A Numerical Scheme for NS-α with greater physical accuracy and increased convergence rates

Leo G. Rebholz* William W. Miles†

Abstract

We study a new numerical scheme for the NS-alpha turbulence model that conserves both energy and helicity. Although most turbulence models (in the continuous case) conserve only energy, NS-alpha is one of only a very few that also conserve helicity. This is one reason why it is becoming accepted as the most physically accurate turbulence model. However, no numerical scheme for NS-alpha, until now, conserved both energy and helicity, and thus the advantage gained in physical accuracy by using NS-alpha could be lost in a discretization. This report presents a finite element numerical scheme, and gives a rigorous analysis of its conservation properties, stability, solution existence, and convergence. A key feature of the analysis is the identification of the discrete energy and energy dissipation norms, and proofs that these norms are equivalent (provided a careful choice of filtering radius) in the discrete space to the usual energy and energy dissipation norms. A generalization of this scheme to a family of high-order NS-alpha-deconvolution models is also discussed. These higher order models combine the attractive physical properties of NS-alpha with the high accuracy gained by using van Cittert approximate deconvolution.

keywords: NS-alpha, Helicity, Conservation, Turbulence

AMS subject classifications: 65M12, 65M60, 76D05

1 Introduction

We study a numerical scheme for the NS-alpha (NS-α) turbulence model that treats energy and helicity ($H = \int_\Omega u\cdot(\nabla\times u)$) in a manner physically consistent with that of true fluid flow. It is a fundamental property of 3d fluid flow (and the NS-α model) that both energy and helicity are created and dissipated through viscous forces, external forces and through boundary conditions; nonlinear effects, in contrast, preserve both energy and helicity and are responsible only for cascading them from the (input) large scales to the (viscosity dominated) small scales [21, 11, 8]. Many numerical schemes enforce a discrete analog of this behavior for energy, which, not coincidentally, often leads to algorithm stability. However, most numerical fluid flow schemes ignore the treatment of (the important rotational quantity) helicity, which can lead to solutions without physical meaning. The scheme developed herein provides balances for energy and helicity that mirror the nondiscretized NS-α: the discrete nonlinearity preserves both energy and helicity, and only viscous and external forces and boundary effects can create and dissipate them.

Development of numerical fluid flow schemes that better match physical processes has been a subject of interest and research (at least) since Arakawa developed a scheme for the 2d Navier Stokes equations (NSE) that accurately treated both energy and enstrophy ($Ens = \frac{1}{2}\int_\Omega |\nabla\times u|^2$ is preserved by the NSE nonlinearity in 2d, but not in 3d). By enforcing energy and enstrophy balances analogous to the true physics, Arakawa's finite difference scheme provided much better results than its contemporaries, especially for simulations done over

*Department of Mathematical Sciences, Clemson University, Clemson SC 29634 (rebholz@clemson.edu)
†Department of Math/CS, Stetson University, DeLand, Florida 32720 (wmiles@stetson.edu)

long-time intervals [3]. The reason for the scheme's success (versus other schemes that ignore enstrophy) is that since its nonlinearity preserved enstrophy, it did not generate small amounts of artificial rotation at each timestep. Thus for long-time simulations, Arakawa's scheme would not be subject to the accumulation of these errors.

With the established superiority of more physically accurate schemes' computational performance, further research yielded an energy and potential enstrophy preserving scheme for the shallow water equations [4], and an energy and enstrophy preserving scheme for atmospheric flows [14], among others. These schemes also showed superiority in long-time simulations versus energy-only preserving schemes.

For the 3d NSE, Liu and Wang developed an energy and helicity preserving scheme (EHPS) for axisymmetric flow [20]. In 3d, enstrophy is not preserved by the nonlinearity; instead, helicity is the preserved rotational quantity of the NSE. By better matching the true physics, the EHPS produced excellent results and eliminated the need for the excessive numerical viscosity often required in such calculations. For the full 3d NSE, a finite element scheme based on rotational form coupled with a projected vorticity in the nonlinearity was developed in [22] that balances discrete energy and helicity analogous to true fluid flow. Preliminary computations showed this scheme treated helicity much more accurately than typical finite element schemes for underresolved computations.

NS-α is one of only a few turbulence models for which an energy and helicity conserving scheme is even possible. In most turbulence models (e.g. Bardina, Leray-α, Smagorinsky, $k-\epsilon$), helicity is not conserved in the model's continuous form, and so developing a discretization for the model that conserves helicity is likely not feasible. The NS-α model, on the other hand, does conserve a model energy and helicity. Moreover, NS-α has other properties that most turbulence models do not, including frame invariance [15], and an energy cascade that matches that of the NSE up to a filtering radius dependent length scale [17]. Indeed, due to these properties and more, NS-α is believed to be one of the most *physically* accurate turbulence models.

The NS-α model, also known as the 3d Camassa-Holm equations [17, 9], is given in nondimensional, rotational form by

$$v_t - \overline{v} \times (\nabla \times v) + \nabla q - \nu \Delta v = f, \tag{1.1}$$

$$\nabla \cdot \overline{v} = 0, \tag{1.2}$$

$$-\alpha^2 \Delta \overline{v} + \overline{v} + \nabla \lambda = v. \tag{1.3}$$

where v is velocity, q is dynamic pressure, ν is kinematic viscosity, f is external force, $\overline{v}$ is the filtered velocity, and α is the filtering radius of the Hemholtz filter (1.3) (hence the name alpha filter and NS-α model). Like true fluid flow, the (continuous) NS-α model conserves both energy and helicity for periodic, inviscid flow [13]. This implies that, again like for true fluid flow, energy and helicity in NS-α are only input by external forces and only dissipated by viscous forces. Hence, the model's nonlinearity neither creates nor dissipates either of these quantities by nonphysical means, but instead cascades them from large to small scales.

However, the excellent physical properties of NS-α can be lost in numerical simulations . Even though NS-α accurately treats helicity in the continuous case, discretizations of NS-α do not necessarily do the same. Typical discretizations of NS-α, such as continuous piecewise polynomial finite element spatial discretizations (see [10, 7]) will conserve energy but not helicity. Hence solutions to such schemes may not contain the physical relevance expected of their continuous counterparts, as helicity and thus the flow's rotational structures are subjected to helicity generation and dissipation through the nonlinearity's discretization (i.e. not through physical means). This deficiency in computing solutions to NS-α motivated our work; the scheme we present herein provides a more physically accurate method to compute solutions for NS-α.

After providing the necessary preliminaries and notation in Section 2, we present the energy and helicity conserving scheme for NS-α, prove its conservation and stability properties, and show that a solution exists in Section 3. Section 4 gives a rigorous convergence analysis for the scheme. A key feature of the analysis is proving that the naturally arising discrete energy and energy dissipation norms are equivalent to the usual ones of fluid flow(i.e. from NSE analysis). Without these norm equivalences, the convergence analysis would have been much more difficult and certainly not as clean as that presented herein.

In Section 5, an extension of this scheme to a very recently introduced family of higher-order accurate NS-α type models, the NS$-\alpha$-deconvolution models [23], is discussed. This is a very interesting family of models because they combine the excellent physical properties of NS-α (e.g. energy and helicity conservation) with the higher order accuracy offered by applying van Cittert approximate deconvolution to filtered terms. We find that for higher order elements $(P_k, P_{k-1}),\ k \geq 3$, the suboptimal convergence of NS-α can be improved to optimal by the use of approximate deconvolution. Conclusions are given in Section 6.

2 Notation and Preliminaries

The domain used throughout this report will be denoted by Ω, and will represent the periodic box $\Omega = (0, L)^3$. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. All other norms will be clearly labeled with subscripts.

Define the velocity and pressure spaces, as usual in this setting, to be

$$X := H^1_{\#}(\Omega) = \left(v \in H^1(\Omega),\ \int_\Omega v = 0,\ v(x + Le_i) = v(x)\right), \tag{2.4}$$

$$Q := L^2_{\#}(\Omega) = \left(q \in L^2(\Omega),\ \int_\Omega q = 0,\ q(x + Le_i) = q(x)\right). \tag{2.5}$$

Note that all of the analysis performed herein can be extended to the case of no slip boundaries on a convex polyhedral domain. In this setting, energy and helicity are conserved by the continuous model when a no-penetration condition is enforced on the vorticity; as this scheme will conserve both energy and helicity, the scheme will implicitly use this condition.

In X, the Poincare-Freidrich's inequality holds: For $v \in X$, $\|v\| \leq C(\Omega)\|\nabla v\|$ [6]. Thus the norms $\|\nabla v\|$ and $\|v\|_{H^1}$ are equivalent on X. We will use the notation $\|\cdot\|_*$ for the norm of the dual space of X, that is,

$$\|f\|_* := \sup_{v \in X} \frac{(f, v)}{\|\nabla v\|}. \tag{2.6}$$

Given a regular elemental discretization $\tau_h(\Omega)$ with maximum element diameter h (and minimum diameter $h_{min} > 0$, define the velocity and pressure finite element spaces $X_h \subset X$ and $Q_h \subset Q$ by

$$X_h := (v \in C_0(\Omega) \cap P_k(e)\ \forall e \in \tau_h), \tag{2.7}$$

$$Q_h := (q \in C_0(\Omega) \cap P_{k-1}(e)\ \forall e \in \tau_h), \tag{2.8}$$

for $k \geq 2$, where P_k denotes degree k polynomials. If $k = 2$, this velocity-pressure element pair is well known as the Taylor-Hood element [6]. Provided $k \geq 2$, this element pair satisfies the discrete inf-sup condition:

$$0 < \beta \leq \inf_{q \in Q_h} \sup_{v \in V_h} \frac{(q, \nabla \cdot v)}{\|q\| \|\nabla v\|} \tag{2.9}$$

Define the discretely divergence free space V_h by

$$V_h := (v \in X_h,\ (\nabla \cdot v, q) = 0\ \forall q \in Q_h). \tag{2.10}$$

It will be notationally convenient to work in V_h whenever possible.

The following is a well known lemma for inverse estimates in X_h, and will be important in analyzing the norms that naturally arise in the analysis of the scheme.

Lemma 2.1 (Inverse estimate). *[6] There exists a constant C dependent on Ω, but independent of h, satisfying*

$$\|\nabla v\| \leq Ch^{-1}\|v\|\ \ \forall v \in X_h. \tag{2.11}$$

The filter used for NS-α is the so-called alpha filter, and there are several ways to discretize it in a finite element setting. We choose to use a discretization that preserves discrete incompressibility and employs the discrete Laplacian. This was found to be the best way for the scheme to retain stability and the desired conservation properties. We will first define the discrete Laplacian, and then the discrete filter.

Definition 2.2 (Discrete Laplacian). *Define the discrete Laplacian operation $\Delta_h : X \to V_h$ by: Given $\phi \in X$, $\Delta_h \phi$ is the unique solution in V_h to*

$$(\Delta_h \phi, v) = -(\nabla \phi, \nabla v) \;\; \forall v \in V_h. \tag{2.12}$$

It is easy to verify the discrete Laplacian is well defined. We are now able to define the discrete filter.

Definition 2.3 (Discrete Filter). *Define the discrete filtering operator $\overline{\cdot}^h : L^2(\Omega) \to V_h$ by: Given $\phi \in L^2(\Omega)$ and $\alpha > 0$, $\overline{\phi}^h$ is the unique solution in V_h to*

$$(\phi, v) = (\overline{\phi}^h, v) - \alpha^2 (\Delta_h \overline{\phi}^h, v) \;\; \forall v \in V_h. \tag{2.13}$$

It is easy to check that the discrete filtering operation is well-defined. Note that the discrete filter commutes with the discrete Laplacian; this property will be used in the later analysis. The filtering radius α is typically chosen to be on the order of the mesh width h. To choose α an order of magnitude smaller than h would be *exact* filtering, and would not sufficiently regularize the discrete model enough to distinguish it from a true NSE scheme. Thus the purpose of using a turbulence model would essentially be defeated. Conversely, choosing α much larger than the mesh width typically would give too much of a smoothing effect. As our analysis is mostly asymptotic, we choose $\alpha = h$ for simplicity, but choosing $\alpha = Ch$ would not change any results. Choosing α not on the order of the mesh size will reduce the asymptotic convergence rates.

We will use the following bounds of filtered quantities extensively in our analysis.

Lemma 2.4. *For $\phi \in X$, we have the following upper bounds:*

$$\|\overline{\phi}^h\| \leq \|\phi\| \tag{2.14}$$
$$\|\nabla \overline{\phi}^h\| \leq \|\nabla \phi\| \tag{2.15}$$

Proof. These results follow by choosing $v = \overline{\phi}^h$ and $v = \Delta_h \overline{\phi}_h$ in (2.13), respectively. ■

For a filtering radius $\alpha > 0$, there are two norms that naturally arise in the analysis of our scheme. Hence for notational convenience, we define the energy norm $\|\cdot\|_E$ and energy dissipation norm $\|\cdot\|_\epsilon$ on V_h to be

$$\|v\|_E := (v, \overline{v}^h)^{1/2} = \left(\|\overline{v}^h\|^2 + \alpha^2 \|\nabla \overline{v}^h\|^2\right)^{1/2} \tag{2.16}$$
$$\|v\|_\epsilon := (\nabla v, \nabla \overline{v}^h)^{1/2} = \left(\|\nabla \overline{v}^h\|^2 + \alpha^2 \|\Delta_h \overline{v}^h\|^2\right)^{1/2}. \tag{2.17}$$

If α is of the order of the mesh width, then these norms are equivalent in V_h to the L^2 and H^1 norms typically used for the analysis of fluid flow schemes. The following lemma proves this norm equivalance.

Lemma 2.5. *For fixed $\alpha = O(h)$, the natural energy norm of NS-α, $\|\cdot\|_E$, is equivalent to the usual L^2 norm in V_h: for $\phi \in V_h$, there exists C_E independent of h, ϕ satisfying*

$$\|\phi\|_E \leq \|\phi\| \leq C_E \|\phi\|_E \tag{2.18}$$

Additionally, the natural energy dissipation norm of NS-α, $\|\cdot\|_\epsilon$, is equivalent to the H^1 (the V_h) norm in V_h: there exists C_ϵ independent of h, ϕ satisfying

$$\|\phi\|_\epsilon \leq \|\nabla \phi\| \leq C_\epsilon \|\phi\|_\epsilon \tag{2.19}$$

Proof. For $\phi \in V_h$, it is easy to see by choosing $v = \overline{\phi}^h$ and $v = \Delta_h \overline{\phi}^h$ in the filter definition (2.13) that

$$\|\phi\|_E \leq \|\phi\| \text{ and } \|\phi\|_\epsilon \leq \|\nabla \phi\|. \tag{2.20}$$

For the reverse inequality for the energy norm, choose $v = \phi$ in (2.13), which gives

$$\|\phi\|^2 = (\phi, \overline{\phi}^h) - \alpha^2(\Delta_h \overline{\phi}^h, \phi). \leq \|\phi\|\|\overline{\phi}^h\| + \alpha^2\|\nabla\overline{\phi}^h\|\|\nabla\phi\| \leq (1 + C\frac{\alpha^2}{h^2})\|\phi\|\|\overline{\phi}^h\|. \tag{2.21}$$

Since $\alpha = h$, Young's inequality and the definition of $\|\cdot\|_E$ shows

$$\|\phi\|^2 \leq \|\overline{\phi}^h\|^2 \leq \|\phi\|_E^2. \tag{2.22}$$

A similar technique, but with $v = \Delta_h \phi$ and using $\|\Delta_h v\| \leq Ch^{-1}\|\nabla v\| \; \forall v \in V_h$, will prove (2.19). ■

The error analysis uses a discrete Gronwall inequality as in [16].

Lemma 2.6 (Discrete Gronwall Lemma). *Let Δt, H, and a_n, b_n, c_n, d_n (for integers $n \geq 0$) be nonnegative numbers such that*

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq \Delta t \sum_{n=0}^{l} d_n a_n + \Delta t \sum_{n=0}^{l} c_n + H \;\; for \; l \geq 0. \tag{2.23}$$

Suppose that $\Delta t d_n < 1 \; \forall n$. Then,

$$a_l + \Delta t \sum_{n=0}^{l} b_n \leq exp\left(\Delta t \sum_{n=0}^{l} \frac{d_n}{1 - \Delta t d_n}\right)\left(\Delta t \sum_{n=0}^{l} c_n + H\right) \;\; for \; l \geq 0. \tag{2.24}$$

The last notation we introduce is the discrete curl function on V_h, $curl_h$. It is defined as the projection of the curl onto the discretely divergence free discrete subspace V_h.

Definition 2.7 (Discrete curl). *The operator $curl_h : X \to V_h$ is defined by, for $\phi \in X$, $curl_h \phi$ is the unique solution in V_h to*

$$(curl_h \phi, v) = (\nabla \times \phi, v) \;\; \forall v \in V_h. \tag{2.25}$$

This definition of the discrete curl is the projection of the usual curl into V_h, and it is easy to see the $curl_h$ operator is well defined.

3 An energy and helicity preserving scheme for NS-α

3.1 The scheme

Section 2 has provided sufficient notation to now present the energy and helicity conserving scheme for NS-α. We use g^n to denote $g(n\Delta t)$. However, we will denote the average of a quantity at the n^{th} and $(n+1)^{st}$ timesteps with the $n + 1/2$ superscipt. That is, $g^{n+1/2} = \frac{g^{n+1}+g^n}{2}$.

Algorithm 3.1 (Energy and helicity conserving scheme for NS-α). *Given initial velocity $u_0 \in X$, forcing term $f \in X' \times (0, T]$, filtering radius $\alpha > 0$, end time T, and time step Δt, set $M = T\Delta t$ and find $u_h^m \in V_h$ for $n = 1, 2, ..., M$ satisfying*

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v) - (\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v) + \nu(\nabla u_h^{n+1/2}, \nabla v) = (f(t^{n+1/2}), v) \; \forall v \in V_h \tag{3.26}$$

$$(u_h^0 - u_0, \chi) = 0 \; \forall \chi \in V_h \tag{3.27}$$

Because the scheme is given in V_h, some of the implementation technicalities are hidden. For example, one would rarely (if ever) compute in V_h; instead one computes the equivalent scheme in (X_h, Q_h). For analysis purposes, however, it is much more convenient (and easier to read!) in the space V_h. Below we illustrate these and other subtleties.

In the continuous NS-α model, it is the filtered velocity that is constrained to be divergence free. However, to achieve stability and physical accuracy, it was found necessary that both velocity and filtered velocity be constrained to discretely divergence free. The discrete divergence free condition for velocity is achieved by restricting the solution space to V_h. Since the velocity solution will be divergence free and the filtering operation maps into V_h, the discretely filtered velocity solution will also be discretely divergence free. Another important technicality is the use of the discrete curl, which is a projection of the usual curl operator into V_h. This is the key feature of the scheme that allows for helicity conservation in addition to energy conservation by the nonlinearity. In the usual rotational form (without the projection), the nonlinearity will not necessarily conserve helicity, and thus in most cases will (nonphysically) create and dissipate it.

The scheme (3.26) is more expensive than the similar, typical Crank-Nicholson scheme without a projected curl or a filter into V_h. More iterations will be required at each Newton step in each time step. Whether the additional physical accuracy (and thus extra stability in some sense) provided by (3.26) is enough to justify the extra work is problem-dependent. Situations in which schemes that conserve additional integral invariants tend to be ones involving long time intervals, when enough time steps are run so that small discretization errors (arising from a nonlinearity creating helicity, for example) are able to grow large enough to alter the solution. Tests are underway to determine the best ways to compute (3.26), and in what situations it would be advantageous to use it.

3.2 Conservation Laws

Conservation laws for energy and helicity, although shown for the inviscid case, are what enforce physically accurate energy and helicity treatment in the viscous case. This is because for energy or helicity to be conserved in the absence of viscous and external forces means that the nonlinearity of the model does not create or dissipate these quantities, but cascades them from the large to the fine scales. This is physically important, as it has long been known that for true fluid flow, the nonlinearity is the mechanism responsible for cascading energy from large to fine scales, and *not* for creating or dissipating it. It has recently been discovered that the nonlinearity treats helicity in almost exactly the same manner, except for some differences between the cascades [11, 8]. Thus for a model and its numerical scheme to conserve these quantities in the inviscid case gives some physical relevance to that scheme's solution in the viscous case.

The discrete scheme (3.26) conserves both a model energy and helicity. The conserved model energy is $E_{NS\alpha} := \frac{1}{2}\|u\|_E^2$. This form of energy is what is conserved in the space filtered Navier-Stokes equations (SFNSE), when the α filter is used. Thus this definition of energy has physical significance if one considers NS-α as a closure model to the SFNSE. Helicity, on the other hand, is conserved by the scheme in its usual form, $H = (u, \nabla \times u)$.

Lemma 3.2. *In the absence of external and viscous forces, the scheme (3.26) conserves model energy and helicity. That is,*

$$E_{NS\alpha}(T) = \frac{1}{2}\|u_h^M\|_E^2 = \frac{1}{2}\|u_h^0\|_E^2 = E_{NS\alpha}(0)$$

and

$$H(T) = (u_h^M, \nabla \times u_h^M) = (u_h^0, \nabla \times u_h^0) = H(0).$$

Proof. We consider here the case of no external or viscous force, so we set $\nu = f = 0$ which reduces the scheme (3.26) to

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v) - (\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v) = 0 \;\; \forall v \in V_h \tag{3.28}$$

To prove energy conservation, set $v = \overline{u_h^{n+1/2}}^h$ in (3.28). This causes the nonlinear term to vanish, since the curl of two vectors is perpendicular to each of them, and yields

$$\frac{1}{\Delta t}(\|u_h^{n+1}\|_E^2 - \|u_h^n\|_E^2) = 0. \tag{3.29}$$

Multiplying through Δt, and summing from $n = 0$ to $M - 1$ completes the energy conservation proof. For helicity conservation, set $v = curl_h u_h^{n+1/2}$ in (3.28). Again the nonlinearity vanishes, and after multiplying through Δt, what remains is

$$(u_h^{n+1} - u_h^n, curl_h u_h^{n+1/2}) = 0. \tag{3.30}$$

Using the definition of $curl_h$ and the fact that u_h^{n+1} and u_h^n are in V_h, we have

$$0 = (u_h^{n+1} - u_h^n, curl_h u_h^{n+1/2}) = (u_h^{n+1} - u_h^n, \nabla \times u_h^{n+1/2}). \tag{3.31}$$

Since the curl operator is linear and is self-adjoint in X, we have

$$(u_h^{n+1}, \nabla \times u_h^{n+1}) = (u_h^n, \nabla \times u_h^n). \tag{3.32}$$

Summing over time steps completes the proof of helicity conservation. ■

3.3 Stability

As is typical in finite element scheme for fluid flow, inviscid energy conservation is the key to nonlinear stability. This scheme is no exception.

Lemma 3.3. *Solutions to the discrete scheme (3.26) satisfy*

$$\|u_h^M\|^2 + \sum_{n=0}^{M-1} \frac{\nu \Delta t}{2} \|\nabla u_h^{n+1/2}\|^2 \leq C(\Omega) \left(\sum_{n=0}^{M-1} \frac{\nu \Delta t}{2} \|f(t^{n+1/2})\|_*^2 + \|u_0\|^2 \right) \tag{3.33}$$

Proof. Setting $v = \overline{u_h(t^{n+1/2})}^h$ in (3.26) vanishes the nonlinearity and yields

$$\frac{1}{\Delta t} \left(\|u_h^{n+1}\|_E^2 - \|u_h^n\|_E^2\right) + \nu \left(\nabla u_h^{n+1/2}, \nabla \overline{u_h^{n+1/2}}^h \right) = \left(f(t^{n+1/2}), \overline{u_h^{n+1/2}}^h \right). \tag{3.34}$$

Bounding the right hand side with the dual norm of X and applying Young's inequality gives the follwoing.

$$\begin{aligned}
\frac{1}{\Delta t} \left(\|u_h(t^{n+1})\|_E^2 - \|u_h(t^n)\|_E^2\right) + \nu \|u_h^{n+1/2}\|_\epsilon^2 &= \left(f(t^{n+1/2}), \overline{u_h^{n+1/2}}^h \right) \\
&= \left(f(t^{n+1/2}), \overline{u_h^{n+1/2}}^h \right) \left(\frac{\|\nabla \overline{u_h^{n+1/2}}^h\|^2}{\|\nabla \overline{u_h^{n+1/2}}^h\|^2} \right) \\
&\leq \|f(t^{n+1/2})\|_* \|\nabla \overline{u_h^{n+1/2}}^h\| \\
&\leq \frac{1}{2\nu} \|f(t^{n+1/2})\|_*^2 + \frac{\nu}{2} \|\nabla \overline{u_h^{n+1/2}}^h\|^2 \\
&\leq \frac{1}{2\nu} \|f(t^{n+1/2})\|_*^2 + \frac{\nu}{2} \|u_h^{n+1/2}\|_\epsilon^2.
\end{aligned}$$

Subtract $\frac{\nu}{2}\|u_h^{n+1/2}\|_\epsilon^2$ from both sides, sum from $n = 0$ to $n = M - 1$ and multiply by Δt to get

$$\|u_h^M\|_E^2 + \sum_{n=0}^{M-1} \frac{\nu \Delta t}{2} \|u_h^{n+1/2}\|_\epsilon^2 \leq \sum_{n=0}^{M-1} \frac{\nu \Delta t}{2} \|f(t^{n+1/2})\|_*^2 + \|u_h^0\|_E^2 \tag{3.35}$$

Now using the equivalence of norms from Lemma 2.5 and that $\|u_h^0\| \leq \|u_0\|$ (by choosing $\chi = u_h^0$) in (3.27) completes the proof. ■

We now have shown that solutions to (3.26) are bounded by the data. Such a result is critical for proving that solutions to fluid flow schemes exist, and, in fact, similar techniques are used in the next subsection, where we prove solutions to (3.26) exist.

3.4 Existence of solutions to the scheme

To show that solutions to the discrete scheme exist, we construct a mapping in such a way that a fixed point of the map is the solution to (3.26) at the $(n+1/2)$ time step; an equivalent way would be for the $(n+1)^{st}$ time step, but the former is simpler to write and follow. We then show that the criteria of the Leray-Schauder fixed point theorem are satisfied. We begin by writing (3.26) in the following equivalent form:

$$\frac{2}{\Delta t}\left(u_h^{n+1/2}, v\right) - \left(\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v\right) + \nu\left(\nabla u_h(t^{n+1/2}), \nabla v\right)$$
$$= \left(f + \frac{2}{\Delta t}u^n, v\right) \quad \forall v \in V^h. \quad (3.36)$$

We construct the map by first considering the linear part of (3.36).

We begin by defining the solution operator to the linear problem equivalent to the linear part of (3.36).

Definition 3.4. *The operator $T(g) : V_H^* \to V^h$ is defined as the solution operator of*

$$\frac{2}{\Delta t}(\phi, v) + \nu(\nabla\phi, \nabla v) = (g, v)\,.$$

The following lemma insures that T is well-defined.

Lemma 3.5. *For $\nu, \Delta t > 0$ and given $g \in V^h$, there exists a unique $\phi \in V^h$ which satsifies*

$$\frac{2}{\Delta t}(\phi, v) + \nu(\nabla\phi, \nabla v) \quad = \quad (g, v) \;\; \forall v \in V_h, \quad (3.37)$$

Proof. Because (3.37) is linear, finite dimensional and symmetric, existence and uniqueness of solutions follows if only if the trivial solution solves the associated homogeneous problem. Indeed, setting $g = 0$, and choosing $v = \phi$ in (3.37) yields

$$\frac{2}{\Delta t}\|\phi\|^2 + \nu\|\nabla\phi\|^2 = 0, \quad (3.38)$$

which implies that $\phi = 0$. Thus zero data produces zero solution, and the proof is complete. ■

We also define the nonlinear operator N, that when composed with T, will create a map that has solutions to (3.36) as fixed points. We prove next that T is linear, bounded and continuous.

Lemma 3.6. *The solution operator T is linear, bounded, and continouous.*

Proof. As T is the solution operator of a linear problem, T is also linear. To determine T is bounded (and thus continuous since T is linear), begin by setting $v = \phi$ in (3.37). Using similar techniques as in the stability proof, we obtain

$$\frac{2}{\Delta t}\|\phi\|^2 + \frac{\nu}{2}\|\nabla\phi\|^2 \leq \frac{1}{\nu}\|g\|_*^2. \quad (3.39)$$

Thus

$$\|T\| = \sup_{g\in V_h^*}\frac{\|T(g)\|}{\|g\|_*} = \sup_{g\in V_h^*}\frac{\|\phi\|_{V_h}}{\|g\|_*} \leq C. \quad (3.40)$$

■

Definition 3.7. *Define the mapping $N(\phi) : V^h \to V_h^*$ by*

$$N(\phi) := f(t^{n+1/2}) + \frac{2}{\Delta t}u^n + \overline{\phi}^h \times curl_h \phi. \tag{3.41}$$

Lemma 3.8. *The operator N is bounded and continuous.*

Proof. To show N is bounded, we expand its definition and use Cauchy-Schwarz and Poincare inequalities. Note the C's below do not necessarily represent the same constant. We have

$$\begin{aligned}
\|N(\phi)\|_* &= \sup_{v\in V_h} \frac{N(\phi)}{\|\nabla v\|} && (3.42)\\
&\le \|f(t^{n+\frac{1}{2}})\|_* + \frac{C}{\Delta t}\|u_h^n\| + C\|\overline{\phi}^h \times curl_h\phi\| && (3.43)\\
&\le \|f(t^{n+\frac{1}{2}})\|_* + \frac{C}{\Delta t}\|u_h^n\| + C\|\phi\|\|\phi\|_\infty. && (3.44)
\end{aligned}$$

Hence, $\|N(\phi)\|_* < C$ since ϕ is fixed in V_h and norms are equivalent in finite dimension. To prove the continuity of N, suppose $\phi_k \to \phi$ in V^h. Then using the equivalence of norms in finite dimension and the fact that $\|\overline{v}^h\| \le \|v\|$ for $v \in V_h$, we have that

$$\begin{aligned}
\|N(\phi) - N(\phi_k)\|_* &\le \|\overline{\phi}^h \times \phi - \overline{\phi_k}^h \times \phi_k\|_* && (3.45)\\
&\le C\left(\|\overline{\phi}^h\|_\infty\|\phi-\phi_k\| + \|\overline{\phi-\phi_k}^h\|\|\phi_k\|_\infty\right) \le && (3.46)\\
&\le C\left(\|\phi-\phi_k\|\right) \to 0. && (3.47)
\end{aligned}$$

Thus the proof is complete. ■

Definition 3.9. *Define the operator $F : V_h \to V_h$ by F(y)=T(N(y)).*

The operator F is well defined because T and N are well defined. It is easy to see F maps as stated using the definitions of T and N. Also, F is compact because T and N are continuous and bounded.

Theorem 3.1. *Assume $\phi \in V_h$ and consider the family of fixed point problems $\phi = \lambda F(\phi),\ 0 \le \lambda \le 1$. A solution ϕ to any of these fixed point problems satisfies $\|\phi\| \le K$, independent of λ. By the Leray-Schauder principle, solutions to (3.37) exist.*

Proof. Suppose $\phi = \lambda F(\phi)$ and consider

$$\phi = \lambda F(\phi) = \lambda T(N(\phi)) = T(\lambda N(\phi)), \tag{3.48}$$

which implies

$$\frac{2}{\Delta t}(\phi, v) - \lambda(\overline{\phi}^h \times \phi, v) + \nu(\nabla\phi, \nabla v) = \lambda(f(t^{n+\frac{1}{2}}), v) + \frac{2\lambda}{\Delta t}(u_h^n, v)\ \forall v \in V_h$$

Choosing $v = \overline{\phi}^h$ gives that

$$\frac{1}{\Delta t}\|\phi\|_E^2 + \frac{\nu}{4}\|\phi\|_\epsilon^2 \le C\nu^{-1}\|f\|_*^2 + \|u_h^n\|_E^2. \tag{3.49}$$

Thus $\|\phi\|_{V_h} \le K$, independent of λ. ■

The scheme (3.26) has now been shown to conserve energy and helicity in the inviscid case, be unconditionally stable, and admit solutions. In the next section we provide a detailed error analysis for the scheme.

4 Convergence

Theorem 4.1. *Suppose (u(t),p(t)) solve the NSE on the periodic box $\Omega \times (0,T]$ with initial condition u_0, timestep Δt chosen sufficiently small, $M := \frac{T}{\Delta t}$, and filtering radius $\alpha = h$. Assume that $u_{tt} \in L^\infty(0,T;H^1)$, $u_{ttt} \in L^\infty(0,T;L^2)$, $u \in L^4(0,T;H^1) \cap L^4(0,T;W_2^3 \cap W_2^{k+1})$, $p \in L^2(0,T;W_1^k)$, and the initial condition $u(0) = u_0 \in W_2^k(\Omega)$. Let $u^{n+1/2} := \frac{u(t^{n+1})+u(t^n)}{2}$, let u_h^n denote the solution of the NS$-\alpha$ scheme (3.26), then the error in the velocity solution of the scheme is bounded by*

$$\|u(T) - u_h^M\|^2 + \nu\Delta t \sum_{n=0}^{M-1} \|\nabla(u^{n+1/2} - u_h^{n+1/2})\|^2 \le C(h^4 + \Delta t^4), \tag{4.50}$$

Remark 4.1. *For $k = 2$, the velocity error bound (4.50) is optimal, since the consistency error from the filtering reduces the error to α^2. The discrete pressure solution can be recovered by using the (X_h, Q_h) form of the scheme (3.26) with the velocity solution. Since the finite element spaces satisfy the inf-sup condition, the pressure solution will also be optimal, i.e. $O(\Delta t^2 + h^2)$. However, for $k > 2$, there is no improvement in convergence rate. This is because the filtering operation is $O(h^2)$, and thus one cannot do better.*

Remark 4.2. *Although for simplicity in the analysis we choose $\alpha = h$, the convergence result (4.50) will hold provided $\alpha = Ch$ where C is a constant independent of h.*

Remark 4.3. *The difference equation for this analysis is created by subtracting the scheme from the NSE at time $t^{n+1/2}$. However, in general $u(t^{n+1/2}) \neq \frac{u(t^{n+1})+u(t^n)}{2}$, and so we will adopt the notation $u^{n+1/2} := \frac{u(t^{n+1})+u(t^n)}{2}$ to allow for a clean analysis.*

Proof. We begin the proof by developing an error equation. This is done by multiplying the NSE at the $t^{n+1/2}$ time step by $v \in V_h$, integrating over the domain, writing the resulting equation in a form to match the scheme (3.26), and finally subtracting it from the scheme (3.26). Writing $e^j = u_h^j - u(t^j)$, we have

$$\frac{1}{\Delta t}(e^{n+1} - e^n, v) + \nu(\nabla e^{n+1/2}, \nabla v) + (\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2} - \overline{u^{n+1/2}}^h \times curl_h u^{n+1/2}, v)$$
$$= Interp(u,p,n,v) \;\; \forall v \in V^h, \tag{4.51}$$

where

$$Interp(u,p,n,v) := (u_t(t^{n+1/2}) - \frac{u(t^{n+1}) - u(t^n)}{\Delta t}, v) + (p(t^{n+1/2}), \nabla \cdot v)$$
$$- (u(t^{n+1/2}) \times (\nabla \times u(t^{n+1/2}) - curl_h u(t^{n+1/2})), v) - ((u(t^{n+1/2}) - \overline{u(t^{n+1/2})}^h) \times (\nabla \times u(t^{n+1/2})), v)$$
$$+ (\nabla(u(t^{n+1/2}) - u^{n+1/2}), \nabla v) + (u(t^{n+1/2}) \times (\nabla \times u(t^{n+1/2}) - u^{n+1/2} \times (\nabla \times u^{n+1/2}), v). \tag{4.52}$$

We decompose the error e^j such that $e^j = \phi_h^j - \eta^j$ where $\phi_h^j \in V_h$ and $\eta^j \in V_h^\perp$. The error equation (4.51) can now be rewritten as

$$\frac{1}{\Delta t}(\phi_h^{n+1} - \phi_h^n, v) + \nu(\nabla \phi_h^{n+1/2}, \nabla v) = \frac{1}{\Delta t}(\eta^{n+1} - \eta^n, v) + \nu(\nabla \eta^{n+1/2}, \nabla v)$$
$$- (\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2} - \overline{u^{n+1/2}}^h \times curl_h u^{n+1/2}, v) + Interp(u,p,n,v) \;\; \forall v \in V^h. \tag{4.53}$$

The nonlinear term in (4.53) is broken into error pieces using $ab - cd = a(b-d) + (a-c)d$. Applying this identity, and then decomposing the resulting error terms as described above yields

$$\frac{1}{\Delta t}(\phi_h^{n+1} - \phi_h^n, v) + \nu(\nabla \phi_h^{n+1/2}, \nabla v) = \frac{1}{\Delta t}(\eta^{n+1} - \eta^n, v) + \nu(\nabla \eta^{n+1/2}, \nabla v)$$
$$+ (\overline{u^{n+1/2}}^h \times curl_h \phi_h^{n+1/2}, v) - (\overline{u^{n+1/2}}^h \times curl_h \eta^{n+1/2}, v) - (\overline{\phi_h^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v)$$
$$+ (\overline{\eta^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v) + Interp(u,p,n,v) \;\; \forall v \in V^h. \tag{4.54}$$

Next we choose $v = \overline{\phi_h^{n+1/2}}^h$, and reduce using the definitions of $\|\cdot\|_E$ and $\|\cdot\|_\epsilon$, as well as the fact that $\overline{\phi_h^{n+1/2}}^h$ is perpendicular to η^{n+1},η^n, and $\overline{\phi_h^{n+1/2}}^h \times curl_h u(t^{n+1/2})$. This gives

$$\frac{1}{\Delta t}(\|\phi_h^{n+1}\|_E^2 - \|\phi_h^n\|_E^2) + \nu\|\phi_h^{n+1/2}\|_\epsilon^2 = \nu(\nabla\eta^{n+1/2}, \nabla\overline{\phi_h^{n+1/2}}^h) + (\overline{u^{n+1/2}}^h \times curl_h\phi_h^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)$$
$$-(\overline{u^{n+1/2}}^h \times curl_h\eta^{n+1/2}, \overline{\phi_h^{n+1/2}}^h) + (\overline{\eta^{n+1/2}}^h \times curl_h u_h^{n+1/2}, \overline{\phi_h^{n+1/2}}^h) + Interp(u,p,n,\overline{\phi_h^{n+1/2}}^h). \tag{4.55}$$

The terms on the right hand side of (4.55) can be bounded above using several common analytical results including:

- Cauchy-Schwarz Inequality
- Young's inequality
- For $a, b, c \in H^1(\Omega)$, $(a \times b, c) \le C(\Omega)\|a\|_{H^1}\|b\|_{H^0}\|c\|_{H^{1/2}}$.
- The equivalnce of norms via Lemma 2.5.
- The Poincare inequality is satisfied for zero mean functions in H^1.

Other standard inequalities are also used. We majorize each of these terms as follows:

$$\begin{aligned}\left|\nu(\nabla\eta^{n+1/2}, \nabla\overline{\phi_h^{n+1/2}}^h)\right| &\le \frac{\nu}{2}\|\nabla\eta^{n+1/2}\|^2 + \frac{\nu}{2}\|\nabla\overline{\phi_h^{n+1/2}}^h\|^2\\ &\le \frac{\nu}{2}\|\nabla\eta^{n+1/2}\|^2 + \frac{\nu}{2}\|\phi_h^{n+1/2}\|_\epsilon^2 \end{aligned} \tag{4.56}$$

$$\left|(\overline{u^{n+1/2}}^h \times curl_h\phi_h^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right|$$
$$\le C\|\nabla\overline{u^{n+1/2}}^h\|\|curl_h\phi_h^{n+1/2}\|\|\overline{\phi_h^{n+1/2}}^h\|_{H^{1/2}}$$
$$\le C\|\nabla u^{n+1/2}\|\|\nabla\phi_h^{n+1/2}\|\|\overline{\phi_h^{n+1/2}}^h\|^{1/2}\|\nabla\overline{\phi_h^{n+1/2}}^h\|^{1/2}$$
$$\le C\|\nabla u^{n+1/2}\|\|\phi_h^{n+1/2}\|_\epsilon\|\phi_h^{n+1/2}\|_E^{1/2}\|\phi_h^{n+1/2}\|_\epsilon^{1/2}$$
$$\le C\|\nabla u^{n+1/2}\|\|\phi_h^{n+1/2}\|_\epsilon^{3/2}\|\phi_h^{n+1/2}\|_E^{1/2} \tag{4.57}$$
$$\le \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-3}\|\nabla u^{n+1/2}\|^4\|\phi_h^{n+1/2}\|_E^2 \tag{4.58}$$

$$\left|(\overline{u^{n+1/2}}^h \times curl_h\eta^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right| \tag{4.59}$$
$$\le C\|\nabla\overline{u^{n+1/2}}^h\|\|curl_h\eta^{n+1/2}\|\|\nabla\overline{\phi_h^{n+1/2}}^h\| \tag{4.60}$$
$$\le \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}\|\nabla u^{n+1/2}\|^2\|\eta^{n+1/2}\|^2 \tag{4.61}$$

The last term in (4.55) takes the most work, as we split it into three terms by adding and subtracting the true solution to the computed solution. Each of these three terms gets analyzed similar to (4.58) and (4.61).

$$\begin{aligned}
&\left|(\overline{\eta^{n+1/2}}^h \times curl_h u_h^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq \left|(\overline{\eta^{n+1/2}}^h \times curl_h \phi_h^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right| + \left|(\overline{\eta^{n+1/2}}^h \times curl_h \eta^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right| \\
&+ \left|(\overline{\eta^{n+1/2}}^h \times curl_h u^{n+1/2}, \overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}\|\nabla\eta^{n+1/2}\|^4 + C\nu^{-1}\|\nabla u^{n+1/2}\|^2\|\eta^{n+1/2}\|^2 \\
&+ C\nu^{-3}\|\nabla\eta^{n+1/2}\|^4\|\phi_h^{n+1/2}\|_E^2 \qquad (4.62)
\end{aligned}$$

Combining (4.56)-(4.62) and (4.55) gives

$$\begin{aligned}
&\frac{1}{\Delta t}(\|\phi_h^{n+1}\|_E^2 - \|\phi_h^n\|_E^2) + \nu\|\phi_h^{n+1/2}\|_\epsilon^2 \leq \\
&\quad C((\nu^{-1}\|\nabla u^{n+1/2}\|^2 + \nu)\|\nabla\eta^{n+1/2}\|^2 + \nu^{-1}\|\nabla\eta^{n+1/2}\|^4 + \nu^{-3}\|\nabla u^{n+1/2}\|^4\|\phi_h^{n+1/2}\|_E^2 \\
&\quad + \nu^{-3}\|\nabla\eta^{n+1/2}\|^4\|\phi_h^{n+1/2}\|_E^2) + \left|Interp(u, p, n, \overline{\phi_h^{n+1/2}}^h)\right|. \qquad (4.63)
\end{aligned}$$

We next bound the Interp terms using standard inequalities. The time derivative term uses just a Taylor series expansion.

$$\begin{aligned}
&\left|(u_t(t^{n+1/2}) - \frac{u(t^{n+1}) - u(t^n)}{\Delta t}, \overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq \|u_t(t^{n+1/2}) - \frac{u(t^{n+1}) - u(t^n)}{\Delta t}\|\|\overline{\phi_h^{n+1/2}}^h\| \\
&\leq C\|u_t(t^{n+1/2}) - \frac{u(t^{n+1}) - u(t^n)}{\Delta t}\|^2 + \|\phi_h^{n+1/2}\|_E^2 \\
&\leq C\Delta t^4\|u_{ttt}\|_{L^\infty(L^2(\Omega)\times(0,T])}^2 + \|\phi_h^{n+1/2}\|_E^2 \qquad (4.64)
\end{aligned}$$

For the pressure term, recall that elements of V_h are discretely divergence free.

$$\begin{aligned}
\left|(p(t^{n+1/2}), \nabla\cdot\overline{\phi_h^{n+1/2}}^h)\right| &\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}\inf_{q\in Q_h}\|p(t^{n+1/2}) - q\|^2 \\
&\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}h^{2k}\left|p(t^{n+1/2})\right|_k^2 \qquad (4.65)
\end{aligned}$$

The third term in Interp is bounded using the fact that $curl_h$ is the projection of the curl operator onto V_h.

$$\begin{aligned}
&\left|(u(t^{n+1/2}) \times (\nabla\times u(t^{n+1/2}) - curl_h u(t^{n+1/2})), \overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}h^{2k+2}\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|_{k+1}^2 \qquad (4.66)
\end{aligned}$$

The fourth term in Interp is majorized using the difference made by the filter operation, which follows from standard finite element analysis.

$$\begin{aligned}
&\left|((u(t^{n+1/2}) - \overline{u(t^{n+1/2})}^h) \times (\nabla\times u(t^{n+1/2})), \overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq C\|\nabla u(t^{n+1/2}) - \overline{u(t^{n+1/2})}^h\|\|\nabla u(t^{n+1/2})\|\|\nabla\overline{\phi_h^{n+1/2}}^h\| \\
&\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}\|\nabla u(t^{n+1/2}) - \overline{u(t^{n+1/2})}^h\|^2\|\nabla u(t^{n+1/2})\|^2 \\
&\leq \frac{\nu}{100}\|\phi_h^{n+1/2}\|_\epsilon^2 + C\nu^{-1}h^4\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|_3^2 \qquad (4.67)
\end{aligned}$$

The fifth term in Interp can be bounded above using Taylor series, Cauchy-Schwarz and Young's inequalities.

$$
\begin{aligned}
&\left|(\nabla(u(t^{n+1/2}) - u^{n+1/2}), \nabla\overline{\phi_h^{n+1/2}}^h)\right| \\
&\leq \|\nabla(u(t^{n+1/2}) - \frac{u(t^n)+u(t^{n+1})}{2})\| \|\phi_h^{n+1/2}\|_\epsilon \\
&\leq C\Delta t^2 \|\nabla u_{tt}\|_{L^\infty(L^2(\Omega)\times(0,T])} \|\phi_h^{n+1/2}\|_\epsilon \\
&\leq C\nu^{-1}\Delta t^4 \|\nabla u_{tt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])} + \frac{\nu}{100}\|\phi_h^{n+1/2}\|^2_\epsilon
\end{aligned} \tag{4.68}
$$

The last term follows using the same tools as the fifth.

$$
\begin{aligned}
&\left|(u(t^{n+1/2}) \times (\nabla \times u(t^{n+1/2}) - u^{n+1/2} \times (\nabla \times u^{n+1/2}), \overline{\phi_h^{n+1/2}})\right| \\
&\leq \frac{\nu}{100} + C\nu^{-1}\Delta t^4 \|\nabla u_{tt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])} \|\nabla u^{n+1/2}\|^2 \\
&+ C\nu^{-1}\Delta t^4 \|\nabla u_{tt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])} \|\nabla u(t^{n+1/2})\|^2.
\end{aligned} \tag{4.69}
$$

Combining (4.63) with the bounds on Interp (4.64)-(4.69), and using the assumption that $\|\nabla u_{tt}\|_{L^\infty(L^2(\Omega)\times(0,T])}$ is bounded, yields

$$
\begin{aligned}
&\frac{1}{\Delta t}(\|\phi_h^{n+1}\|^2_E - \|\phi_h^n\|^2_E) + \nu\|\phi_h^{n+1/2}\|^2_\epsilon \leq \\
&\quad C((\nu^{-1}\|\nabla u^{n+1/2}\|^2 + \nu)\|\nabla\eta^{n+1/2}\|^2 + \nu^{-1}\|\nabla\eta^{n+1/2}\|^4 + \nu^{-1}h^{2k}\left|p(t^{n+1/2})\right|^2_k \\
&\quad + \nu^{-1}h^{2k+2}\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_{k+1} + \nu^{-1}h^4\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_3 \\
&\quad + \Delta t^4\|u_{ttt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])} + (\nu^{-3}\|u^{n+1/2}\|^4 + \nu^{-3}\|\nabla\eta^{n+1/2}\|^4 + 1)\|\phi_h^{n+1/2}\|^2_E \\
&\quad + \nu^{-1}\Delta t^4 + \nu^{-1}\Delta t^4\|\nabla u^{n+1/2}\|^2 + \nu^{-1}\Delta t^4\|\nabla u(t^{n+1/2})\|^2),
\end{aligned} \tag{4.70}
$$

which reduces to

$$
\begin{aligned}
&\frac{1}{\Delta t}(\|\phi_h^{n+1}\|^2_E - \|\phi_h^n\|^2_E) + \nu\|\phi_h^{n+1/2}\|^2_\epsilon \leq \\
&\quad C((\nu^{-1}\|\nabla u^{n+1/2}\|^2 + \nu)h^{2k}\left|u^{n+1/2}\right|^2_{k+1} + \nu^{-1}h^{4k}\left|u^{n+1/2}\right|^4_{k+1} + \nu^{-1}h^{2k}\left|p(t^{n+1/2})\right|^2_k \\
&\quad + \nu^{-1}h^{2k+2}\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_{k+1} + \nu^{-1}h^4\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_3 \\
&\quad + \Delta t^4\|u_{ttt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])} + (\nu^{-3}\|\nabla u^{n+1/2}\|^4 + \nu^{-3}h^{4k}\left|u^{n+1/2}\right|^4_{k+1} + 1)\|\phi_h^{n+1/2}\|^2_E \\
&\quad + \nu^{-1}\Delta t^4 + \nu^{-1}\Delta t^4\|\nabla u^{n+1/2}\|^2 + \nu^{-1}\Delta t^4\|\nabla u(t^{n+1/2})\|^2).
\end{aligned} \tag{4.71}
$$

Summing from $n = 0...M-1$ and multiplying through Δt gives

$$
\begin{aligned}
&\|\phi_h^M\|^2_E + \nu\Delta t\sum_{n=0}^{M-1}\|\phi_h^{n+1/2}\|^2_\epsilon \leq \|\phi_h^0\|^2_E + C\Delta t\sum_{n=0}^{M-1}\Delta t^4(\nu^{-1} + \|u_{ttt}\|^2_{L^\infty(L^2(\Omega)\times(0,T])}) \\
&\quad Ch^{2k}\Delta t\sum_{n=0}^{M-1}\left((\nu^{-1}\|\nabla u^{n+1/2}\|^2 + \nu)\left|u^{n+1/2}\right|^2_{k+1} + \nu^{-1}\left|p(t^{n+1/2})\right|^2_k\right) \\
&+ C\Delta t\sum_{n=0}^{M-1}\nu^{-1}h^{2k+2}\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_{k+1} + C\Delta t\sum_{n=0}^{M-1}\nu^{-1}h^4\|\nabla u(t^{n+1/2})\|^2\left|u(t^{n+1/2})\right|^2_3 \\
&+ C\Delta t\sum_{n=0}^{M-1}\nu^{-1}h^{4k}\left|u^{n+1/2}\right|^4_{k+1} + C\Delta t\sum_{n=0}^{M-1}(\nu^{-3}\|\nabla u^{n+1/2}\|^4 + \nu^{-3}h^{4k}\left|u^{n+1/2}\right|^4_{k+1} + 1)\|\phi_h^{n+1/2}\|^2_E).
\end{aligned} \tag{4.72}
$$

Holder's inequality, the smoothness assumptions on the true solution, assuming without loss of generality that $h < 1$ and choosing Δt small enough to apply the discrete Gronwall inequality reduces the error equation to

$$\|\phi_h^M\|_E^2 + \nu\Delta t \sum_{n=0}^{M-1} \|\phi_h^{n+1/2}\|_\epsilon^2 \leq C(\nu, T, u, p) \cdot (h^4 + \Delta t^4). \tag{4.73}$$

By Lemma 2.5, $||\cdot||$ is equivalent to $||\cdot||_E$ and $||\cdot||_{V_h}$ is equivalent to $||\cdot||_\epsilon$, independently of h in (4.73). Therefore by the triangle inequality,

$$\|u(T) - u_h^M\|^2 + \nu\Delta t \sum_{n=0}^{M-1} \|\nabla(u^{n+1/2} - u_h^{n+1/2})\|^2 \leq C(h^4 + \Delta t^4) \tag{4.74}$$

This completes the proof. ■

5 Extension to higher order NS-α deconvolution models

Recently a generalization of NS-α has been proposed in [23] that applies van Cittert approximate deconvolution to the NS-α model to create a family of models with arbitrarily high order consistency error. These models could also be considered as helicity-corrected Leray-deconvolution models, or Leray-deconvolution models corrected to restore frame invariance [24] (for more on Leray-deconvolution models, see [18]). The NS-α-deconvolution family of models are given by

$$\begin{aligned} v_t - D_N\overline{v} \times v + \nabla q - \nu\Delta v &= f, && (5.75)\\ \nabla \cdot D_N\overline{v} &= 0, && (5.76)\\ A\overline{v} := -\alpha^2\Delta\overline{v} + \overline{v} &= v && (5.77) \end{aligned}$$

where A is the inverse of the alpha filter, and D_N is the N^{th} van Cittert approximate deconvolution operator defined by

$$D_N\overline{\phi} := \sum_{n=0}^{N} (I - A^{-1})^n \overline{\phi}. \tag{5.78}$$

Note that the first three approximate deconvolution operators are

$$\begin{aligned} D_0\overline{\phi} &= \overline{\phi} \\ D_1\overline{\phi} &= 2\overline{\phi} - \overline{\overline{\phi}} \\ D_2\overline{\phi} &= 3\overline{\phi} - 3\overline{\overline{\phi}} + \overline{\overline{\overline{\phi}}} && (5.79)\\ & && (5.80) \end{aligned}$$

and from here it can be seen that van Cittert approximate deconvolution is actually extrapolation via the filtering operation. The following results about the operator D_N are known from [12, 5].

Lemma 5.1. *The N^{th} van Cittert deconvolution operator D_N is positive, self adjoint, and satisfies*

$$\|\phi - D_N\overline{\phi}\| \leq C(\Omega)\alpha^{2N+2}\|\phi\|_{H^{2N+2}}. \tag{5.81}$$

Thus for smooth ϕ, applying van Cittert approximate deconvolution to $\overline{\phi}$ recovers ϕ to a very high order. Note that if $N = 0$, (5.75) reduces to the NS-α model. For larger N, it can be inferred from Lemma 5.1 that the NS-α-deconvolution model can be a better approximation to true fluid flow, as the consistency error is $O(\alpha^{2N+2})$.

The idea of using van Cittert approximate deconvolution in turbulence modeling was pioneered by Stolz and Adams in [1, 2]. It was also used in [19] to create a higher-order Leray-α model, and its use was found to increase the order of convergence as well as produce more accurate simulations of transitional flow over a forward and backward facing step. NS-α, like Leray-α, is limited in its accuracy immediately by an $O(\alpha^2)$ filtering operation on the velocity term in its nonlinearity. Since NS-α is believed superior to Leray-α due to its excellent physical properties (e.g. the Leray-α model does not conserve helicity [23] and is not frame invariant [15]), adding deconvolution to NS-α will combine its excellent physical properties with the high accuracy gained by the use of van Cittert approximate deconvolution.

Because NS-α-deconvolution conserves both a model energy $\frac{1}{2}(v, D_N\overline{v})$ and helicity $(v, \nabla \times v)$, and takes a form similar to NS-α, the energy and helicity scheme presented and analyzed herein for NS-α can be extended to NS-α-deconvolution. We will present this scheme after we introduce some necessary notation. Define the N^{th} discrete approximate deconvolution operator, D_N^h by

$$D_N^h v = \sum_{n=0}^{N}(I - A_h^{-1})^n v, \tag{5.82}$$

where A_h^{-1} is a second, more convenient, notation for the discrete filter: $A_h^{-1}v = \overline{v}^h$. Then an energy and helicity conserving scheme for NS-α-deconvolution is given by

Algorithm 5.2 (Energy and helicity conserving scheme for NS-α-deconvolution). *Given initial velocity $u_0 \in X$, forcing $f \in X' \times (0,T]$, filtering radius $\alpha > 0$, end time T, and time step Δt, set $M = T\Delta t$ and find $u_h^m \in V_h$ for $n = 1, 2, ..., M$ satisfying*

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v) - (D_N^h\overline{u_h^{n+1/2}}^h \times curl_h u_h^{n+1/2}, v) + \nu(\nabla u_h^{n+1/2}, \nabla v) \;=\; (f(t^{n+1/2}), v)\ \forall v \in V_h \tag{5.83}$$

$$(u_h^0 - u_0, \chi) \;=\; 0\ \forall \chi \in V_h \tag{5.84}$$

Remark 5.3. *Note that this scheme is identical to that for NS-α except for the discrete deconvolution operator. The different divergence-free constraint of NS-α-deconvolution is handled without additional constraints since the filtering operation maps into V_h, and deconvolution operators are polynomials in the filter.*

For the $N = 0$ case, we have already proven that the scheme conserves energy and helicity, is stable, admits solutions, and converges optimally for Taylor-Hood elements. For each of these results, there are analogous results for $N > 0$. We begin with the conservation properties. We define the model's energy by $E_{D_N^h} := \frac{1}{2}(u_h, D_N^h\overline{u_h}^h)$ and energy dissipation $\epsilon_{D_N^h} := (\nabla u_h, \nabla D_N^h\overline{u_h}^h)$. Since $v \approx D_N^h\overline{v}^h$, it is reasonable to define the model's energy and energy dissipation in this manner (aside from the fact that these are the natural forms that arise in the analysis). We define now the associated model energy and energy dissipation norms.

Definition 5.4. *Define the operator $\|\cdot\|_{E;D_N^h}$ and $\|\cdot\|_{\epsilon;D_N^h}$ by*

$$\|v\|_{E;D_N^h} \;:=\; (v, D_N^h\overline{v}^h)^{1/2} \tag{5.85}$$

$$\|v\|_{\epsilon;D_N^h} \;:=\; (\nabla v, \nabla D_N^h\overline{v}^h)^{1/2} \tag{5.86}$$

Like in the $N = 0$ case, the naturally arising energy and energy dissipation norms are equivalent in V_h to the $L^2(\Omega)$ and X norms. We prove this now, as this result will make later analysis cleaner.

Lemma 5.5. *The operators $\|\cdot\|_{E;D_N^h}$ and $\|\cdot\|_{\epsilon;D_N^h}$ define norms on V_h equivalent to the L^2 and H^1 norms on V_h: there exists a constants C_0, C_1, C_2, C_3 independent of h satisfying*

$$C_0\|v\| \le \;\|v\|^2_{E;D_N^h}\; \le C_1\|v\| \tag{5.87}$$

$$C_2\|\nabla v\| \le \;\|v\|^2_{\epsilon;D_N^h}\; \le C_3\|\nabla v\| \tag{5.88}$$

Proof. To show that $\|\cdot\|_{E;D_N^h}$ defines a norm, consider first $(v, (I - A_h^{-1})^n A_h^{-1}v)$ for $n \ge 0$ and $v \in V_h$. If n is even, then

$$(v, (I - A_h^{-1})^n A_h^{-1}v) = ((I - A_h^{-1})^{n/2}v, (I - A_h^{-1})^{n/2}A_h^{-1}v) = \|(I - A_h^{-1})^{n/2}v\|_E^2. \tag{5.89}$$

If n is odd, we split the $(I - A_h^{-1})^n$ operator again, and use that A_h^{-1} is self-adjoint to get

$$
\begin{aligned}
&(v, (I - A_h^{-1})^n A_h^{-1} v) = ((I - A_h^{-1})^{(n-1)/2} v, (I - A_h^{-1})(I - A_h^{-1})^{(n-1)/2} A_h^{-1} v) \\
&= \|(I - A_h^{-1})^{(n-1)/2} v\|_E^2 - ((I - A_h^{-1})^{(n-1)/2} v, A_h^{-1}(I - A_h^{-1})^{(n-1)/2} A_h^{-1} v) \\
&= \|(I - A_h^{-1})^{(n-1)/2} v\|_E^2 - \|(I - A_h^{-1})^{(n-1)/2} A_h^{-1} v\|^2 \\
&= \alpha^2 \|\nabla (I - A_h^{-1})^{(n-1)/2} A_h^{-1} v\|^2.
\end{aligned} \tag{5.90}
$$

Since $\|v\|^2_{E;D_N^h} := (v, D_N^h \overline{v}^h)^{1/2} = \sum_{n=0}^N (v, (I - A_h^{-1})^n v$, we have that if N is odd,

$$
\|v\|^2_{E;D_N^h} = \sum_{n=0}^{(N-1)/2} \|(I - A_h^{-1})^n v\|^2 + 2\alpha^2 \|\nabla (I - A_h^{-1})^n v\|^2, \tag{5.91}
$$

and if N is even

$$
\|v\|^2_{E;D_N^h} = \sum_{n=0}^{N/2-1} \left(\|(I - A_h^{-1})^n v\|^2 + 2\alpha^2 \|\nabla (I - A_h^{-1})^n v\|^2 \right) + \|(I - A_h^{-1})^{N/2} v\|^2 + \alpha^2 \|\nabla (I - A_h^{-1})^{N/2} v\|^2. \tag{5.92}
$$

From (5.91)-(5.92) and the fact that A_h^{-1} is a positive operator we have that $\|v\|_{E;D_N^h}$ is in fact a norm. To prove it is equivalent to the L^2 norm on V_h, first we write it as

$$
\|v\|^2_{E;D_N^h} = \|v\|_E^2 + \sum_{n=1}^{N} (v, (I - A_h^{-1})^n A_h^{-1} v) \geq \|v\|_E^2, \tag{5.93}
$$

and so by Lemma 2.5,

$$
\|v\|^2_{E;D_N^h} \geq \|v\|_E^2 \geq C\|v\|^2, \tag{5.94}
$$

For the reverse inequality, since $\|A_h^{-1} v\| \leq \|v\|$, we have that

$$
(v, (I - A_h^{-1})^n A_h^{-1} v) = \|(I - A_h^{-1})^{n/2} v\|_E^2 \leq C \|(I - A_h^{-1})^{n/2} v\|^2 \leq C(N)\|v\|^2, \tag{5.95}
$$

and thus $\|v\|^2_{E;D_N^h} \leq C(N)\|v\|^2$. That $\|\cdot\|^2_{E;D_N^h}$ is a norm equilalent to $\|\cdot\|_{H^1}$ on V_h follows in an analogous manner, except uses that Δ_h commutes with A_h^{-1} and thus D_N^h. This completes the proof. ■

In the same manner as for the NS-α scheme (3.26), we can show the scheme for NS-α-deconvolution conserves energy and helicity, is stable, and admits solutions. The results follow analogously to the $N = 0$ case, and thus we state them without proof.

Lemma 5.6. *In the absence of viscous and external forces, solutions to the NS-α-deconvolution scheme (5.83) conserve a model energy and helicity. That is,*

$$
E_{D_N^h}(T) = \|u_h^M\|_{E;D_N^h} = \|u_h^0\|_{E;D_N^h} = E_{D_N^h}(0), \tag{5.96}
$$

$$
H(T) = (u_h^M, \nabla \times u_h^M) = (u_h^0, \nabla \times u_h^0) = H(0). \tag{5.97}
$$

The stability result for the NS-α-deconvolution scheme is the same as that for NS-α except the right hand side constant now depends on N (which is typically chosen less than five or six). The proof for this lemma follows exactly as in the $N = 0$ case (the NS-α scheme) except using NS-α-deconvolution's energy and energy dissipation norms. That solutions exist also follows in exactly the same manner, with the key step being that the scheme is stable.

Lemma 5.7. *Solutions to (5.83) exist, satisfy the a priori bound.*

$$
\|u_h^M\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \|u_h^{n+1/2}\|^2 \leq C(\Omega, N) \left(\|u_0\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \|f(t^{n+1/2})\|_*^2 \right) \tag{5.98}
$$

The advantage of using NS-α-deconvolution is to improve accuracy, and its advantage can be seen in the convergence result. When comparing the NS-α scheme ($N = 0$) to NS-α-deconvolution, there is only one term that is different. Hence the convergence analysis will be performed in exactly the same manner, except that instead of the term $\left|((u(t^{n+1/2}) - \overline{u(t^{n+1/2})}^h) \times (\nabla \times u(t^{n+1/2}), \overline{\phi_h^{n+1/2}}^h)\right|$, we must analyze $\left|((u(t^{n+1/2}) - D_N^h \overline{u(t^{n+1/2})}^h) \times (\nabla \times u(t^{n+1/2}), \overline{\phi_h^{n+1/2}}^h)\right|$ instead. A bound on the size of $\|\nabla(w - D_N^w \overline{w}^h)\|$ is given in [19], provided sufficient smoothness:

Lemma 5.8. *Let $w \in H^{2N+3}(\Omega) \cap H^{k+1}(\Omega)$. Then there exists C dependent on $|w|_{k+1}$ and $\|w\|_{H^{2N+3}}$ but independent of h satisfying*

$$\|\nabla(w - D_N^h \overline{w})\| \leq C(\alpha^{2N+2} + h^{k+1}) \tag{5.99}$$

This lemma provides the key step in the convergence analysis for the NS-α-deconvolution scheme (5.83). We state the convergence result now.

Theorem 5.1. *Let (u(t),p(t)) be NSE solutions on the periodic box $\Omega \times (0, T]$ satisfying, for N a nonnegative integer, initial condition u_0, timestep Δt chosen sufficiently small, $M := \frac{T}{\Delta t}$, and filtering radius $\alpha = h$. Assume $u_{ttt} \in L^\infty(0, T; L^2)$, $u_{tt} \in L^\infty(0, T; H^1)$, $u \in L^4(0, T; H^1) \cap L^4(0, T; W_2^3 \cap W_2^{k+1}) \cap L^{infty}(0, T; H^{2N+3})$, $p \in L^2(0, T; W_1^k)$, and the initial condition $u(0) = u_0 \in W_2^k(\Omega)$. Let u_h^n denote the solution of the NS$-\alpha$-deconvolution scheme (5.83), then the error in the velocity solution of the scheme is bounded by*

$$\|u(T) - u_h^M\|^2 + \nu \Delta t \sum_{n=0}^{M-1} \|\nabla(u^{n+1/2} - u_h^{n+1/2})\|^2 \leq C(h^k + \Delta t^4 + h^{2N+2}), \tag{5.100}$$

The advantage to NS-α-deconvolution is quite clear in the convergence theorem. For higher order elements ($k \geq 3$), NS-α-deconvolution ($N \geq 1$) will converge with higher order accuracy than NS-α ($N = 0$). For example, if $k = 4$, NS-α is limited to suboptimal $O(h^2 + \Delta t^2)$ accuracy, but NS-α-deconvolution with $N = 1$ for $k = 4$ achieves optimal $O(h^4 + \Delta t^2)$ accuracy.

6 Conclusions

In this article we have presented and analyzed a trapezoidal in time finite element scheme for computing solutions to the NS-α turbulence model. The scheme preserves the desirable physical properties of NS-α in that it conserves both energy and helicity. A key feature of the analysis for this model was that the naturally arising norms of the scheme are equivalent (in a finite element space) to the usual norms of finite element fluid flow schemes. Once this fact was established, it was shown that the scheme is unconditionally stable, admits solutions, and converges to a Navier-Stokes solution with optimal accuracy for Taylor-Hood elements.

A generalization of the scheme to the higher-order accurate NS-α-deconvolution model was also studied, whose base case (i.e. zeroth order model) is the energy and helicity preserving scheme for NS-α. Analogous to the base case, this model lent itself to analysis in naturally arising energy and energy dissipation norms. These norms were identified, and proven to be equivalant in the finite element space to the usual energy and energy dissipation norms. With the use of these norms, the generalized scheme was shown to conserve a model energy and helicity, be unconditionally stable, and admit solutions. The convergence analysis for this scheme shows that with the careful choice of filtering radius, for higher order elements, (P_k, P_{k-1}) with $k \geq 3$, optimal convergence rates can be obtained by appropriately increasing the order of the discrete deconvolution by balancing the model and discretization errors. This is an improvment to the $N = 0$ case (NS-α), where only suboptimal convergence rates could be achieved for higher order elements.

References

[1] N. A. Adams and S. Stolz. On the Approximate Deconvolution procedure for LES. *Phys. Fluids*, 2:1699–1701, 1999.

[2] N. A. Adams and S. Stolz. Deconvolution methods for subgrid-scale approximation in large eddy simulation. *Modern Simulation Strategies for Turbulent Flow*, 2001.

[3] A. Arakawa. Computational design for long-term numerical integration of the equations of fluid motion: two dimensional incompressible flow, part i. *J. Comput. Phys.*, 1:119–143, 1966.

[4] A. Arakawa and V. Lamb. A potential enstophy and energy conserving scheme for the shallow water equations. *Monthly Weather Review*, 109:18–36, 1981.

[5] L.C. Berselli, T. Iliescu, and W. J. Layton. *Mathematics of Large Eddy Simulation of Turbulent Flows*. Scientific Computation. Springer, 2006.

[6] S. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, 1994.

[7] A. Caglar. A finite element approximation of the Navier-Stokes-Alpha model. Technical report, University of British Columbia, Pacific Institute for the Mathematical Sciences, 2003.

[8] Q. Chen S. Chen and G. Eyink. The joint cascade of energy and helicity in three dimensional turbulence. *Physics of Fluids*, 15(2):361–374, 2003.

[9] S. Chen, C. Foias, D. Holm, E. Olson, E. Titi, and S. Wynne. The Camassa-Holm equations as a closure model for turbulent channel and pipe flow. *Phys. Rev. Lett.*, 81:5338–5341, 1998.

[10] J. Connors. Numerical analysis of a finite element scheme for the NS-alpha model of turbulence. Technical report, University of Pittsburgh, 2007.

[11] P. Ditlevson and P. Guiliani. Cascades in helical turbulence. *Physical Review E*, 63, 2001.

[12] A. Dunca and Y. Epshteyn. On the Stolz-Adams deconvolution model for the Large-Eddy simulation of turbulent flows. *SIAM J. Math. Anal.*, 37(6):1890–1902, 2005-2006.

[13] C. Foias, D. Holm, and E. Titi. The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory. *Journal of Dynamics and Differential Equations*, 14:1–35, 2002.

[14] M. Gavrilov and Z. Janji. Computed rotational energy spectra of two energy and enstrophy conserving schemes on semi-staggered grids. *Meteorology and Atmospheric Physics*, 41(1):1–4, 1989.

[15] J. L. Guermond, J. T. Oden, and S. Prudhomme. An interpretation of the Navier-Stokes-alpha model as a frame-indifferent Leray regularization. *Physica D Nonlinear Phenomena*, 177:23–30, mar 2003.

[16] J. Heywood and R. Rannacher. Finite element approximation of the nonstationary navier-stokes problem. part iv: Error analysis for the second order time discretization. *SIAM J. Numer. Anal.*, 2:353–384, 1990.

[17] C. Foias D. Holm and E. Titi. The Navier-Stokes-alpha model of fluid turbulence. *Physica D*, pages 505–519, May 2001.

[18] W. Layton and R. Lewandowski. A high accuracy Leray-deconvolution model of turbulence and its limiting behavior. *Analysis and Applications*, 6:1–27, 2008.

[19] W. Layton, C. Manica, M. Neda, and L. Rebholz. Numerical analysis and computational testing of a high-accuracy Leray-deconvolution model of turbulence. *Numerical Methods for Partial Differential Equations*, 24(2):555–582, 2008.

[20] J. Liu and W. Wang. Energy and helicity preserving schemes for hydro and magnetohydro-dynamics flows with symmetry. *Journal of Computational Physics*, 200:8–33, 2004.

[21] H. Moffatt and A. Tsoniber. Helicity in laminar and turbulent flow. *Annual Review of Fluid Mechanics*, 24:281–312, 1992.

[22] L. Rebholz. An energy and helicity conserving finite element scheme for the Navier-Stokes equations. *SIAM Journal on Numerical Analysis*, 45(4):1622–1638, 2007.

[23] L. Rebholz. A family of new high order NS-alpha models arising from helicity correction in Leray turbulence models. *Journal of Mathematical Analysis and Applications*, 342(1):246–254, 2008.

[24] L. Rebholz and S. Sussman. On the ns-alpha-deconvolution model of turbulent fluid flow. Technical report, University of Pittsburgh, 2008.