^n \right\|^2.
\]
\[ \leq \frac{\lambda M}{2} \left( \frac{d}{6} \| \nabla \cdot \mathbf{E}^{n-1} \|_{\infty} + M \| \mathbf{F}^n \| + \| \mathbf{F}^n \| \right), \text{ using (3.2)}. \]

Next, with \( \tilde{R} = (1 - Re_m/Re_M) \),

\[ 2\alpha \tilde{R} |(D(\mathbf{E}^n), \mathbf{F}^n)| \leq 2\alpha \tilde{R} ||D(\mathbf{E}^n)|| ||\mathbf{F}^n|| \]
\[ \leq 4\alpha^2 \tilde{R}^2 \epsilon_2 ||D(\mathbf{E}^n)||^2 + \frac{1}{4\epsilon_2} ||\mathbf{F}^n||^2. \]

Then

\[ 2\alpha |(D(\mathbf{E}^n), \nu \mathbf{F}_u^n)| \leq 2\alpha ||D(\mathbf{E}^n)|| ||\nu \mathbf{F}_u^n|| \]
\[ \leq 4\alpha^2 \epsilon_3 ||D(\mathbf{E}^n)||^2 + \frac{\nu^2}{4\epsilon_3} ||\mathbf{F}_u^n||^2. \]

Also,

\[ |(\mathbf{F}^n, \nu \mathbf{F}_u^n)| = \nu |(\mathbf{F}^n, \mathbf{F}_u^n)| \]
\[ \leq \nu ||\mathbf{F}^n|| ||\mathbf{F}_u^n|| \]
\[ \leq ||\mathbf{F}^n||^2 + \frac{\nu^2}{4} ||\mathbf{F}_u^n||^2. \]

Thus, for the first summation on the right hand side of (4.56), we have

\[ \Delta t \sum_{n=1}^{l} \left\{ 2\alpha \left| c(Re \mathbf{u}_h^{n-1}, \mathbf{E}^n, \mathbf{E}^n) \right| + \left| b(\lambda \mathbf{u}_h^{n-1}, \mathbf{F}^n, \mathbf{F}^n) \right| + 2\alpha \nu |(D(\mathbf{E}^n), \mathbf{F}_u^n)| \right\} \]
\[ -2\alpha \tilde{R} |(D(\mathbf{E}^n), \mathbf{F}^n)| + \nu |(\mathbf{F}^n, \mathbf{F}_u^n)| \}
\[ \leq \Delta t \sum_{n=1}^{l} \left( 4\alpha^2 (C_K^2 Re_M^2 \epsilon_1 + \tilde{R}^2 \epsilon_2 + \epsilon_3) ||D(\mathbf{E}^n)||^2 \right) \]
\[ + \Delta t \sum_{n=1}^{l} \left( \frac{d K^2}{4\epsilon_1} ||\mathbf{E}^n||^2 \right) \]
\[ + \Delta t \sum_{n=1}^{l} \left( \frac{\lambda M}{2} \left( \frac{d}{6} \| \nabla \cdot \mathbf{E}^{n-1} \|_{\infty} + M \right) + \frac{1}{4\epsilon_2} + 1 \right) ||\mathbf{F}^n||^2 \]
\[ + \Delta t \sum_{n=1}^{l} \left( \frac{\nu^2}{4\epsilon_3} + \frac{\nu^2}{4} \right) ||\mathbf{F}_u^n||^2 \] (4.60)

Next we consider \( \mathcal{F}_1(\mathbf{E}^n) \).

\[ |(p^n, \nabla \cdot \mathbf{E}^n)| = |(p^n - \mathcal{P}^n, \nabla \cdot \mathbf{E}^n)| \]
\[ \leq \|p^n - \mathcal{P}^n\| d^n \|\nabla \mathbf{E}^n\| \]
\[
\begin{align*}
\leq C_K^2 \varepsilon_5 \|D(E^n)\|^2 + \frac{d}{4\varepsilon_5} \|p^n - P^n\|, \\
(\text{using Korn's lemma})
\end{align*}
\]

\[
\begin{align*}
| (Re \, d_t \Lambda^n, E^n) | & \leq Re_M \|E^n\| \|d_t \Lambda^n\| \\
& \leq Re_M^2 \|E^n\|^2 + \frac{1}{4} \|d_t \Lambda^n\|^2.
\end{align*}
\]

\[
\begin{align*}
|c(Re \, u_h^{n-1}, \Lambda^n, E^n)| & \leq Re_M \|E^n\| \|u_h^{n-1}\| \nabla \Lambda^n \| \\
& \leq Re_M \|E^n\| \|u_h^{n-1}\| \|d_t \Lambda^n\|. \\
& \leq Re_M \|E^n\|^2 + \frac{K^2 d}{4} \|\nabla \Lambda^n\|^2, \quad \text{using (IH1).}
\end{align*}
\]

\[
\begin{align*}
2(1 - \alpha) \|(D(\Lambda^n), D(E^n))\| & \leq (1 - \alpha) \varepsilon_6 \|D(E^n)\|^2 + \frac{1 - \alpha}{2\varepsilon_6} \|D(\Lambda^n)\|^2. \\
\|(\Gamma^n, D(E^n))\| & \leq \|D(E^n)\| \|\Gamma^n\| \\
& \leq \varepsilon_7 \|D(E^n)\|^2 + \frac{1}{4\varepsilon_7} \|\Gamma^n\|^2.
\end{align*}
\]

For the \( R_1(E^n) \) terms we have:

\[
\begin{align*}
| (Re \, d_t u^n, E^n) - (Re \, u_t^n, E^n) | & \leq Re_M^2 \|E^n\|^2 + \frac{1}{4} \|d_t u^n - u_t^n\|^2 \\
|c(Re \, u_h^{n-1}, u^n, E^n) - c(Re \, u^n, u^n, E^n)| & = |c(Re (u_h^{n-1} - \mathcal{U}^{n-1}), u^n, E^n) \\
& \quad + c(Re (\mathcal{U}^{n-1} - u^{n-1}), u^n, E^n) \\
& \quad + c(Re (u^{n-1} - u^n), u^n, E^n)| \\
& \leq Re_M \|E^{n-1}\| \nabla u^n \|E^n\| \\
& \quad + Re_M \|\Lambda^{n-1}\| \nabla u^n \|E^n\| \\
& \quad + Re_M \|(u^n - u^{n-1}) \nabla u^n\| \|E^n\| \\
& \leq Re_M \mathring{d} M \|E^{n-1}\| \|E^n\| \\
& \quad + Re_M \mathring{d} M \|\Lambda^{n-1}\| \|E^n\| \\
& \quad + Re_M \mathring{d} M \|(u^n - u^{n-1})\| \|E^n\|, \text{using (4.29)} \\
& \leq \frac{Re_M^2 d^2 M^2}{2} \|E^{n-1}\|^2 \\
& \quad + \frac{3}{2} \|E^n\|^2 + \frac{Re_M^2 d^2 M^2}{2} \|\Lambda^{n-1}\|^2 \\
& \quad + \frac{Re_M^2 d^2 M^2}{2} \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|^2 \, dt.
\end{align*}
\]
Combining (4.61)-(4.66) we have the following estimate for $\mathcal{F}_1(E^n)$:

$$
|2\alpha \mathcal{F}_1(E^n)| \leq 2\alpha(C_K^2 \epsilon_5 + (1 - \alpha)\epsilon_6 + \epsilon_7) \| D(E^n) \|^2 + 2\alpha \left( 3Re_M^2 + \frac{3}{2} \right) \| E^n \|^2 \\
+ \alpha R e_M^2 d^2 M^2 \| E^{n-1} \|^2 + 2\alpha \frac{d}{4\epsilon_5} \| (p^n - P^n) \|^2 + \alpha Re_M^2 d^2 M^2 \| A^{n-1} \|^2 \\
+ 2\alpha \left( \frac{K^2d}{4} + \frac{1 - \alpha}{4\epsilon_6} \right) \| \nabla A^n \|^2 + \frac{1}{2} \| d_t A^n \|^2 + \frac{1}{2} \| \Gamma^n \|^2 \\
+ \alpha \frac{1}{2} \| d_t u^n - u^n_t \|^2 + \alpha Re_M^2 d^2 M^2 \Delta t \int_{t_{n-1}}^{t_{n}} \| u_t \|^2 \, dt. 
$$

(4.67)

Next we consider the terms in $\mathcal{F}_2(F^n)$.

$$
|\langle \lambda d_t \Gamma^n, F^n \rangle | \leq \lambda_M^2 \| F^n \|^2 + \frac{1}{4} \| d_t \Gamma^n \|^2. 
$$

(4.68)

$$
|b(\lambda u^n_h^{-1}, \Gamma^n, F^n) + b(\lambda u^n_h^{-1}, \Gamma^n, \nu F^n_u)| \\
\leq \lambda_M \| u^n_h^{-1} \cdot \nabla \Gamma^n \| \| F^n \| + \lambda_M \| u^n_h^{-1} \cdot \nabla F^n_u \| \| v F^n_u \|
\leq \lambda_M d^2 \| u^n_h^{-1} \| \| \nabla \Gamma^n \| \| F^n \| + \lambda_M d^2 \| u^n_h^{-1} \| \| \nabla \Gamma^n \| \| \nu F^n_u \|
\leq \lambda_M^2 \| F^n \|^2 + \nu^2 \| F^n_u \|^2 + \frac{dK^2}{2} \| \nabla \Gamma^n \|^2.
$$

(4.69)

$$
2\alpha \left| \langle D(A^n), \tilde{F}^n \rangle \right| = 2\alpha \left| \langle D(A^n), F^n \rangle + (D(A^n), \nu F^n_u) \rangle \right|. \\
\leq \| F^n \|^2 + \nu^2 \| F^n_u \|^2 + 2\alpha^2 \| \nabla A^n \|^2. 
$$

(4.70)

$$
\left| \langle \Gamma^n, \tilde{F}^n \rangle \right| = \left| \langle \Gamma^n, F^n \rangle + \nu \langle \Gamma^n, \nu F^n_u \rangle \right|
\leq \| F^n \|^2 + \nu^2 \| F^n_u \|^2 + \frac{1}{2} \| \Gamma^n \|^2.
$$

(4.71)

For the terms making up $R_2(F^n)$ we have:

$$
|\langle \lambda d_t \tau^n, F^n \rangle - \langle \lambda \tau^n, F^n \rangle | \leq \| \lambda F^n \| \| d_t \tau^n - \tau^n_t \|
\leq \lambda_M^2 \| F^n \|^2 + \frac{1}{4} \| d_t \tau^n - \tau^n_t \|^2. 
$$

(4.72)

$$
|\langle \lambda \tau^n, \nu F^n_u \rangle | = \left| \langle \lambda \tau^n, \nu u^n_h^{-1} \cdot \nabla F^n \rangle \right|
\leq \left| B(\lambda \nu u^n_h^{-1}, F^n, \tau^n) \right|
\leq \left| B(\lambda \nu u^n_h^{-1}, \tau^n, F^n) \right| + \left| \left( \nabla \cdot u^n_h^{-1} \lambda \nu F^n, \tau^n \right) \right|
\quad (\text{using (3.23)})
\leq \lambda_M \nu \| u^n_h^{-1} \cdot \nabla \tau^n \| \| F^n \|
\quad + \left| \left( \nabla \cdot (u^n_h^{-1} - U^{n-1}) \lambda \nu F^n, \tau^n \right) \right|
$$

(4.74)
\[
\begin{align*}
&+ \left| \nabla \mathcal{U}^{n-1} \lambda \nu \mathbf{F}^n, \tau^t_n \right| \\
\leq \lambda_M \nu \left\| \mathbf{u}^{n-1}_h \right\|_\infty d_\lambda \left\| \nabla \tau^t_n \right\| \left\| \mathbf{F}^n \right\| \\
&+ \lambda_M \nu \left\| \nabla \left( \mathbf{u}^{n-1}_h - \mathcal{U}^{n-1} \right) \right\|_\infty \left\| \mathbf{F}^n \right\| \left\| \tau^t_n \right\| \\
&+ \left\| \nabla \mathcal{U}^{n-1} \right\|_\infty \lambda_M \nu \left\| \mathbf{F}^n \right\| \left\| \tau^t_n \right\| \\
\leq \lambda_M^2 \left( 2 + d \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \left\| \mathbf{F}^n \right\|^2 \\
&+ \frac{\nu^2}{4} \left( d^2 M^2 + d \left\| \nabla \mathbf{E}^{n-1} \right\|_\infty \right) \left\| \tau^t_n \right\|^2 \\
&+ \frac{\nu^2}{4} R^2 d \left\| \nabla \tau^t_n \right\|^2 , \text{ (using (3.2) and (III)) \ . (4.75)}
\end{align*}
\]

\[
\left| b(\lambda \mathbf{u}^{n-1}_h, \tau^n, \mathbf{F}^n) - b(\lambda \mathbf{u}^n, \tau^n, \mathbf{F}^n) \right| = \left| \left( \lambda (\mathbf{u}^{n-1}_h - \mathbf{u}^n) \cdot \nabla \tau^n, \mathbf{F}^n \right) \right| \\
\leq \left\| (\mathbf{u}^{n-1}_h - \mathbf{u}^n) \cdot \nabla \tau^n \right\| \lambda \mathbf{F}^n \\
\leq \frac{\lambda_M^2 \nu}{2} \left\| \mathbf{F}^n \right\|^2 + \frac{1}{2} d^3 \left\| \nabla \tau^n \right\|_\infty^2 \left\| \mathbf{u}^{n-1}_h - \mathbf{u}^n \right\|^2 \\
\leq \frac{\lambda_M^2 \nu}{2} \left\| \mathbf{F}^n \right\|^2 + \frac{\lambda_M^2 \nu^2}{2} \left\| \mathbf{F}^n \right\|^2 \\
&+ \frac{1}{2} d^3 M^2 \left\| \mathbf{E}^{n-1} - \mathbf{U}^{n-1} + \mathbf{u}^{n-1} - \mathbf{u}^n \right\|^2 \\
\leq \frac{\lambda_M^2 \nu}{2} \left\| \mathbf{F}^n \right\|^2 + \frac{\lambda_M^2 \nu^2}{2} \left\| \mathbf{F}^n \right\|^2 + \frac{3}{2} d^3 M^2 \left\| \mathbf{E}^{n-1} \right\|^2 \\
&+ \frac{3}{2} d^3 M^2 \left\| \mathbf{A}^{n-1} \right\|^2 \\
&+ \frac{3}{2} d^3 M^2 \Delta t \int_{\tau^n_{n-1}}^{\tau^n_t} \left\| \mathbf{u}_r \right\|^2 \, dt \ . (4.76)
\]

In order to estimate the \( g_a \) terms in \( \mathcal{F}_2(\cdot) \) note that

\[
\lambda \left( g_a \left( \tau^{n-1}_h, \nabla \mathbf{u}_h^{n-1} \right) - g_a \left( \tau^n, \nabla \mathbf{u}^n \right) \right) = \lambda \left( g_a \left( \tau^{n-1}_h, \nabla (\mathbf{u}_h^{n-1} - \mathcal{U}^{n-1}) \right) \right) \\
+ g_a \left( \tau^{n-1}_h, \nabla (\mathcal{U}^{n-1} - \mathbf{u}^{n-1}) \right) \\
+ g_a \left( \tau^{n-1}_h, \nabla (\mathbf{u}^{n-1} - \mathbf{u}^n) \right) \\
+ g_a \left( \tau^{n-1}_h - \tau^{n-1}, \nabla \mathbf{u}^n \right) \\
+ g_a \left( \tau^{n-1} - \tau^n, \nabla \mathbf{u}^n \right) \\
+ g_a \left( - \tau^{n-1}, \nabla \mathbf{E}^{n-1} \right) \\
- g_a \left( \tau^{n-1}_h, \nabla \mathbf{A}^{n-1} \right)
\]
Bounding each of the terms on the right hand side of (4.77) we obtain

\[ -g_a \left( \tau_h^{n-1}, \nabla (u^n - u^{n-1}) \right) \]
\[ -g_a \left( F^{n-1}, \nabla u^n \right) \]
\[ -g_a \left( \Gamma^{n-1}, \nabla u^n \right) \]
\[ -g_a \left( \tau^n - \tau^{n-1}, \nabla u^n \right) \]. 

(4.77)

\[ \left\| \left( \lambda g_a \left( \tau_h^{n-1}, \nabla E^{n-1} \right), \tilde{F}^n \right) \right\| \leq \left\| g_a \left( \tau_h^{n-1}, \nabla E^{n-1} \right) \right\| \left\| \lambda \tilde{F}^n \right\| \]
\[ \leq 4\dot{d} \left\| \tau_h^{n-1} \right\|_\infty \left\| \nabla E^{n-1} \right\|_\lambda M \left\| \tilde{F}^n \right\| \]
\[ \leq \epsilon_8 \left\| \nabla E^{n-1} \right\|^2 + \frac{8d^2K^2\lambda_M^2}{\epsilon_8} \left\| F^n \right\|^2 
+ \frac{8d^2K^2\lambda_M^2}{\epsilon_8} \nu^2 \left\| F^n_u \right\|^2, \]

(4.78)

\[ \left\| \left( \lambda g_a \left( \tau_h^{n-1}, \nabla \Lambda^{n-1} \right), \tilde{F}^n \right) \right\| \leq 8d^2K^2 \left\| \nabla \Lambda^{n-1} \right\|^2 + \lambda_M^2 \left\| F^n \right\|^2 + \lambda_M^2 \nu^2 \left\| F^n_u \right\|^2, \]

(4.79)

\[ \left\| \left( \lambda g_a \left( \tau_h^{n-1}, \nabla (u^n - u^{n-1}) \right), \tilde{F}^n \right) \right\| \leq 8d^2K^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \nabla u_t \right\|^2 \, dt + \lambda_M^2 \left\| F^n \right\|^2 
+ \lambda_M^2 \nu^2 \left\| F^n_u \right\|^2, \]

(4.80)

\[ \left\| \left( \lambda g_a \left( \Gamma^{n-1}, \nabla u^n \right), \tilde{F}^n \right) \right\| \leq 8d^2M^2 \left\| \Gamma^{n-1} \right\|^2 + \lambda_M^2 \left\| F^n \right\|^2 + \lambda_M^2 \nu^2 \left\| F^n_u \right\|^2, \]

(4.81)

\[ \left\| \left( \lambda g_a \left( \tau^n - \tau^{n-1}, \nabla u^n \right), \tilde{F}^n \right) \right\| \leq 8d^2M^2 \Delta t \int_{t^{n-1}}^{t^n} \left\| \tau_t \right\|^2 \, dt + \lambda_M^2 \left\| F^n \right\|^2 
+ \lambda_M^2 \nu^2 \left\| F^n_u \right\|^2. \]

(4.82)

Combining the estimates in (4.68)-(4.76), (4.78)-(4.83), we obtain the following estimate for \( \mathcal{F}_2(F^n) \):

\[ \left| \mathcal{F}_2(F^n) \right| \leq \epsilon_8 \left\| \nabla E^{n-1} \right\|^2 + \nu^2 \left\| F^n_u \right\|^2 \left( 6\lambda_M^2 + \frac{3 + 8d^2K^2\lambda_M^2}{\epsilon_8} \right) \]
\[ + \left\| F^n \right\|^2 \left( 11\lambda_M^2 + 2 + \frac{8d^2K^2\lambda_M^2}{\epsilon_8} + \dot{d} \left\| \nabla E^{n-1} \right\|_\infty \right) \]
\[ + \left\| E^{n-1} \right\|^2 \left( \frac{3}{2}d^2M^2 \right) + \left\| F^{n-1} \right\|^2 \left( 8d^2M^2 \right) \]
\[ + 2\alpha^2 \left\| \nabla \Lambda^n \right\|^2 + \left\| \nabla \Gamma^n \right\|^2 \left( \frac{dK^2}{2} \right) + \left\| \Gamma^n \right\|^2 \left( \frac{1}{2} \right) + \left\| \Gamma^n_t \right\|^2 \left( \frac{1}{4} \right) \]
Then, from (4.56) we have that

\[
\alpha \text{Re}_m \left(\|E\|^2 - \|E^0\|^2\right) + \frac{\lambda_m}{2} \left(\|F\|^2 - \|F^0\|^2\right) + 4\alpha(1 - \alpha) \sum_{n=1}^{l} \frac{\text{Re}_m \Delta t}{\text{Re}_M} \left\|D\left(E^n\right)\right\|^2 \\
+ \nu \sum_{n=1}^{l} \left(\frac{2\lambda_n}{\lambda_M} \|F^n\|^2 + \sum_{n=1}^{l} \frac{\lambda_n \Delta t}{\lambda_M} \left\|\Gamma^n\right\|^2 \right) \\
\leq \Delta t \sum_{n=1}^{l} \left\{4\alpha^2 \left(C_K^2 \text{Re}_M^2 \epsilon_1 + \tilde{R}^2 \epsilon_2 + \epsilon_3 \right) + 2\alpha \left(C_K^2 \epsilon_5 + (1 - \alpha) \epsilon_6 + \epsilon_7 \right) \right\} \left\|D\left(E^n\right)\right\|^2 \\
+ \Delta t \sum_{n=1}^{l} C_K^2 \epsilon_8 \left\|D\left(E^n-1\right)\right\|^2 + \Delta t \sum_{n=1}^{l} \left(\frac{\dot{K}^2}{4\epsilon_1} + 2\alpha \left(3\text{Re}_M + \frac{3}{2}\right) \right) \|E^n\|^2 \\
+ \Delta t \sum_{n=1}^{l} \left(\frac{3\dot{d}^2 M^2}{2} + \alpha \text{Re}_M^2 \dot{d}^2 M^2 \right) \|E^n-1\|^2 \\
+ \Delta t \sum_{n=1}^{l} \left(\frac{\lambda_M}{2} + 1\right) \dot{d} \left\|\nabla E^n-1\right\| + \frac{\lambda_M \dot{d} M}{2} + \frac{1}{4\epsilon_2} + 3 + 11\lambda_M + \frac{8\dot{d}^2 K^2 \lambda_M^2}{\epsilon_8} \right\} \|F^n\|^2 \\
+ \Delta t \sum_{n=1}^{l} 8\dot{d}^2 M^2 \|E^n-1\|^2 \\
+ \Delta t \sum_{n=1}^{l} \nu^2 \left(\frac{1}{4\epsilon_3} + \frac{1}{4} + 6\lambda_M^2 + 3 + \frac{8\dot{d}^2 K^2 \lambda_M^2}{\epsilon_8} \right) \|F^n\|^2 \\
+ \Delta t \sum_{n=1}^{l} \frac{2\alpha \dot{d}}{4\epsilon_5} \left\|p^n - P^n\right\|^2 + \Delta t \sum_{n=1}^{l} \left(\alpha \text{Re}_M^2 \dot{d}^2 M^2 + \frac{3\dot{d}^2 M^2}{2} \right) \|A^n-1\|^2 \\
+ \Delta t \sum_{n=1}^{l} 2\alpha \left(\frac{\dot{K}^2}{4} + \frac{1 - \alpha}{\epsilon_6} + 2\alpha^2 \right) \|\nabla A^n\|^2 + \Delta t \sum_{n=1}^{l} 8\dot{d}^2 K^2 \|\nabla A^n-1\|^2 \\
+ \Delta t \sum_{n=1}^{l} \frac{\alpha}{2} \left\|d_t A^n\right\|^2 + \Delta t \sum_{n=1}^{l} \frac{\alpha}{4} \left\|d_t F^n\right\|^2 + \Delta t \sum_{n=1}^{l} \left(\frac{\alpha}{2\epsilon_7} + \frac{1}{2}\right) \|\Gamma^n\|^2 \\
+ \Delta t \sum_{n=1}^{l} \frac{\dot{d} K^2}{2} \|\nabla \Gamma^n\|^2 + \Delta t \sum_{n=1}^{l} 8\dot{d}^2 M^2 \|\Gamma^n-1\|^2 \\
+ \Delta t \sum_{n=1}^{l} \frac{\alpha}{2} \left\|d_t u^n - u^n\right\|^2 + \Delta t \sum_{n=1}^{l} \frac{1}{4} \left\|d_t \tau^n - \tau^n\right\|^2 \\
+ \Delta t \sum_{n=1}^{l} \nu^2 \left(\frac{\dot{d}^2 M^2 + \dot{d} \left\|\nabla E^n-1\right\|}{\infty} \right) \|\tau^n\|^2 
\]
\[ + \Delta t \sum_{n=1}^{l} \left( \alpha Re_M d^2 M^2 \Delta t + \frac{3}{2} \beta^2 M^2 \Delta t \right) \int_{t_{n-1}}^{t_n} \| \mathbf{u} \|^2 dt \]

\[ + \Delta t \sum_{n=1}^{l} \frac{\nu^2}{4} K^2 \dot{d} \| \nabla \tau^n \|^2 + \Delta t \sum_{n=1}^{l} 8 \beta^2 M^2 \Delta t \| \tau \|^2 dt \]

\[ + \Delta t \sum_{n=1}^{l} 8 \beta^2 K^2 \Delta t \int_{t_{n-1}}^{t_n} \| \nabla \mathbf{u} \|^2 dt. \] (4.85)

With the following choices:

\[ \epsilon_1 = \frac{Re_m(1 - \alpha)}{14 C^2_K Re^3_M \alpha}, \quad \epsilon_2 = \frac{Re_m(1 - \alpha)}{14 \beta^2 Re_M \alpha}, \quad \epsilon_3 = \frac{Re_m(1 - \alpha)}{14 Re_M \alpha}, \]

\[ \epsilon_5 = \frac{Re_m(1 - \alpha)}{7 C^2_K Re_M}, \quad \epsilon_6 = \frac{Re_m}{7 Re_M}, \quad \epsilon_7 = \frac{Re_m}{7 Re_M}, \]

\[ \epsilon_8 = \frac{Re_m 2 \alpha (1 - \alpha)}{7 Re_M C^2_K}, \quad \mathbf{u}^0 = \mathbf{U}^0 (\Rightarrow \mathbf{E}^0 = \mathbf{0}), \quad \tau^0 = \mathbf{T}^0 (\Rightarrow \mathbf{F}^0 = \mathbf{0}), \]

substituting into (4.85)

\[ \alpha Re_m \| \mathbf{E} \|^2 + \frac{\lambda m}{2} \| \mathbf{F} \|^2 + 2 \alpha(1 - \alpha) \frac{Re_m}{Re_M} \Delta t \sum_{n=1}^{l} \| D(\mathbf{E}) \|^2 \]

\[ + \left( \nu \frac{\lambda^2}{\lambda_M} - \nu^2 \left( \frac{7 \beta^2 Re_M \alpha}{2 Re_m(1 - \alpha)} + \frac{28 \beta^2 C^2_K K^2 \lambda^2_M Re_M}{\alpha(1 - \alpha) Re_m} + \frac{13}{4} + 6 \lambda^2_M \right) \right) \Delta t \sum_{n=1}^{l} \| \mathbf{F}_u \|^2 \]

\[ + \frac{\lambda m}{\lambda_M} \Delta t \sum_{n=1}^{l} \| \mathbf{F} \|^2 \leq C_1 \Delta t \sum_{n=1}^{l} \| \mathbf{E} \|^2 + \Delta t \sum_{n=1}^{l} \left( \frac{C_2}{2} \left( 1 + \| \nabla \mathbf{E} \|_{\infty} \right) \right) \| \mathbf{F} \|^2 \]

\[ + C_3 \Delta t \sum_{n=1}^{l} \| \mathbf{A} \|^2 + C_4 \Delta t \sum_{n=1}^{l} \| \nabla \mathbf{A} \|^2 + \frac{\alpha}{2} \Delta t \sum_{n=1}^{l} \| \dot{d} \mathbf{A} \|^2 \]

\[ + \frac{1}{4} \Delta t \sum_{n=1}^{l} \| \dot{d} \mathbf{E} \|^2 + C_5 \Delta t \sum_{n=1}^{l} \| \mathbf{F} \|^2 + C_6 \Delta t \sum_{n=1}^{l} \| \nabla \mathbf{F} \|^2 \]

\[ + \alpha \Delta t \sum_{n=1}^{l} \| \dot{d} \mathbf{u} - \mathbf{u} \|^2 + \frac{1}{4} \Delta t \sum_{n=1}^{l} \| \dot{d} \mathbf{r} - \mathbf{r} \|^2 \]

\[ + \Delta t \sum_{n=1}^{l} \frac{\nu^2}{4} \left( d^2 M^2 + \dot{d} \| \nabla \mathbf{E} \|_{\infty} \right) \| \mathbf{E} \|^2 \]

\[ + \Delta t \sum_{n=1}^{l} \left( \frac{\alpha Re_M d^2 M^2}{4} + \frac{3}{2} \beta^2 M^2 \right) \| \mathbf{u} \|^2_{0,0} \]

\[ + 2 \alpha \frac{\dot{d}}{4 \epsilon_5} \Delta t \sum_{n=1}^{l} \| p^n - P^n \|^2 + K^2 \frac{\nu^2}{4} \| \nabla \mathbf{u} \|^2_{0,0} \]

\[ + 8 \beta^2 M^2 \| \Delta t \|^2 \| \tau \|^2_{0,0} \]

\[ + 8 \beta^2 K^2 \| \Delta t \|^2 \| \nabla \mathbf{u} \|^2_{0,1}. \] (4.86)
We now apply the interpolation properties of the approximating spaces to estimate the terms on the right hand side of (4.86). Using elements of order \( k \) for velocity, elements of order \( m \) for stress, and elements of order \( q \) for pressure, we have

\[
\sum_{n=1}^{l} \Delta t \| \nabla \mathbf{A}^n \|^2 + \sum_{n=1}^{l} \Delta t \| \nabla \mathbf{\Gamma}^n \|^2 \leq C \left( h^{2k} \sum_{n=1}^{l} \Delta t \| \mathbf{u}^n \|^2_{k+1} + h^{2m} \sum_{n=1}^{l} \Delta t \| \mathbf{\tau}^n \|^2_{m+1} \right) \leq C \left( h^{2k} \| \mathbf{u} \|_{0,k+1}^2 + h^{2m} \| \mathbf{\tau} \|_{0,m+1}^2 \right),
\]

(4.87)

\[
\sum_{n=1}^{l} \Delta t \| \mathbf{A}^n \|^2 + \sum_{n=1}^{l} \Delta t \| \mathbf{\Gamma}^n \|^2 + \sum_{n=1}^{l} \Delta t \| p - \mathcal{P}^n \|^2 \leq C \left( h^{2k+2} \sum_{n=1}^{l} \Delta t \| \mathbf{u}^n \|^2_{k+1} + h^{2m+2} \sum_{n=1}^{l} \Delta t \| \mathbf{\tau}^n \|^2_{m+1} + h^{2q+2} \sum_{n=1}^{l} \Delta t \| p^n \|^2_{q+1} \right) \leq C \left( h^{2k+2} \| \mathbf{u} \|_{0,k+1}^2 + h^{2m+2} \| \mathbf{\tau} \|_{0,m+1}^2 + h^{2q+2} \| p \|_{0,q+1}^2 \right),
\]

(4.88)

\[
\sum_{n=1}^{l} \Delta t \| d_t \mathbf{A}^n \|^2 = \sum_{n=1}^{l} \Delta t \left| \int_{t_{n-1}}^{t_n} \frac{1}{\Delta t} \frac{\partial \mathbf{A}}{\partial t} \, dt \right|^2 \leq \sum_{n=1}^{l} \Delta t \left( \frac{1}{\Delta t} \right)^2 \int_{\Omega} \left( \int_{t_{n-1}}^{t_n} 1 \, dt \right) \left( \int_{t_{n-1}}^{t_n} \left( \frac{\partial \mathbf{A}}{\partial t} \right)^2 \, dt \right) \, dx \leq Ch^{2k+2} \| \mathbf{u} \|_{0,k+1}^2,
\]

(4.89)

and similarly,

\[
\sum_{n=1}^{l} \Delta t \| d_t \mathbf{\Gamma}^n \|^2 \leq Ch^{2m+2} \| \mathbf{\tau} \|_{0,m+1}^2.
\]

(4.90)

Note that \( d_t \mathbf{u}^n - \mathbf{u}^n_t \) may be expressed as

\[
d_t \mathbf{u}^n - \mathbf{u}^n_t = \frac{1}{2 \Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}^n_t (\cdot, t) (t_{n-1} - t) \, dt.
\]

Also,

\[
\left( \frac{1}{2 \Delta t} \int_{t_{n-1}}^{t_n} \mathbf{u}^n_t (\cdot, t) (t_{n-1} - t) \, dt \right)^2 \leq \frac{1}{4 |\Delta t|^2} \int_{t_{n-1}}^{t_n} \mathbf{u}^n_t (\cdot, t)^2 \, dt \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2 \, dt = \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}^n_t (\cdot, t)^2 \, dt.
\]

Therefore it follows that

\[
\sum_{n=1}^{l} \Delta t \| d_t \mathbf{u}^n - \mathbf{u}^n_t \|^2 \leq \sum_{n=1}^{l} \Delta t \int_{\Omega} \frac{1}{12} \Delta t \int_{t_{n-1}}^{t_n} \mathbf{u}^n_t (\cdot, t) \, dt \, dx
\]
\[
\sum_{n=1}^{l} \Delta t \|d_{t}\tau^{n} - \tau^{n}_{t}\|^2 \leq \frac{1}{12} |\Delta t|^2 \|\tau^{n}_{t}\|_{0,0}^2. \tag{4.92}
\]

In view of (4.87)-(4.92), our induction hypotheses (IH1), (IH2), and with \(\nu\) chosen such that

\[
\nu \leq \frac{1}{2} \frac{\lambda_{m}^{2}}{\lambda_{M}} \left( \frac{7 \tilde{R}^{2} R_{e,M} \alpha}{2 \bar{R}_{e,m}(1-\alpha)} + \frac{28d^{2} C_{K}^{2} K^{2} \lambda_{M}^{2} R_{e,M}}{\alpha(1-\alpha) R_{e,M}} + \frac{13}{4} + 6 \lambda_{M}^{2} \right)^{-1} \tag{4.93}
\]

from (4.86) we obtain

\[
\alpha R_{e,m} \|E^{l}\|^{2} + \frac{\lambda_{m}}{2} \|F^{l}\|^{2} + 2 \alpha(1-\alpha) R_{e,m} \Delta t \sum_{n=1}^{l} \|D(E^{n})\|^{2} + \frac{\nu}{2} \frac{\lambda_{m}^{2}}{\lambda_{M}} \sum_{n=1}^{l} \Delta t \|F^{n}_{u}\|^{2}
\]

\[
\leq \frac{C}{2} \sum_{n=1}^{l} \Delta t \left( \|E^{n}\|^{2} + \|F^{n}\|^{2} \right) + C \sum_{n=1}^{l} \Delta t \left\| \nabla E^{n-1} \right\|_{\infty} \|\tau^{n}_{t}\|^{2} + C \Delta t^{2} \left( \|u_{t}\|_{0,1}^{2} + \|\tau\|_{0,0}^{2} + \|u_{t}\|_{0,0}^{2} + \|\tau_{t}\|_{0,0}^{2} \right)
\]

\[
+ C h^{2k-2} \|u\|_{0,k+1}^{2} + Ch^{2m+2} \|\tau\|_{0,m+1}^{2} + Ch^{2q+2} \|p\|_{0,q+1}^{2} + Ch^{2k} \|u\|_{0,k+1}^{2}
\]

\[
+ Ch^{2k+2} \|u_{t}\|_{0,k+1}^{2} + Ch^{2m} \|\tau\|_{0,m+1}^{2} + Ch^{2m+2} \|\tau_{t}\|_{0,m+1}^{2}, \tag{4.94}
\]

where the \(C\)'s denote constants independent of \(l, \Delta t, h, \nu\). Applying Gronwall's lemma and (IH2) to (4.94), the estimate given in (4.45) follows.

\[\Box\]

**Step 2.** We now show that (IH1) and (IH2) are true.

**Verification of (IH1)**

Assume that (IH1) holds true for \(n = 1, 2, \ldots, l-1\). By interpolation properties, inverse estimates and (4.45), we have that

\[
\left\|u_{h}'\right\|_{\infty} \leq \left\|u_{h}' - u'\right\|_{\infty} + \left\|u'\right\|_{\infty}
\]

\[
\leq \left\|E_{l}\right\|_{\infty} + \left\|A_{l}\right\|_{\infty} + M
\]

\[
\leq Ch^{-\frac{d}{2}} \left\|E_{l}\right\|_{0} + Ch^{-\frac{d}{2}} \left\|A_{l}\right\|_{0} + M
\]

\[
\leq C \left( |\Delta t| h^{-\frac{d}{2}} + vh^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{n-\frac{d}{2}} + h^{q+1-\frac{d}{2}} + h^{k+1-\frac{d}{2}} \right) + M. \tag{4.95}
\]
Note that the expression \( C \left( |\Delta t| h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} + h^{k+1-\frac{d}{2}} \right) \) is independent of \( l \). Hence, if we set \( k, m \geq \frac{d}{2}, q \geq \frac{d}{2} - 1 \), and choose \( h, \Delta t, \nu \) such that

\[
h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \leq \frac{1}{C}, \quad \Delta t, \nu \leq \frac{h^{\frac{d}{2}}}{C},
\]

then from (4.95)

\[
\| u^l_h \| \leq M + 6.
\]

Similarly it follows that \( \| \tau^l_h \|_\infty \leq M + 6. \)

**Verification of (IH2)**

Assume that (IH2) is true for \( n = 1, 2, \ldots, l - 1 \). Equations (4.45), (4.94), and Korn’s inequality imply

\[
\sum_{n=1}^{l} \Delta t \| \nabla E^n \|_0^2 \leq C \left( h^{2k} + h^{2m} + h^{2q+2} + |\Delta t|^2 + \nu^2 \right). \tag{4.97}
\]

Applying the inverse estimate and using the inequality

\[
\sum_{n=1}^{l} a_n \leq \sqrt{l} \left( \sum_{n=1}^{l} a_n^2 \right)^{\frac{1}{2}},
\]

from (4.97) we obtain

\[
\sum_{n=1}^{l} \Delta t \| \nabla E^n \|_\infty \leq Ch^{-\frac{d}{2}} \sum_{n=1}^{l} \Delta t \| \nabla E^n \|
\]

\[
\leq Ch^{-\frac{d}{2}} \sqrt{\Delta t} \sqrt{l} \left( \sum_{n=1}^{l} \Delta t \| \nabla E^n \|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \tilde{C} \left( \Delta t h^{-\frac{d}{2}} + \nu h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}} \right),
\]

where \( \tilde{C} = C \sqrt{T} \) is a constant independent of \( l, h, \Delta t, \) and \( \nu \). Hence when

\[
\nu, \Delta t \leq \frac{h^{\frac{d}{2}}}{5C}, \tag{4.98}
\]

and

\[
h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \leq \frac{1}{5C},
\]
\((IH2)\) holds.

**Step 3.** We derive the error estimate in (4.41), (4.42).

**Proof of the Theorem 4.**

Using estimates (4.45) and (approximation properties), we have

\[
\|u - u_h\|_{\infty,0}^2 + \|\tau - \tau_h\|_{\infty,0}^2 \leq \|E\|_{\infty,0}^2 + \|A\|_{\infty,0}^2 + \|F\|_{\infty,0}^2 + \|\Gamma\|_{\infty,0}^2
\]

\[
\leq G(\Delta t, h, \nu) + C\left(h^{2k+2} \|u\|_{\infty,k+1}^2 + h^{2m+2} \|\tau\|_{\infty,m+1}^2\right).
\]

Note the restrictions on \(\nu\) from (4.93), (4.96), (4.98), and on \(\Delta t\) from (4.96), (4.98).

To establish (4.42), from (4.45), (4.94), we have

\[
\|\nabla E\|_{0,0}^2 \leq C(T + 1)G(\Delta t, h, \nu)
\]

and

\[
\|E\|_{0,0}^2 + \|F\|_{0,0}^2 \leq TG(\Delta t, h, \nu).
\]

Hence

\[
\|\nabla E\|_{1,0}^2 + \|F\|_{0,0}^2 \leq \tilde{C}G(\Delta t, h, \nu).
\]

**4.5 Numerical Results**

In this section, we present a numerical simulation of viscoelastic fluid flow involving two immiscible fluids. For a discussion on the numerical implementation of the continuum surface force model see [41].

Let \( \Omega := (0, 1) \times (0, 1) \). At \( t = 0 \), let \( \Omega_1 := \left\{(x, y) : \frac{(x-0.5)^2}{0.35^2} + \frac{y-0.5)^2}{0.25^2} < 1\right\} \), \( \mathcal{I} := \left\{(x, y) : \frac{(x-0.5)^2}{0.35^2} + \frac{(y-0.5)^2}{0.25^2} = 1\right\} \), and \( \Omega_2 = \Omega \setminus (\Omega_1 \cup \mathcal{I}) \). Initially, both fluids are at rest, \( u(x, 0) = 0 \). For the results given, \( Re_1 = Re_2 = 1.0 \) and \( \lambda_1 = \lambda_2 = 0.1 \). It is common in polymer processing that two fluids have very similar properties so the above assumptions are reasonable. Also, the coefficient of interfacial tension is assumed to be constant, \( \sigma = 5.0 \).
From a minimum energy argument, we have that the interfacial forces will drive $\Omega_1$ from its initial elliptical profile to a circular orientation.

In the computations we use for $\nu$, the SUPG coefficient, $\nu = 0.6 \times h$, and take $\Delta t = h/2$. To approximate the velocity and pressure we use the Taylor-Hood approximation pair (continuous piecewise quadratics for velocity, continuous piecewise linears for pressure) and use a continuous piecewise linear approximation for the polymeric stress.

Presented in Figures 4.1, 4.2, 4.3, and 4.4 is the velocity field and the interface $\mathcal{I}$ at times $t = 0.00, 0.11, 0.55, \text{ and } 3.54$, for the grid with $h = 1/64$.

Figure 4.1  Initial velocity field

Figure 4.2  Velocity field after 10 time steps

Figure 4.3  Velocity field after 50 time steps

Figure 4.4  Velocity field after 320 time steps
In table 4.1, we list $\|u_h\|_{0,1}$ and $\|\tau_h\|_{0,0}$ at time $T = 3.536$, together with their experimental convergence rates. The experimental convergence rate for $\|u_h\|_{0,1}$ was computed as follows. From Theorem 3, the choice of approximating elements used, $\nu = 0.6 \, h$, and $\Delta t = h/2$, we have

$$\|u_h\|_{0,1} - \|u\|_{0,1} \leq \|u_h - u\|_{0,1} \leq C_v \, h.$$  \hspace{1cm} (4.102)

Using $\|u_{1/64}\|_{0,1} = C_v \frac{1}{64}$ and $\|u_{1/48}\|_{0,1} - \|u\|_{0,1} = C_v \frac{1}{48}$, we obtain an estimate for $\|u\|_{0,1} \sim \|u_\infty\| = 0.455327$ and an estimate for $C_v = 1.072320$.

Using $\|u_\infty\|_{0,1}$ we then compute the experimental convergence rates for $\|u_h\|_{0,1}$ given in Table 4.1. The experimental convergence rates for $\|\tau_h\|_{0,0}$ are computed analogously. From Theorem 4 and (4.102), we have that the theoretical asymptotic convergence rates for $\|u_h\|_{0,1}$ and $\|\tau_h\|_{0,0}$ is 1.

<table>
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<tr>
<th>$h$</th>
<th>$|u_h|_{0,1}$</th>
<th>Exp. conv. rate</th>
<th>$|\tau_h|_{0,0}$</th>
<th>Exp. conv. rate</th>
</tr>
</thead>
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<tr>
<td>1/32</td>
<td>.422046</td>
<td>1.00</td>
<td>.463853</td>
<td>1.00</td>
</tr>
<tr>
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<td>.428505</td>
<td>1.00</td>
<td>.458594</td>
<td>1.00</td>
</tr>
<tr>
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<td>1.00</td>
<td>.456062</td>
<td>1.00</td>
</tr>
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<td>.454650</td>
<td>1.00</td>
</tr>
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<td>1.01</td>
<td>.454567</td>
<td>0.99</td>
</tr>
<tr>
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<td>1.01</td>
<td>.453620</td>
<td>0.98</td>
</tr>
<tr>
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<td>1.00</td>
<td>.449975</td>
<td>1.00</td>
</tr>
<tr>
<td>$\infty$</td>
<td>.455327</td>
<td>.435950</td>
<td>$\infty$</td>
<td>.435950</td>
</tr>
</tbody>
</table>

Table 4.1  Experimental Rates of Convergence
CHAPTER 5
NUMERICAL RESULTS

In this chapter, we describe the specific numerical implementation which was used and give some results of various fluid flow simulations.

5.1 The Numerical Implementation

The following procedure is used to solve for velocity, pressure, and stress.

**Step 1: Load the Mesh**

The computational mesh is loaded via a mesh file. This contains all information with regards to the triangulation: node coordinates, mid-edge coordinates if needed, triangle nodes, and boundary node indicators. This mesh, which will be referred to as the VPS mesh, is used to solve for velocity, pressure, and stress.

**Step 2: Construct the Submesh**

A subgrid is constructed in which each triangle of the VPS mesh is divided into four sub-triangles by connecting the midpoints of the triangle edges. We call this submesh the *color mesh* because all color-related variables are computed using this submesh.

**Step 3: Build Neighbor Matrix**

A map is built to indicate which triangles in the color mesh share vertices. This allows the user, if desired, to solve for color on only a neighborhood about the interface, as opposed to solving for the color throughout the entire domain.

**Step 4: Initialize**

All variables are initialized and boundary values applied.

**Step 5: Constitutive Solve**

Solve for stress on the VPS mesh at time \( n \), using the known velocity at time \( n - 1 \). That is, solve

\[
\left(1 + \frac{\lambda}{\Delta t}\right)\tau^n, \tilde{\psi} + \left(\lambda\mathbf{u}^{n-1} : \nabla\tau^n, \tilde{\psi}\right) = \left(\frac{\lambda}{\Delta t}\tau^{n-1}, \tilde{\psi}\right) + \left(\lambda(\nabla\mathbf{u}^{n-1} \cdot \tau^{n-1} + \tau^{n-1} \cdot (\nabla\mathbf{u}^{n-1})^T), \tilde{\psi}\right) + \left(\beta(\nabla\mathbf{u}^{n-1} + (\nabla\mathbf{u}^{n-1})^T), \tilde{\psi}\right)
\]

where \( \tilde{\psi} = \psi + \nu \frac{u_{n-1}^n}{l_{n-1}} \cdot \nabla \psi, \quad \psi \in S_h \), and \( S_h \) is the trial space for the stress.

**Step 6: VP Solve**

Solve for velocity and pressure on the VPS mesh at time \( n \), using the stress which was found in the previous step. That is, solve

\[
\left( \frac{Re}{\Delta t} u^n, v \right) - (p^n \mathbb{I}, \nabla v) + (\nabla u^n, \nabla v) = \left( \frac{Re}{\Delta t} u^{n-1}, v \right) - (\tau^n, \nabla v) + (STF, v), \quad v \in X_h
\]
\[
(\nabla \cdot u^n, q) = 0, \quad q \in Q_h
\]

where \( STF \) represents the interfacial tension forces, calculated as prescribed in Section 2.3. The spaces \( X_h \) and \( Q_h \) denote the trial spaces for velocity and pressure, respectively. We have used a “creeping flow” assumption so that inertial terms are not included.

**Step 7: Temperature Solve**

Solve for the temperature at time \( n \), using the velocity and stress at time \( n \).

\[
\left( \frac{1}{\Delta t} T^n, r \right) + (u^n \cdot \nabla T^n, r) + \left( \frac{1}{Re} Pr \nabla T^n, \nabla r \right) = \left( \frac{Br}{Re} Pr \tau^n : \nabla u^n, r \right), \quad r \in R_h,
\]

where \( R_h \) is the trial space for the temperature.

**Step 8: Advect the Color**

Solve for the new color function as described in section 2.2.

**Step 9: Defect Correction**

If more than .5% of the mass is lost to diffusion in Step 8, employ defect correction. (For more on defect correction, see Appendix A.)

**Step 10: Correct the Color**

Re-establish the distance property of the color function as described in section 2.2.

**Step 11: Defect Correction**

If more than .5% of the mass is lost to diffusion, employ defect correction.

**Step 12: Correct the Level Set**

If the defect correction did not preserve mass, then use the bisection method to find a level set which contains the correct mass. Reassign this to be the zero level set. This is a last resort within the method to assure that mass is preserved.

**Step 13: Diﬀuse the Color**
Solve for a “smoothed” version of the color function to be used in calculating curvature.

**Step 14: Return to Step 5**

The user has the option to use different element types for velocity, stress, pressure, and color. The color function should be at least quadratic. If SUPG is to be used, the stress elements should be continuous. Velocity and pressure must satisfy the discrete $inf$-$sup$ condition. The user may also choose to use a variety of linear solvers. We currently have available GMRES and stabilized BiCG as well as diagonal and incomplete LU preconditioners.

5.2 Numerical Results

Here we present some of the results of various simulations which were run.

**Evolving Ellipse**

The first problem which was solved is the following. An elliptically shaped “blob” of fluid 1 is placed in a still melt of fluid 2. Since the fluids will try to achieve a position of minimum energy, fluid 1 should align itself into a circular shape. The only forces in the system are those due to interfacial tension. Figures 5.1 - 5.6 present the velocities, pressure and stress approximations at $t = .221$, $t = 1.105$, and $t = 3.315$. 
Figure 5.1  Pressure and Velocity at $t = 0.221$.

Figure 5.2  Pressure and Velocity at $t = 1.105$.

Figure 5.3  Pressure and Velocity at $t = 3.315$. 
Figure 5.4 Stress at $t = 0.221$.

Figure 5.5 Stress at $t = 1.105$.

Figure 5.6 Stress at $t = 3.315$. 
Chaotic Mixing Cavity

The next problem places a circular blob of fluid 1 in a melt of fluid 2. The top wall is set in horizontal motion to the right with a constant velocity. The top wall moves for a prescribed time. Once the top wall stops, the lower wall moves at the same speed in the opposite direction for the same amount of time. This mixing process is highly deforming and can cause fibrous formation of fluid 1 within fluid 2. For (4.36), (4.37), this problem is not well defined because there exist singularities in stress where the moving wall meets the stationary walls. This infinite stress violates the boundedness hypothesis of Theorem 4. Because of these singularities, the approximation diverges under grid refinement. Below we present the normal stress in the $x$-direction for two separate triangulations. Note the dramatic increase in the value of stress at the corners. These singularities cause divergence of the approximate solution.

![Figure 5.7](image1.png) Figure 5.7 $\tau_{11}$ after 20 time steps on a mesh with $h = 0.07071$.  

![Figure 5.8](image2.png) Figure 5.8 $\tau_{11}$ after 20 time steps on a mesh with $h = 0.04714$.

Circular Mixing Cavity

To avoid the singularities in stress mentioned in the previous example, we changed to a circular geometry. This is more physical in nature and there are no discontinuities in the velocity boundary conditions. We plot below a sample of velocity, stress, pressure for the case where the inner wall (or rod) is spinning uniformly in the counter-clockwise direction with a magnitude of one.
Figure 5.9  Velocity and Pressure after 100 time steps

Figure 5.10  Stress after 100 time steps

Dual Movement Circular Cavity

Example 1
In this example, the inner rod of a circular cavity moves in a counter clockwise direction while the outer wall of the cavity moves in a clockwise direction simultaneously. Both inner and outer velocities have magnitude of one. This generates a deforming velocity field. Figures 5.11 and 5.12 give an indication of the role that interfacial tension can play in the mixing process. The initial blob is circular, with radius 0.2 and center \((0, -0.67)\). Figures 5.11 and 5.12 present the interfacial shape at time \(t = 1.5\) for varying values of \(\sigma\). In this example, \(\sigma\) is considered constant.

![Figure 5.11 Interface after 150 time steps: \(\sigma = 0.3\)](image1)

![Figure 5.12 Interface after 150 time steps: \(\sigma = 1.0\)](image2)

**Example 2**

In this simulation, we begin with two regions of the minor phase fluid into the dual movement circular cavity. Both the inner rod and the outer wall are spun in the clockwise direction with magnitude 1. The inner rod is assumed to be held at a temperature of 1.0, and the outer cylinder at temperature 0.0, thereby inducing a temperature gradient through the domain. The coefficient of interfacial tension, \(\sigma\), is assumed to vary linearly with temperature. Of interest is the time it takes for the two regions to collide and merge. In Figures 5.13 - 5.17, we present the results when the surface gradient of \(\sigma\), \(\nabla_s \sigma\), is included with the results obtained when the gradient term is neglected.
Figure 5.13  Interface at $t = 0$ with varying $\sigma$

Figure 5.14  Interface at $t = 0.51257$: no surface gradient

Figure 5.15  Interface at $t = 0.53307$: no surface gradient

Figure 5.16  Interface at $t = 0.48181$: with surface gradient

Figure 5.17  Interface at $t = 0.50232$: with surface gradient
CHAPTER 6
ONGOING/FUTURE WORK

The following are topics of ongoing and future investigation.

- Investigation of the effect of a varying coefficient of surface tension in a more physical setting.
- Extend the theoretical results to permit point singularities in the stress.
- Extend the theoretical results to include the energy equation.
- Investigate parallelization of the C++ simulation software.
- Investigate more efficient Stokes’ solvers and preconditioners.
- Improve the color advection scheme.
APPENDICES
Appendix A
Other Interface Tracking Schemes

Flux-Corrected Transport

Flux-Corrected Transport is a volume of fluid (VOF) method which attempts to maintain sharp interfaces. Hirt and Nichols developed some of the early ideas of the method [20], and it has since been improved upon by several others [44, 33]. As with other VOF methods, the goal is to use a color function defined over the entire domain. The color is advected via

\[ \frac{\partial C}{\partial t} + \nabla \cdot (uC) = 0. \]  \hspace{1cm} (A.1)

For explanatory purposes, we begin in one dimension. Consider the interval \([0, 1]\) on the real axis, partitioned into several subintervals. Denote by \(C^n_i\) the percentage of the minor phase fluid in subinterval \(i\) at time \(n \Delta t\). The initial color profile consists of zeros and ones throughout the computational domain (see Figure A.1 below). A finite difference scheme is the usual choice of discretization for this method because of the adjustments which need to be made across each cell boundary. Such adjustments will be discussed below.

If a low-order difference scheme is used to model (A.1), then we gain stability at the cost of numerical diffusion (see Figure A.1).

Figure A.1
However, if higher-order difference schemes are used, stability is lost and non-physical oscillations may occur (see Figure A.2).

Thus, we need a “compromise” which will allow the best of each of the schemes. A scheme which does this is the Flux-Correct Transport algorithm given in [44]. The method of Flux-Corrected Transport proceeds as follows:

1-D Implementation

Step 1:
Compute the low-order flux across each cell boundary. Denote this flux by $F^L_{i+\frac{1}{2}}$. Rudman[34] does this via

$$F^L_{i+\frac{1}{2}} = \begin{cases} u_{i+\frac{1}{2}} C_i \Delta t & ; u_{i+\frac{1}{2}} \geq 0, \\ u_{i+\frac{1}{2}} C_{i+1} \Delta t & ; u_{i+\frac{1}{2}} < 0. \end{cases}$$

Step 2:
Compute a high-order flux across each cell boundary. Denote this flux by $F^H_{i+\frac{1}{2}}$.

Step 3:
Advect the color using the low-order fluxes.

$$C^{tmp}_i = C^n_i - \left( \frac{F^L_{i+\frac{1}{2}} - F^L_{i-\frac{1}{2}}}{\Delta x} \right)$$
Step 4:

Define the antidiffusive flux

\[ A_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}^H - F_{i+\frac{1}{2}}^L. \]

Limit \( A_{i+\frac{1}{2}} \) so that applying it to the approximation will not introduce any new extrema. That is, determine constants \( f_{i+\frac{1}{2}} \) such that

\[ A_{i+\frac{1}{2}} = f_{i+\frac{1}{2}} A_{i+\frac{1}{2}} ; 0 \leq f_{i+\frac{1}{2}} \leq 1. \]

Step 5:

Compute the new corrected approximation

\[ C_{i+1}^{m+1} = C_{i+1}^{tmp} - \left( \frac{A_{i+\frac{1}{2}}^{fc} - A_{i-\frac{1}{2}}^{fc}}{\Delta x} \right). \]

The corrective constants used in Step 4 are computed as follows. Introduce

\[
\begin{align*}
P^+_i & := \text{sum of the anti-diffusive fluxes entering cell } i \\
& = \max(0, A_{i-\frac{1}{2}}) - \min(0, A_{i+\frac{1}{2}}) \\
Q^+_i & := (C_{i+1}^{max} - C_{i+1}^{tmp}) \Delta x \\
R^+_i & := \begin{cases} 
\min \left( 1, \frac{Q^+_i}{P^+_i} \right) ; & P^+_i > 0 \\
0 ; & P^+_i = 0 
\end{cases}
\end{align*}
\]

Note that \( R^+_i \) is the largest constant by which we can multiply the incoming fluxes and still insure that no new maximum is created. Likewise

\[
\begin{align*}
P^-_i & := \text{sum of the anti-diffusive fluxes exiting cell } i \\
& = \max(0, A_{i+\frac{1}{2}}) - \min(0, A_{i-\frac{1}{2}}) \\
Q^-_i & := (C_{i+1}^{tmp} - C_{i+1}^{min}) \Delta x \\
R^-_i & := \begin{cases} 
\min \left( 1, \frac{Q^-_i}{P^-_i} \right) ; & P^-_i > 0 \\
0 ; & P^-_i = 0 
\end{cases}
\end{align*}
\]
Here, $R_i^-$ is the largest constant by which we can multiply the incoming fluxes and still ensure that no new minimum is created. Finally, we define $f_{i+\frac{1}{2}}$ by

$$
f_{i+\frac{1}{2}} = \begin{cases} 
\min(R_{i+1}^+, R_i^-); & A_{i+\frac{1}{2}} \geq 0 \\
\min(R_i^+, R_{i+1}^-); & A_{i+\frac{1}{2}} < 0 
\end{cases},
$$

and the corrected flux is given by $A_{i+\frac{1}{2}}^{fc} = f_{i+\frac{1}{2}} A_{i+\frac{1}{2}}$. Rudman defines the bounding constants $C_{i+\frac{1}{2}}^{max}, C_{i+\frac{1}{2}}^{min}$ by using the color values at the current time level and the low-order color values at the next time level. Specifically,

$$
C_{i}^a = \max(C_i^a, C_i^{tmp}), \\
C_{i}^b = \min(C_i^b, C_i^{tmp}), \\
C_{i}^{max} = \max(C_{i-1}^a, C_i^a, C_{i+1}^a), \\
C_{i}^{min} = \min(C_{i-1}^b, C_i^b, C_{i+1}^b).
$$

**Numerical Results**

Some results for one dimensional problems are presented below.

**Test Problem 1**

The left picture in Figure A.3 shows the unit pulse advected without flux correction (i.e. using a low-order method). The right picture shows the same pulse advected using the corrective fluxes.

![Figure A.3 Velocity is nondeforming and positive](image)
Test Problem 2

The second test problem is purely mathematical in nature. The left picture in Figure A.4 shows the result of low-order advection while the right picture shows the result of advection using the flux correction.

![Diagram of test problem results]

Figure A.4  Velocity is nondeforming and positive

2-D Implementation

The FCT concept generalizes to higher dimensions. In two dimensions, color may enter and exit a cell through any of four cell boundaries (assuming the grid is rectangular). This amounts to computing fluxes in two directions (along the x and along y directions). Then the anti-diffusive fluxes must be defined along all four sides of the computational cell. Correspondingly, there are more calculations necessary to determine the flux correction constants. The analogous computations are presented below [34].

\[
C_{i,j}^{\text{temp}} = C_{i,j}^n - \frac{F_{i+\frac{1}{2},j}^L - F_{i-\frac{1}{2},j}^L + G_{i,j+\frac{1}{2}}^L - G_{i,j-\frac{1}{2}}^L}{\Delta x \Delta y}
\]

where \(G_{i,j+\frac{1}{2}}^L\) represents the flux in the y direction at \((i, j + \frac{1}{2})\). Likewise, the anti-diffusive fluxes are given by

\[
A_{i+\frac{1}{2},j} = F_{i+\frac{1}{2},j}^H - F_{i+\frac{1}{2},j}^L
\]
A_{i,j+\frac{1}{2}} = G_{i,j+\frac{1}{2}}^H - G_{i,j+\frac{1}{2}}^L
A_{i+\frac{1}{2},j}^{fc} = f_{i+\frac{1}{2},j} A_{i+\frac{1}{2},j}
A_{i,j+\frac{1}{2}}^{fc} = f_{i,j+\frac{1}{2}} A_{i,j+\frac{1}{2}}

The limited anti-diffusive fluxes are determined by

\begin{align*}
P_{i,j}^+ &= \max(0, A_{i-\frac{1}{2},j}) - \min(0, A_{i+\frac{1}{2},j}) + \max(0, A_{i,j-\frac{1}{2}}) - \max(0, A_{i,j+\frac{1}{2}}) \\
Q_{i,j}^+ &= (C_{i,j}^{max} - C_{i,j}^{tmp}) \Delta x_i \Delta y_j \\
R_{i,j}^+ &= \begin{cases} 
\min \left( 1, \frac{Q_{i,j}^+}{P_{i,j}} \right) ; & P_{i,j} > 0 \\
0 ; & P_{i,j} = 0 
\end{cases} \\
P_{i,j}^- &= \max(0, A_{i+\frac{1}{2},j}) - \min(0, A_{i-\frac{1}{2},j}) + \max(0, A_{i,j+\frac{1}{2}}) - \max(0, A_{i,j-\frac{1}{2}}) \\
Q_{i,j}^- &= (C_{i,j}^{tmp} - C_{i,j}^{min}) \Delta x_i \Delta y_j \\
R_{i,j}^- &= \begin{cases} 
\min \left( 1, \frac{Q_{i,j}^-}{P_{i,j}} \right) ; & P_{i,j} > 0 \\
0 ; & P_{i,j} = 0 
\end{cases} \\
f_{i+\frac{1}{2},j} &= \begin{cases} 
\min \left( R_{i+1,j}^+, R_{i,j}^- \right) ; & A_{i+\frac{1}{2},j} \geq 0 \\
\min \left( R_{i,j}^+, R_{i+1,j}^- \right) ; & A_{i+\frac{1}{2},j} < 0 
\end{cases}
\end{align*}
\( f_{i,j+\frac{1}{2}} = \begin{cases} 
\min \left( R_{i,j+1}^+, R_{i,j}^- \right); & A_{i,j+\frac{1}{2}} \geq 0 \\
\min \left( R_{i,j}^+, R_{i,j+1}^- \right); & A_{i,j+\frac{1}{2}} < 0 
\end{cases} \)

\[ C_{i,j}^a = \max \left( C_{i,j}^m, C_{i,j}^{tmp} \right) \]

\[ C_{i,j}^{max} = \max \left( C_{i-1,j}^a, C_{i,j}^a, C_{i,j-1}^a, C_{i,j+1}^a \right) \]

\[ C_{i,j}^b = \min \left( C_{i,j}^m, C_{i,j}^{tmp} \right) \]

\[ C_{i,j}^{min} = \min \left( C_{i-1,j}^b, C_{i,j}^b, C_{i,j-1}^b, C_{i,j+1}^b \right) \]

\[ A_{i+\frac{1}{2},j}^{fc} = f_{i+\frac{1}{2},j} A_{i+\frac{1}{2},j} \]

\[ A_{i,j+\frac{1}{2}}^{fc} = f_{i,j+\frac{1}{2}} A_{i,j+\frac{1}{2}} \]

The “corrected” approximation is then given by

\[ C_{i,j}^{n+1} = C_{i,j}^{tmp} - \frac{A_{i+\frac{1}{2},j}^{fc} - A_{i-\frac{1}{2},j}^{fc} + A_{i,j+\frac{1}{2}}^{fc} - A_{i,j-\frac{1}{2}}^{fc}}{\Delta x \Delta y}. \]

2-D Numerical Results

A numerical simulation using a notched cylinder as the original color profile was performed. The notch enables the rotation of the cylinder to be clearly observed. A nondeforming rotational velocity is imposed.
Conclusions on FCT

On structured meshes, the FCT method works very well. Mass is conserved and computational run time is short. However, the ad hoc “fixes” implemented to avoid spurious oscillations are somewhat problematic and mesh specific. In addition, because of the ad hoc nature of the correction, a rigorous mathematical analysis of the method is not possible.
Finite Element Method with Defect Correction

In this section we describe a method which uses the Finite Element Method (FEM). The FEM is attractive because it allows for more complex geometries regarding the domain and the possibility of adaptive refinement. The method employed here is still technically a front-capturing scheme since the interface is not solved for explicitly, but rather imbedded in a characteristic variable of the fluid throughout the domain. Again, we call the characteristic variable color. As with FCT, the general idea is to stably advect the color with a low-order (smooth) scheme and then correct the approximation to obtain a higher order approximation. For the finite element formulation, the low order approximation is obtained by adding some artificial diffusion to the numerical implementation which smooths the approximation. A *defect correction* method is then used to improve the low order approximation. For the implementation, the following assumptions are used.

(i) Regular triangular elements are used.

(ii) The approximation is constructed using basis elements which are continuous, piecewise linear in space and discontinuous, piecewise constant in time.

Recall the problem

$$\frac{\partial C}{\partial t} + \nabla \cdot (uC) = 0.$$  \hspace{1cm} (A.2)

Artificial diffusion is applied by adding $-\epsilon \Delta C$ to left side of (A.2). Then the problem becomes

$$\frac{\partial C}{\partial t} - \epsilon \Delta C + \nabla \cdot (uC) = 0.$$  \hspace{1cm} (A.3)

Assume that $C(x, t)$ is of the form

$$C = \sum_{j,k} a_{jk} w_k(t) \varphi_j(x).$$

Define basis functions for time, $w_k(t)$, and space, $\varphi(x)$. For spatial test/trial functions, piecewise, continuous linear elements are used; and the temporal test/trial functions are piecewise, discontinuous constants. The choice of discontinuous elements in time actually adds another degree of freedom to the problem. Using these test/trial functions, the weak
form of (A.3) is given by

\[
\int_{I_n} M(t) \frac{\partial C}{\partial t} w_l(t) \, dt + \int_{I_n} A(t) C(t) w_l(t) \, dt \\
+ M(t) \left( C(t_{n-1}^+) - C(t_{n-1}^-) \right) w_l(t) t_{n-1} = \int_{I_n} F(t) w_l(t) \, dt ,
\]

(A.4)

where

\[
M_{i,j}(t) = \int_{\Omega} \phi_j \phi_i \, dV \\
A_{i,j}(t) = \int_{\Omega} \nabla \phi_j \cdot \nabla \phi_i + (\mathbf{u} \cdot \nabla \phi_j) \phi_i + (\nabla \cdot \mathbf{u}) \phi_j \phi_i \, dV \\
F_i(t) = \int_{\Omega} f(\mathbf{x}, t) \phi_i \, dV.
\]

For our purposes, \( f(\mathbf{x}, t) = 0 \). If \( \epsilon = 0 \), the problem has been shown to be unstable (see Figure A.9. The addition of a small amount of diffusion adds stability and produces a

![Figure A.9](image)

**Figure A.9** Left: Initial Profile Right: Profile after 5 time steps \((\Delta t = 0.005)\)

smooth but smeared interface (see Figure A.10.

The artificial diffusion solution behaves much like the low order solution of the FCT. To improve upon this diffusive solution, define

\[
L_\epsilon(C) = \int_{I_n} M(t) \frac{\partial C}{\partial t} w_l(t) \, dt + \int_{I_n} A(t) C(t) w_l(t) \, dt \\
+ M(t) \left( C(t_{n-1}^+) - C(t_{n-1}^-) \right) w_l(t) t_{n-1}.
\]
In general, the original problem is given by $L_0(C) = b$. Solving this directly gives undesirable results and thus the artificial diffusion is added the equation to be solved becomes $L_\varepsilon(C^*) = B$. To improve the solution, calculate the residual as

$$r = b - L_0(C^*) = L_0(C) - L_0(C^*) = L_0(C - C^*).$$

Note that, theoretically, the problem to be solved is

$$L_0(\xi) = r.$$ \hfill (A.5)

The true solution would be found via

$$C_{\text{new}} = C^* + \xi.$$

However, (A.5) is an unstable problem. Instead, solve

$$L_\varepsilon(\xi) = r$$

and obtain an improved as

$$C_{\text{new}} = C^* + \xi.$$
This *residual defect-correction* may be done several times during each time step. The optimal number of correction steps has not been investigated in this work. In our numerical examples, the number of correction steps does not exceed two.

**Numerical Results for FEM-DC**

To begin with, a simple example is done to show the effectiveness of the defect correction. This result is to be compared to Figure 9.

![Artificial Diffusion With No Correction](image1.png) ![Artificial Diffusion With Correction](image2.png)

Figure A.11

Consider the same notched cylinder which was used in the FCT examples. The results are given below.
Clearly these numerical results are not nearly as good as those produced via FCT. However, there are improvements which could be made. Upwinding schemes could be employed and higher order elements could be used. The method is attractive because of its flexibility on nonuniform grids. In addition, a rigorous mathematical analysis is possible for this
method. The main drawback of this method is the large amount of time required to compute approximations (easily an order of magnitude larger than that needed for FCT calculations). The use of finite elements in both time and space yields several solves of large (although sparse) linear systems. An alternative approach is to use finite differences in time and finite elements in space. One such approach is the level-set method discussed in Chapter 2.


